AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **72(2)** (2018), Pages 350–356

A lower bound for the minimal counter-example to Frankl's conjecture

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Abstract

Frankl's conjecture, from 1979, states that any finite union-closed family, containing at least one non-empty member set, must have an element which belongs to at least half of the member-sets. In this paper we show that if the minimum cardinality of $\bigcup\mathcal{A}$ over all counter-examples is q, then any counter-example family must contain at least $4q+1$ sets. As a consequence, we show that a minimal counter-example must contain at least 53 sets.

1 Introduction

A family of sets $\mathcal{A}$ is said to be union-closed if the union of any two member sets is also a member of $\mathcal{A}$. Frankl's conjecture (or the union-closed sets conjecture) states that if $\mathcal{A}$ is finite, then some element must belong to at least half of the sets in $\mathcal{A}$, provided at least one member set is non-empty. Although the origin of this conjecture is not explicit, it is generally attributed to Frankl (1979) following [5]. A detailed discussion and current standing of the conjecture can be found in [1].

In [3], Roberts and Simpson showed that if q is the minimum cardinality of $\bigcup\mathcal{A}$ over all counter-examples, then any counter-example $\mathcal{A}$ must satisfy the inequality $|\mathcal{A}| \geq 4q-1$. In this paper, we expand the ideas presented in [3] to find an improved lower bound $4q+1$. In [4], it was proved that a minimal counter-example must contain at least 13 elements in $\bigcup\mathcal{A}$. Hence we show that the minimal counter-example family must contain at least 53 sets.

ISSN: 2202-3518

2 Main results

2.1 Preliminary lemmas

Throughout this paper, $\mathcal{A}$ will denote a minimal counter-example with $|\bigcup\mathcal{A}| = q$, the minimum number of constituent elements across all counter-examples. Here $|\mathcal{A}|$ must be odd, because if it is even we can remove a *basis set* (a set that *cannot* be obtained by the union of any two other sets of $\mathcal{A}$) to generate a counter-example with $|\mathcal{A}| - 1$. Let $|\mathcal{A}| = 2n + 1$.

We denote the family of sets in $\mathcal{A}$ containing an element x by $\mathcal{A}_x$.

The universal set for $\mathcal{A}$ is defined by $S := \bigcup\mathcal{A}$. Thus $|S| = q$.

We define $\mathcal{A}_{\overline{x}} := \{A \in \mathcal{A} : x \notin A\}$. Let $C_x := \bigcup\mathcal{A}_{\overline{x}}$. We denote the family containing all such C_x by $\mathcal{C}$:

$$\mathcal{C} := \{C_x : x \in S\}.$$

For any x we define the family $\mathcal{D}_x$ to be

$$\mathcal{D}_x := \mathcal{A}_x \setminus \{S\} \setminus \mathcal{C}.$$

We now define and note the difference between the terms *abundant* and *abundance*. We call an element x *abundant* in a family $\mathcal{F}$ if $2|\mathcal{F}_x| \geq |\mathcal{F}|$. (By definition, our counterexample $\mathcal{A}$ cannot contain any abundant element.) On the other hand, we define *abundance* of x in $\mathcal{F}$ simply as $|\mathcal{F}_x|$.

Next, we define and distinguish the terms *mutually dominant* and *dominant*. We say that two elements a and b are *mutually dominating* if a and b always appear together in the member sets of $\mathcal{A}$. We say a *dominates* b if $\mathcal{A}_b \subset \mathcal{A}_a$ and $|\mathcal{A}_a| > |\mathcal{A}_b|$. Our counter-example family $\mathcal{A}$ *cannot* contain any *mutually dominating* pair of elements, since they can be replaced by a single element which in turn would violate the minimality of q. Therefore, for any $a, b \in S$, if $a \neq b$, then $C_a \neq C_b$. However, $\mathcal{A}$ may contain elements which *dominate* other elements.

Definition 1. *We define the sets I and J by:*

$$\begin{aligned} I &:= \{a \in S : a \text{ is } NOT \text{ dominated by any other element in } S\}; \\ J &:= \{b \in S : b \text{ is dominated by some other element in } S\}. \end{aligned}$$

If an element is present in n sets of $\mathcal{A}$, then it cannot be dominated by any other element. Hence they must be present in I. We know from [2] that $\mathcal{A}$ must contain at least three elements with abundance n. Thus $|I| \geq 3$. Note that every non-empty set in $\mathcal{A}$ must contain at least one element from I.

We now prove slightly modified versions of two lemmas from [3].

Lemma 1. *Let a be an element of S. If $a \notin I$ then $I \subseteq C_a$, and if $a \in I$ then $I \setminus \{a\} \subseteq C_a$.*

Proof. When $a \notin I$, let $y \in I$. Since a cannot dominate y, there must exist a set containing y but not a. So $y \in C_a$.

When $a \in I$, let $z \in I$ and $z \neq a$. Since a cannot dominate z, there must exist a set containing z but not a. So $z \in C_a$. But $a \notin C_a$ because $\bigcup \mathcal{A}_{\overline{a}}$ cannot contain a. □

So we conclude that if $a \in I$, then it must be present in $q - 1$ sets of $\mathcal{C}$.

Lemma 2. *For any a, C_a cannot be a basis set of $\mathcal{A}$.*

Proof. Let C_a be a basis. So we can remove C_a to get a new union-closed $\mathcal{A}'$ with $|\mathcal{A}'| = |\mathcal{A}| - 1$.

If $a \notin I$, then $I \subseteq C_a$ (Lemma 1). Since I must contain all elements with abundance n, removing C_a would generate another counter-example $\mathcal{A}'$ with $|\mathcal{A}'| < |\mathcal{A}|$, which is a contradiction.

If $a \in I$, then $I \setminus \{a\} \subseteq C_a$ (Lemma 1). Let B_a be a basis set containing a. Removing B_a and C_a from $\mathcal{A}$ we get $\mathcal{A}''$ with $|\mathcal{A}''| = |\mathcal{A}| - 2 = 2n - 1$, and no element is contained in more than $n - 1$ sets. Hence $\mathcal{A}''$ is also a counter-example, which is again a contradiction. □

Definition 2. *We say that elements a and b are mutually abundant if $2|\mathcal{A}_a \cap \mathcal{A}_b| \geq |\mathcal{A}_a|$ and $2|\mathcal{A}_a \cap \mathcal{A}_b| \geq |\mathcal{A}_b|$.*

Definition 3. *For every element a, we define the sets H_a and L_a as follows:*

$$\begin{aligned} H_a &:= \{b \in S : b \text{ is abundant in } \mathcal{A}_{\overline{a}}\}; \\ L_a &:= \{c \in S : c \text{ is abundant in } \mathcal{A}_a\}. \end{aligned}$$

We now prove a few lemmas which will be used repeatedly in the next section.

Lemma 3. *If $a, b \in I$, $b \in H_a$ and $\mathcal{D}_a \cap \mathcal{D}_b \neq \emptyset$, then $|\mathcal{A}| \geq 4q + 3$.*

Proof. Since $b \in H_a$, it must be present in at least $(n + 1)/2$ sets of $\mathcal{A}_{\overline{a}}$. Also $b \in S$ and b must be in $q - 2$ sets of $\mathcal{C} \setminus \{C_a\}$. It must also be present in at least one set of $\mathcal{D}_a$, since $\mathcal{D}_a \cap \mathcal{D}_b \neq \emptyset$. So we have

$$\frac{(n + 1)}{2} + 1 + (q - 2) + 1 \leq n,$$

which yields $|\mathcal{A}| \geq 4q + 3$. □

Lemma 4. *If $|\mathcal{A}_x| = |\mathcal{A}_y| = n$, $x \neq y$, then $y \in H_x$ or $y \in L_x$, but $y \notin H_x \cap L_x$.*

Proof. Suppose $y \notin H_x$ and $y \notin L_x$. Let us assume that n is even (say $n = 2k$). Since $y \notin L_x$, we have $|\mathcal{A}_x \cap \mathcal{A}_y| \leq k - 1$. Since $y \notin H_x$, we have $|\mathcal{A}_{\overline{x}} \cap \mathcal{A}_y| \leq k$. So $|\mathcal{A}_y| \leq k - 1 + k = n - 1$, a contradiction.

On the other hand, if n is odd (say $n = 2k+1$), since $y \notin L_x$, we have $|\mathcal{A}_x \cap \mathcal{A}_y| \leq k$. Since $y \notin H_x$, we have $|\mathcal{A}_{\overline{x}} \cap \mathcal{A}_y| \leq k$. So $|\mathcal{A}_y| \leq k + k = n - 1$, a contradiction again.

The case $y \in H_x \cap L_x$ is not possible because it will render y abundant in $\mathcal{A}$. □

Lemma 5. *If $|\mathcal{A}_x| = |\mathcal{A}_y| = n$ and $y \in H_x$, then $x \in H_y$.*

Proof. Since $y \in H_x$, we have $y \notin L_x$ from Lemma 4. So x and y cannot be mutually abundant (because $|\mathcal{A}_x| = |\mathcal{A}_y| = n$). Hence $x \notin L_y$. Thus, from Lemma 4, we have $x \in H_y$. □

Definition 4. *For any $x, y \in S$, we define*

$$\mathcal{A}_{\overline{xy}} := \mathcal{A}_{\overline{x}} \cap \mathcal{A}_{\overline{y}}; \qquad E_{xy} := \cup \mathcal{A}_{\overline{xy}}.$$

Note that $\mathcal{A}_{\overline{xy}}$ is union-closed.

Lemma 6. *If $x, y \in I$, then $E_{xy} \notin \mathcal{C}$.*

Proof. From Lemma 1, any $C_a \in \mathcal{C}$ must contain either I or $I \setminus \{a\}$. But E_{xy} can contain at most $I \setminus \{x\} \setminus \{y\}$. Hence $E_{xy} \notin \mathcal{C}$. □

As a corollary to the above lemma, note that $\mathcal{A}_{\overline{xy}}$ cannot contain any set from $\mathcal{C}$ when $x, y \in I$. Also $S \notin \mathcal{A}_{\overline{xy}}$, since S must contain both x and y.

Now we prove our central result, $|\mathcal{A}| \geq 4q + 1$. To do so, we divide the proof into the following two cases.

2.2 The case when $C_x \neq S \setminus \{x\}$ for some x

Theorem 1. *If there exists $x \in I$ such that $|\mathcal{A}_x| < n$, then $|\mathcal{A}| \geq 4q + 1$.*

Proof. We have $|\mathcal{A}_{\overline{x}}| \geq n + 2$. There must exist $y \in I$ abundant in $\mathcal{A}_{\overline{x}}$ (for if y is dominated by some z, then z would also be abundant in $\mathcal{A}_{\overline{x}}$ and we would then choose z instead of y). Hence y must be in at least $(n+2)/2$ sets of $\mathcal{A}_{\overline{x}}$. Since $y \in I$, y must be in $q - 2$ sets of $\mathcal{C} \setminus \{C_x\}$. Also $y \in S$. So we have

$$\frac{n+2}{2} + (q - 2) + 1 \leq n$$

which yields $|\mathcal{A}| \geq 4q + 1$. □

Theorem 2. *If $|\mathcal{A}_x| = n$ for all $x \in I$, then $|\mathcal{A}| \geq 4q + 1$.*

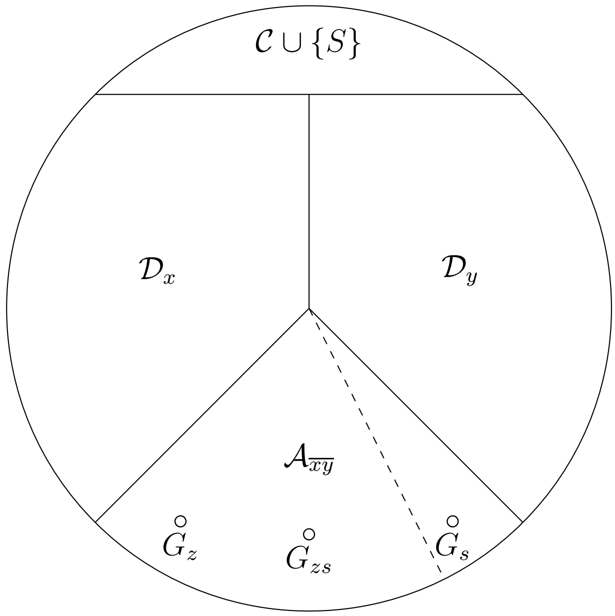

Figure 1: Representation of $\mathcal{A}$

Proof. Let $y \in I$ and $y \in H_x$. If $\mathcal{D}_x \cap \mathcal{D}_y \neq \emptyset$, then we immediately have $|\mathcal{A}| \geq 4q+3$ from Lemma 3. So let $\mathcal{D}_x \cap \mathcal{D}_y = \emptyset$. Then $|\mathcal{A}_{\overline{xy}}| = q$ (since $|\{S\}| = 1$, $|\mathcal{C}| = q$, $|\mathcal{D}_x| = |\mathcal{D}_y| = n - q$).

Since $\mathcal{A}_{\overline{xy}}$ is union closed, there must exist some $z \in I$ abundant in $\mathcal{A}_{\overline{xy}}$. We choose z as the element with *maximum abundance* in $\mathcal{A}_{\overline{xy}}$. If z is present in all q sets of $\mathcal{A}_{\overline{xy}}$, then we have $|\mathcal{A}_z| \geq 2q$ (since z must be in q sets of $\mathcal{C} \cup \{S\}$). This yields $|\mathcal{A}| \geq 4q + 1$.

So let z be present in at most $q - 1$ sets of $\mathcal{A}_{\overline{xy}}$. Hence there must exist $s \in I$ present in $\mathcal{A}_{\overline{xy}} \setminus \mathcal{A}_z$. Consequently, there exists $G_s \in \mathcal{A}_{\overline{xy}}$ such that $s \in G_s$ and $z \notin G_s$. Since z is maximal in $\mathcal{A}_{\overline{xy}}$, s must also be present in at most $q - 1$ sets of $\mathcal{A}_{\overline{xy}}$. So there must exist $G_z \in \mathcal{A}_{\overline{xy}}$ such that $z \in G_z$ and $s \notin G_z$. Also, since $\mathcal{A}_{\overline{xy}}$ is union-closed, there exists $G_{zs} \in \mathcal{A}_{\overline{xy}}$ such that $z \in G_{zs}$ and $s \in G_{zs}$. We summarize this as follows.

$$\begin{aligned} z \in G_z \quad &\text{and} \quad s \notin G_z; \\ s \in G_s \quad &\text{and} \quad z \notin G_s; \\ s \in G_{zs} \quad &\text{and} \quad z \in G_{zs}; \end{aligned}$$

where $G_z, G_s, G_{zs} \in \mathcal{A}_{\overline{xy}}$.

Our set-up is depicted in Figure 1.

By the hypothesis of this theorem, we have $|\mathcal{A}_x| = |\mathcal{A}_y| = |\mathcal{A}_z| = n$. Therefore, applying Lemma 4, we have the following three sub-cases:

(a) $z \in H_x$:

We consider the family $\mathcal{A}_{\overline{sy}}$. There exists a basis B_x, where $x \in B_x$ and $s \notin B_x$, since s cannot dominate x. Since $\mathcal{D}_x \cap \mathcal{D}_y = \emptyset$, $y \notin B_x$. Hence $B_x \in \mathcal{A}_{\overline{sy}}$. Since $G_z \in \mathcal{A}_{\overline{xy}}$, $y \notin G_z$. Also $s \notin G_z$. Therefore $G_z \in \mathcal{A}_{\overline{sy}}$.

Since B_x and G_z are in $\mathcal{A}_{\overline{sy}}$, we have $x, z \in E_{sy}$. From Lemma 6, $E_{sy} \notin \mathcal{C}$. Hence $E_{sy} \in \mathcal{D}_x \cap \mathcal{D}_z$. Thus, since $\mathcal{D}_x \cap \mathcal{D}_z \neq \emptyset$ and $z \in H_x$, we have $|\mathcal{A}| \geq 4q + 3$ from Lemma 3.

(b) $z \in H_y$:

The proof is similar to case (a), but with the roles of x and y reversed.

(c) $z \in L_x$ and $z \in L_y$:

Here $z \in L_x$ implies $x \in L_z$, since $|\mathcal{A}_x| = |\mathcal{A}_z| = n$. Similarly, since $z \in L_y$, we have $y \in L_z$. Therefore, we have $x, y \notin H_z$ from Lemma 4. Since $x, y \notin H_z$, let $r \in I$ be an element of H_z.

If r is present in any set of $\mathcal{A}_{\overline{xy}}$, then we have a set $G_{rz} \in \mathcal{A}_{\overline{xy}}$ containing both r and z, since $\mathcal{A}_{\overline{xy}}$ is union-closed. Since $G_{rz} \notin \mathcal{C}$, we have $G_{rz} \in \mathcal{D}_r \cap \mathcal{D}_z$. Therefore we have $|\mathcal{A}| \geq 4q + 3$ from Lemma 3, since $r \in H_z$ and $\mathcal{D}_r \cap \mathcal{D}_z \neq \emptyset$.

Let us assume that r is not in any sets of $\mathcal{A}_{\overline{xy}}$. So $D_r \subset \mathcal{D}_x \cup \mathcal{D}_y$. Since r cannot be dominated by s, there must exist a basis B_r such that $r \in B_r$ and $s \notin B_r$.

If $B_r \in \mathcal{D}_x$, then $B_r \in \mathcal{A}_{\overline{sy}}$ (because $y \notin B_r$, since $\mathcal{D}_x \cap \mathcal{D}_y = \emptyset$). Also, $G_z \in \mathcal{A}_{\overline{sy}}$. So $z, r \in E_{sy} \notin \mathcal{C}$.

If $B_r \in \mathcal{D}_y$, then $B_r \in \mathcal{A}_{\overline{sx}}$ (because $x \notin B_r$, since $\mathcal{D}_x \cap \mathcal{D}_y = \emptyset$). Also, $G_z \in \mathcal{A}_{\overline{sx}}$. So $z, r \in E_{sx} \notin \mathcal{C}$.

So at least one of E_{sx} and E_{sy} must be present in $\mathcal{D}_r \cap \mathcal{D}_z$. Therefore, we have $|\mathcal{A}| \geq 4q + 3$ from Lemma 3, since $r \in H_z$ and $\mathcal{D}_r \cap \mathcal{D}_z \neq \emptyset$. ☐

2.3 The case when $C_x = S \setminus \{x\}$ for all x

In this case, no element can be dominated by any other element. Thus all elements must be present in $q - 1$ sets of $\mathcal{C}$.

Theorem 3. *If there exists x such that $|\mathcal{A}_x| < n$, then $|\mathcal{A}| \geq 4q + 1$.*

Proof. The proof is similar to that of Theorem 1. We have $|\mathcal{A}_{\overline{x}}| \geq n+2$. Let $y \in H_x$. So y must be in at least $(n + 2)/2$ sets of $\mathcal{A}_{\overline{x}}$. It must be in $q - 2$ sets of $\mathcal{C} \setminus \{C_x\}$. Also, $y \in S$. So $(n + 2)/2 + (q - 2) + 1 \leq n$, which yields $|\mathcal{A}| \geq 4q + 1$. ☐

Theorem 4. *If for all x, $|\mathcal{A}_x| = n$, then $|\mathcal{A}| \geq 4q + 1$.*

Proof. Since $|\mathcal{A}_x| = n$ for all x, no element can dominate any other element. Therefore $I = S$. Since, in the proof of Theorem 2, we did not consider any element from J, this just becomes a special case of Theorem 2. ☐

Corollary 1. *The minimal counter-example to Frankl's conjecture must contain at least* 53 *sets.*

Proof. Combining Theorems 1, 2, 3 and 4, we obtain $|\mathcal{A}| \geq 4q + 1$. Since it is shown in [4] that $q \geq 13$, we have $|\mathcal{A}| \geq 53$. □

Acknowledgements

I deeply appreciate the valuable suggestions from the referees and editors.

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(Received 21 Feb 2018; revised 18 June 2018)