AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **72(2)** (2018), Pages 369–375

An m-ary partition generalization of a past Putnam problem

Timothy B. Flowers

Department of Mathematics
Indiana University of Pennsylvania
Indiana, PA 15705
U.S.A.
flowers@iup.edu

Scott Neville

Department of Mathematics
University of Utah
Salt Lake City, UT 84112
U.S.A.
drop.scott.n@gmail.com

James A. Sellers

Department of Mathematics
The Pennsylvania State University
University Park, PA 16802
U.S.A.
sellersj@psu.edu

Abstract

The following problem appeared in the 44th William Lowell Putnam Mathematical Competition: "For positive integers n, let $C(n)$ be the number of representations of n as a sum of nonincreasing powers of 2, where no power can be used more than three times ... Prove or disprove that there is a polynomial $P(x)$ such that $C(n) = \lfloor P(n) \rfloor$ for all positive integers n." We use generating functions to generalize this problem to enumerating a two-parameter family of m-ary integer partitions, $b^*_{m,j}(n)$. In addition, we use generating functions and a bijection to give an identity between $b^*_{m,j}(n)$ and another family of m-ary partitions.

ISSN: 2202-3518

1 Introduction

Problem B2 of the 44th Putnam Exam in 1983 asked participants to find a function to count the number of ways of representing n as a sum of powers of 2 with no power being used more than three times [1]. To restate in terms of integer partitions, the challenge was to find an expression for the number of binary partitions of n wherein each part is used at most three times.

It is natural to extend the Putnam question to partitions into powers of m, although there are several options for how one might pose an analogous restriction on the maximum number of times each part is allowed to appear. With this in mind, we define a two-parameter family of m-ary partition functions, $b^*_{m,j}(n)$, which enumerates the partitions of n where each part is a power of m and each part is used at most $m^j - 1$ times. This choice is motivated by the following observation of Rucci [3] in her 2016 Master's thesis:

$$b^*_{m,2}(n) = \left\lfloor \frac{n}{m} \right\rfloor + 1 \ . \tag{1}$$

To see why this is true, let $B^*_{m,2}(q)$ be the generating function for $b^*_{m,2}(n)$,

$$B^*_{m,2}(q) = \sum_{n=0}^{\infty} b^*_{m,2}(n) q^n = \prod_{k=0}^{\infty} \left(1 + (q^{m^k}) + (q^{m^k})^2 + \ldots + (q^{m^k})^{m^2-1}\right) . \tag{2}$$

Now, we rewrite the generating function as

$$\begin{aligned} B^*_{m,2}(q) &= \prod_{k=0}^{\infty} \frac{1-(q^{m^k})^{m^2}}{1-q^{m^k}} \\ &= \frac{1-q^{m^2}}{1-q^{m^0}} \cdot \frac{1-q^{m^3}}{1-q^{m^1}} \cdot \frac{1-q^{m^4}}{1-q^{m^2}} \cdot \frac{1-q^{m^5}}{1-q^{m^3}} \cdot \ldots \\ &= \frac{1}{1-q} \cdot \frac{1}{1-q^m} \\ &= \left(1+q+q^2+q^3+\ldots\right) \cdot \left(1+q^m+q^{2m}+q^{3m}+\ldots\right) . \end{aligned}$$

When the right hand side is expanded, there will exist an integer a such that the degree n term is

$$q^n \cdot 1 + q^{n-m} \cdot q^m + q^{n-2m} \cdot q^{2m} + q^{n-3m} \cdot q^{3m} + \ldots + q^{n-am} \cdot q^{am} = (a+1)q^n \ .$$

Here, a will be as large as possible such that $n - am \geq 0$, so we may conclude that $a = \lfloor n/m \rfloor$. Since the coefficient of q^n is $b^*_{m,2}(n)$, this verifies the claim (1) above.

Using (1), we note that the answer to the Putnam problem is

$$b^*_{2,2}(n) = \left\lfloor \frac{n}{2} \right\rfloor + 1 \ .$$

Although this approach is different than the published solution technique [1], the generating functions above are suggestive of a further extension. Thus, we add the second parameter j and now consider the function $b^*_{m,j}(n)$, which we defined above. In Section 2, we use generating functions to show that these partition functions relate to another family of m-ary partitions. We present a straightforward bijection of this relationship in Section 3 and provide illustrative examples. We close in Section 4 by returning to the Putnam problem and generalizing Rucci's result (1).

2 An m-ary identity from generating functions

Rather than restricting m-ary partitions by the number of times a part may be used, we may instead bound the sizes of parts that may be used. Rødseth and Sellers [2] defined $b_{m,j}(n)$ to be the number of m-ary partitions of n with the largest part at most m^{j-1}. While proving congruences for this partition function, they note that $B_{m,j}(q)$, the generating function for $b_{m,j}(n)$, is

$$B_{m,j}(q) = \sum_{n=0}^{\infty} b_{m,j}(n)q^n = \prod_{0\le k<j} \frac{1}{1-q^{m^k}}. \tag{3}$$

Following an argument similar to that in the previous section, we have the following identity:

Theorem 1. *Let $m \ge 2$ and $j \ge 1$. Then, for all n, $b^*_{m,j}(n) = b_{m,j}(n)$.*

Proof. We begin with the generating function $B^*_{m,j}(q)$ from (2). Note that

$$\begin{aligned}
B^*_{m,j}(q) &= \prod_{k=0}^{\infty}\left(1+(q^{m^k})+(q^{m^k})^2+\ldots+(q^{m^k})^{m^2-1}\right)\\
&= \left(1+q+q^2+\ldots+q^{m^j-1}\right)\cdot\left(1+(q^m)+(q^m)^2+\ldots+(q^m)^{m^j-1}\right)\cdot\\
&\quad \left(1+(q^{m^2})+(q^{m^2})^2+\ldots+(q^{m^2})^{m^j-1}\right)\cdot\ldots\\
&= \frac{1-q^{m^j}}{1-q}\cdot\frac{1-(q^m)^{m^j}}{1-q^m}\cdot\frac{1-(q^{m^2})^{m^j}}{1-q^{m^2}}\cdot\ldots\\
&= \prod_{0\le k<j}\frac{1}{1-q^{m^k}}\\
&= B_{m,j}(q).
\end{aligned}$$

Thus, since the generating functions are equal, we conclude that $b^*_{m,j}(n) = b_{m,j}(n)$ for all $n \ge 0$, $m \ge 2$, and $j \ge 1$. □

3 A bijective proof of Theorem 1

While the generating function proof of Theorem 1 mentioned above is satisfying, it would be more illuminating to provide a bijective proof of Theorem 1. We now provide such a proof.

Proof. Let

$$f_0^* m^0 + f_1^* m^1 + \ldots + f_r^* m^r \tag{4}$$

be a partition of n counted by $b^*_{m,j}(n)$ where f_i^* is the frequency each part appears, $0 \le f_i^* \le m^j - 1$. Note that the largest part that may be used in any such partition is m^r where $r = \lfloor \log_m(n) \rfloor$. Also, each frequency f_i^* may be written in base m using no more than j digits (up to the m^{j-1} power). So, we write each f_i^* in base m as follows:

$$\begin{aligned}
f_0^* &= a_{01}m^0 + a_{02}m^1 + \ldots + a_{0j}m^{j-1} \\
&\vdots \\
f_i^* &= a_{i1}m^0 + a_{i2}m^1 + \ldots + a_{ij}m^{j-1} \\
&\vdots \\
f_r^* &= a_{r1}m^0 + a_{r2}m^1 + \ldots + a_{rj}m^{j-1}
\end{aligned}$$

where $0 \le a_{i\ell} \le m - 1$.

Next, we form the matrix A, an $(r+1) \times j$ matrix with the $a_{i\ell}$ coefficients, so that

$$A = \begin{pmatrix} a_{01} & a_{02} & \ldots & a_{0j} \\ a_{11} & a_{12} & \ldots & a_{1j} \\ & & \vdots & \\ a_{r1} & a_{r2} & \ldots & a_{rj} \end{pmatrix}.$$

This means each row of A gives the base m digits (in reverse order) of the frequency that a part appears in the partition (4) above. Thus, each of the partitions enumerated by $b^*_{m,j}(n)$ may be uniquely expressed as an $(r+1) \times j$ matrix with all entries between 0 and $m-1$ (although, not all matrices of this type give such a partition of n).

Now, we take n as written above

$$\begin{aligned}
n = {} & f_0^* m^0 + f_1^* m^1 + \ldots + f_r^* m^r \\
= {} & (a_{01}m^0 + a_{02}m^1 + \ldots + a_{0j}m^{j-1}) \cdot m^0 \\
& + (a_{11}m^0 + a_{12}m^1 + \ldots + a_{1j}m^{j-1}) \cdot m^1 \\
& + \ldots \\
& + (a_{r1}m^0 + a_{r2}m^1 + \ldots + a_{rj}m^{j-1}) \cdot m^r
\end{aligned}$$

and rearrange as

$$\begin{aligned}
n = {} & (a_{01}m^0 + a_{11}m^1 + a_{21}m^2 + \ldots + a_{r1}m^r) \cdot m^0 \\
& + (a_{02}m^0 + a_{12}m^1 + a_{22}m^2 + \ldots + a_{r2}m^r) \cdot m^1 \\
& + \ldots \\
& + (a_{0j}m^0 + a_{1j}m^1 + a_{2j}m^2 + \ldots + a_{rj}m^r) \cdot m^{j-1} \\
= {} & f_0 m^0 + f_1 m^1 + \ldots + f_{j-1} m^{j-1},
\end{aligned}$$

where we define $f_{i-1} = a_{0i}m^0 + a_{1i}m^1 + \ldots + a_{ri}m^r$ for i from 1 to j. This gives an m-ary partition of n, with new frequencies for the parts and the largest part at most m^{j-1}, meaning this is one of the partitions counted by $b_{m,j}(n)$. We take these new frequencies in base m, with digits in reverse order, and put these base m coefficients into a matrix:

$$\begin{pmatrix} a_{01} & a_{11} & \ldots & a_{r1} \\ a_{02} & a_{12} & \ldots & a_{r2} \\ & & \vdots & \\ a_{0j} & a_{1j} & \ldots & a_{rj} \end{pmatrix}.$$

We note that this matrix is of size $j \times (r+1)$ and is equal to A^t. Further, we observe that any m-ary partition of n with largest part at most m^{j-1} may be uniquely expressed by a $j \times (r+1)$ matrix of this form with entries between 0 and $m-1$.

Thus, each partition of n counted by $b^*_{m,j}(n)$ has a corresponding matrix which is mapped via matrix transposition to a matrix corresponding to a partition of the same weight n counted by $b_{m,j}(n)$. This mapping is injective and is invertible since matrix transposition is invertible. Therefore, there is a bijection between the two types of partitions, which means $b^*_{m,j}(n) = b_{m,j}(n)$. □

We now provide two examples to demonstrate the ideas of this bijective proof of Theorem 1.

Example 1. *Let $m = 2$, $j = 2$, and $n = 17$.*

We consider a partition of 17 into powers of 2 with each part used at most $3 = 2^2 - 1$ times. So, $r = \lfloor \log_2(17) \rfloor = 4$. We choose the particular partition

$$17 = 1 \cdot 2^0 + 0 \cdot 2^1 + 2 \cdot 2^2 + 1 \cdot 2^3 + 0 \cdot 2^4. \tag{5}$$

Each frequency is then written into base 2 and we form matrix A with the coefficients:

$$\begin{matrix} 1 & = & 1 \cdot 2^0 + 0 \cdot 2^1 \\ 0 & = & 0 \cdot 2^0 + 0 \cdot 2^1 \\ 2 & = & 0 \cdot 2^0 + 1 \cdot 2^1 \\ 1 & = & 1 \cdot 2^0 + 0 \cdot 2^1 \\ 0 & = & 0 \cdot 2^0 + 0 \cdot 2^1 \end{matrix} \quad \Longrightarrow \quad A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Next, we transpose the matrix, and interpret the rows as frequencies written in base 2, with digits in reverse order:

$$A^t = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \quad \Longrightarrow \quad \begin{matrix} 1 \cdot 2^0 + 0 \cdot 2^1 + 0 \cdot 2^2 + 1 \cdot 2^3 + 0 \cdot 2^4 & = & 9 \\ 0 \cdot 2^0 + 0 \cdot 2^1 + 1 \cdot 2^2 + 0 \cdot 2^3 + 0 \cdot 2^4 & = & 4 \end{matrix}.$$

We now write the binary partition

$$17 = 9 \cdot 2^0 + 4 \cdot 2^1, \tag{6}$$

which is a partition into powers of 2, with largest part at most $2^{2-1} = 2$. Thus, the bijection maps the partition (5) to the partition (6).

Example 2. *Let $m = 5$, $j = 3$, and $n = 845$.*

Here, $r = \lfloor \log_5(845) \rfloor = 4$. We begin with a partition of 845 into powers of 5 with each part used at most $5^3 - 1 = 124$ times. In particular,

$$845 = 5 \cdot 5^0 + 3 \cdot 5^1 + 8 \cdot 5^2 + 0 \cdot 5^3 + 1 \cdot 5^4. \tag{7}$$

Next, write each frequency into base 5 and use the coefficients to form matrix A:

$$\begin{array}{rcl} 5 &=& 0 \cdot 5^0 + 1 \cdot 5^1 + 0 \cdot 5^2 \\ 3 &=& 3 \cdot 5^0 + 0 \cdot 5^1 + 0 \cdot 5^2 \\ 8 &=& 3 \cdot 5^0 + 1 \cdot 5^1 + 0 \cdot 5^2 \\ 0 &=& 0 \cdot 5^0 + 0 \cdot 5^1 + 0 \cdot 5^2 \\ 1 &=& 1 \cdot 5^0 + 0 \cdot 5^1 + 0 \cdot 5^2 \end{array} \implies A = \begin{pmatrix} 0 & 1 & 0 \\ 3 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Now we transpose the matrix, and interpret the rows as frequencies written in base 2, with digits in reverse order:

$$A^t = \begin{pmatrix} 0 & 3 & 3 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \implies \begin{array}{rcl} 0 \cdot 5^0 + 3 \cdot 5^1 + 3 \cdot 5^2 + 0 \cdot 5^3 + 1 \cdot 5^4 &=& 715 \\ 1 \cdot 5^0 + 0 \cdot 5^1 + 1 \cdot 5^2 + 0 \cdot 5^3 + 0 \cdot 5^4 &=& 26 \\ 0 \cdot 5^0 + 0 \cdot 5^1 + 0 \cdot 5^2 + 0 \cdot 5^3 + 0 \cdot 5^4 &=& 0. \end{array}$$

This gives the frequencies for a 5-ary partition of 845 with largest part at most $5^{3-1} = 25$. In fact,

$$845 = 715 \cdot 5^0 + 26 \cdot 5^1 + 0 \cdot 5^2. \tag{8}$$

Thus, (8) corresponds with the partition (7) under the bijection.

4 Closing Remarks

We close this note by considering whether we can provide an expression for $b^*_{m,j}(n)$ to generalize the solution of the Putnam problem which motivated this work.

It is clear that $b^*_{m,1}(n) = 1$ and we know

$$b^*_{m,2}(n) = \left\lfloor \frac{n}{m} \right\rfloor + 1$$

from Section 1. Next, let $j = 3$. Using the generating function (3) above,

$$\begin{aligned} B_{m,3}(q) &= B_{m,2}(q) \cdot \frac{1}{1 - q^{m^2}} \\ &= B_{m,2}(q) \cdot \left(1 + (q^{m^2}) + (q^{m^2})^2 + (q^{m^2})^3 + \dots\right) \\ &= B_{m,2}(q) + q^{m^2} \cdot B_{m,2}(q) + q^{2m^2} \cdot B_{m,2}(q) + q^{3m^2} \cdot B_{m,2}(q) + \dots . \end{aligned}$$

We extract the coefficient of q^n from both sides and apply Theorem 1 to get back to b^*:

$$b^*_{m,3}(n) = b^*_{m,2}(n) + b^*_{m,2}(n - m^2) + b^*_{m,2}(n - 2m^2) + b^*_{m,2}(n - 3m^2) + \dots .$$

Since there will always be a point after which all terms of this sum will be 0, we may truncate and conclude that

$$b^*_{m,3}(n) = \sum_{k=0}^{\lfloor \frac{n}{m^2} \rfloor} b^*_{m,2}(n - km^2) = \sum_{k=0}^{\lfloor \frac{n}{m^2} \rfloor} \left\lfloor \frac{n - km^2}{m} \right\rfloor + 1 \ .$$

Now, we may follow similar reasoning for any $j \geq 3$. Thus,

$$\begin{aligned} B_{m,j}(q) &= B_{m,j-1}(q) \cdot \frac{1}{1 - q^{m^{j-1}}} \\ &= B_{m,j-1}(q) \cdot \left(1 + (q^{m^{j-1}}) + (q^{m^{j-1}})^2 + (q^{m^{j-1}})^3 + \dots\right) \\ &= B_{m,j-1}(q) + q^{m^{j-1}} \cdot B_{m,j-1}(q) + q^{2m^{j-1}} \cdot B_{m,j-1}(q) + q^{3m^{j-1}} \cdot B_{m,j-1}(q) + \dots \end{aligned}$$

which implies that

$$\begin{aligned} &b^*_{m,j}(n) \\ &\quad = b^*_{m,j-1}(n) + b^*_{m,j-1}(n - m^{j-1}) + b^*_{m,j-1}(n - 2m^{j-1}) + b^*_{m,j-1}(n - 3m^{j-1}) + \dots \\ &\quad = \sum_{k=0}^{\lfloor \frac{n}{m^{j-1}} \rfloor} b^*_{m,j-1}(n - km^{j-1}). \end{aligned}$$

It is now possible to iterate the above process until we write $b^*_{m,j}(n)$ in terms of values of $b^*_{m,2}(n)$, giving a generalization that is reminiscent of the solution to the original Putnam problem.

References

[1] L. F. Klosinski, G. L. Alexanderson and A. P. Hillman, The William Lowell Putnam Mathematical Competition, *Amer. Math. Monthly* **91** (8) (1984), 487–495.

[2] Ø. Rødseth and J. Sellers, On m-ary partition function congruences: a fresh look at a past problem, *J. Number Theory* **87** (2) (2001), 270–281.

[3] L. Rucci, The kth m-ary partition function, *Theses and Dissertations*, Indiana University of Pennsylvania, Indiana, PA, 2016.

(Received 8 Mar 2018; revised 29 June 2018)