AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **72(3)** (2018), Pages 421–445

The multivariate avalanche polynomial*

Demara Austin

Department of Mathematics, Kansas State University
Manhattan, KS 66506, U.S.A.
draustin@math.ksu.edu

Megan Chambers

Department of Mathematics, North Carolina State University
Raleigh, NC 27695-8205, U.S.A.
mjchambe@ncsu.edu

Rebecca Funke

Department of Computer Science, University of Southern California
Los Angeles, CA 90007, U.S.A.
rfunke@usc.edu

Luis David García Puente

Department of Mathematics and Statistics, Sam Houston State University
Huntsville, TX 77341-2206, U.S.A.
lgarcia@shsu.edu

Lauren Keough†

Department of Mathematics, Grand Valley State University
Allendale, MI 49401, U.S.A.
keoulaur@gvsu.edu

Abstract

The (univariate) avalanche polynomial of a graph, introduced by Cori, Dartois and Rossin in 2004, captures the distribution of the length of (principal) avalanches in the abelian sandpile model. This polynomial

* This research was supported by NSF grants DMS-1045082, DMS-1045147 , and NSA grant H98230-14-1-0131. The fourth author was also partially supported by Simons Foundation Collaboration grant 282241.

† Corresponding author.

ISSN: 2202-3518

has been used to show that the avalanche distribution in the sandpile model on a multiple wheel graph does not follow a power law. In this article, we introduce the *multivariate avalanche polynomial* that enumerates the toppling sequences of all principal avalanches. This polynomial generalizes the univariate avalanche polynomial and encodes more information. In particular, the avalanche polynomial of a tree uniquely identifies the underlying tree. In this paper, the avalanche polynomial is characterized for trees, cycles, wheels, and complete graphs.

1 Introduction

A dynamical system that has a critical point as an attractor is said to display *self-organized criticality*. This concept was first introduced in 1987 with the seminal work of Bak, Tang and Wiesenfeld [4]. Self-organized criticality is thought to be present in a large variety of physical systems like earthquakes [12, 30], forest fires and in stock market fluctuations [2]. This property is considered to be one of the mechanisms by which *complexity* arises in nature [3] and has been extensively studied in the statistical physics literature during the last three decades [22, 26, 29].

In [4], Bak, Tang and Wiesenfeld conceived a *cellular automaton* model as a paradigm of self-organized criticality. This model is defined on a rectangular grid of cells as shown in Figure 1.

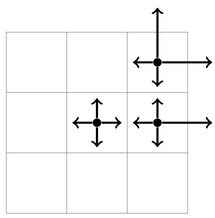


Figure 1: The Bak, Tang and Wiesenfeld model of self-organized criticality.

The system evolves in discrete time such that at each time step a sand grain is dropped onto a random grid cell. When a cell amasses four grains of sand, it becomes *unstable*. It relaxes by *toppling* whereby four sand grains leave the site, and each of the four neighboring sites gets one grain. If the unstable cell is on the boundary of the grid then, depending on whether the cell is a corner or not, either one or two sand grains fall off the edge and disappear. As the sand percolates over the grid in this fashion, adjacent cells may accumulate four grains of sand and become unstable causing an *avalanche*. This settling process continues until all cells are *stable*. Then another cell is picked randomly, the height of the sand on that grid cell is increased by one, and the process is repeated.

Imagine starting this process on an empty grid. At first there is little activity, but as time goes on, the size (the total number of topplings performed) and extent of the avalanche caused by a single grain of sand becomes hard to predict. Figure 2 shows the distribution of avalanches in a computational experiment performed on a 20×20 grid. Starting with the *maximal stable configuration*, i.e., the configuration with three grains of sand at each site, a total of $100,000$ sand grains were added at random, allowing the configuration to stabilize in between. In Figure 2, s denotes the size of the avalanche caused by adding a grain at random, and $D(s)$ is the number of avalanches of size s that occur throughout the simulation. We have displayed this data on a logarithmic scale for both parameters. This simulation suggest that this distribution follows a powe law. In fact, many authors have studied the distribution of avalanche sizes for this model. While it was first believed that this distribution was a power law, new studies suggest that avalanche statistics are significantly more complicated, see [23] and references therein.

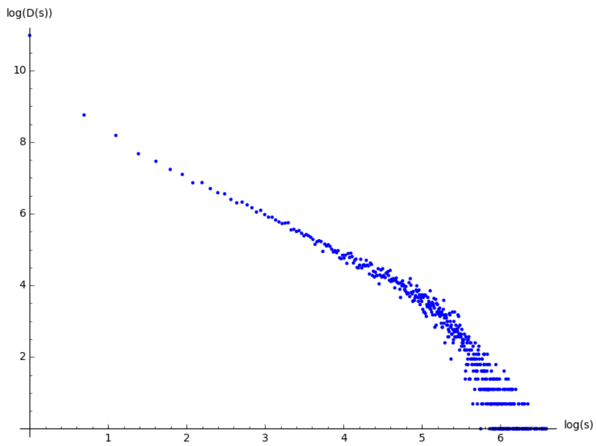


Figure 2: Distribution of avalanches $D(s)$ as a function of the size s on a 20×20 grid.

In 1990, Dhar generalized the Bak, Tang and Wiesenfeld model replacing the rectangular grid with an arbitrary combinatorial graph [18]. In this model, known as the *(abelian) sandpile model*, the sand grains are placed at the vertices of the graph and the toppling threshold depends on the degree (outdegree) of each vertex. Certain conditions on the graph (the existence of a *global sink vertex*) ensure that any avalanche terminates after a finite number of topplings. The sandpile model was also considered by combinatorialists as a game on a graph called the *chip firing game* [11, 10, 9].

The long-term behavior of the abelian sandpile model on a graph is encoded by the *recurrent configurations*, also known as *critical configurations*. These recurrent configurations have connections to *parking functions* [7], to the *Tutte polynomial* [27], and to the lattices of *flows* and *cuts* of a graph [1]. Among other properties, the recurrent configurations have the structure of a *finite abelian group*. This group has been discovered in several different contexts and received many names: the *sandpile group* [18, 21], the *critical group* [9], the *group of bicycles* [8], the *group of components* [25], the *Picard group* [1], and the *Jacobian* of a graph [5].

The study of the distribution of avalanche sizes on the grid has been a focal point from the statistical physics perspective. A natural question is what type of distributions do we get in the more general setting introduced by Dhar. In this paper, we focus on this question. In fact, we go one step beyond just finding such distributions. We actually describe the combinatorial structure of each avalanche for certain families of graphs.

As stated in [13], experiments on the distribution of sizes of the avalanches have been mostly restricted to the cases of rectangular grids and some classes of regular graphs. However, very little is known for arbitrary graphs. In order to address this issue, Cori, Dartois, and Rossin [13] introduced the (univariate) *avalanche polynomial* that encodes the sizes of *principal avalanches*, that is, avalanches resulting from adding a single grain of sand to a recurrent configuration. These authors completely describe the avalanche polynomials for trees, cycles, complete and lollypop graphs. Moreover, using these polynomials, they show that the resulting sandpile models on these graphs do not obey a power law. In 2003, Dartois and Rossin obtained exact results for the avalanche distribution on wheel graphs [16]. In 2009, Cori, Micheli and Rossin studied further properties of the avalanche polynomial on plane trees [14]. In particular, they show that the avalanche polynomial of a tree does not uniquely characterize the tree. They also give closed formulas for the average and variance of the avalanche distribution on trees.

In this paper we introduce the *multivariate avalanche polynomial*, i.e., a multivariate polynomial encoding the toppling sequences of all principal avalanches. This polynomial generalizes the univariate avalanche polynomial and encodes more information. In Section 2, we describe the sandpile model on an undirected graph and introduce both the univariate and multivariate avalanche polynomials. We also present some particular evaluations of the latter polynomial. In particular, one such evaluation gives rise to the unnormalized distribution of burst sizes, that is, the number of grains of sand that fall into the sink in a principal avalanche. In Section 3, we give recurrence relations to compute the avalanche polynomial of a tree. We also prove in Theorem 3.4 that this polynomial uniquely characterizes its underlying tree. In Section 4, we compute the multivariate avalanche polynomial for cycle graphs and in Section 5, we compute this polynomial for complete graphs using the bijection between recurrent configurations and parking functions. Our arguments fix some details in the proof of Proposition 5 in [13] that enumerates the number of principal avalanches of positive sizes in the complete graph. In Section 6, we compute the avalanche polynomial for wheel graphs. Our methods simplify the arguments in [16] where the authors use techniques from *regular languages*, *automatas* and *transducers* to characterize the recurrent configurations and determine the exact distribution of avalanche lengths in the wheel graph.

2 The Abelian Sandpile Model

The *abelian sandpile model* is defined both on directed and undirected graphs, but here we focus on families of undirected multigraphs without loops.

Definition 2.1. An *(undirected) graph* G is an ordered pair (V, E), where V is a finite set and E is a finite multiset of the set of 2-element subsets of V. The elements of V are called *vertices* and the elements of E are called *edges*. Given an undirected graph $G = (V, E)$, the *degree* d_v of a vertex $v \in V$ is the number of edges $e \in E$ with $v \in e$. For a pair of vertices $u, v \in V$, the weight $\mathrm{wt}(u, v)$ is the number of edges between u and v. We say u and v are *adjacent* if $\mathrm{wt}(u, v) > 0$.

Here, we will always assume that our graphs are *connected*. Moreover, given a graph $G = (V, E)$ we will distinguish a vertex $s \in V$ and call it a *sink*. The resulting graph will be denoted $G = (V, E, s)$. We will also denote the set of all non-sink vertices by $\widetilde{V} = V \setminus \{s\}$.

Definition 2.2. A *configuration* c on $G = (V, E, s)$ is a function $c : \widetilde{V} \to \mathbb{Z}_{\geq 0}$ from the non-sink vertices of G to the set of non-negative integers, where $c(v)$ represents the number of grains of sand at vertex v. We call v *unstable* if $c(v) \geq d_v$. An unstable vertex v can *topple*, resulting in a new configuration c' obtained by moving one grain of sand along each of the d_v edges emanating from v; that is, $c'(w) = c(w) + \mathrm{wt}(v, w)$ for all $w \neq v$ and $c'(v) = c(v) - d_v$. A configuration is *stable* if $c(v) < d_v$ for every non-sink vertex v and *unstable* otherwise.

The following proposition justifies the name "abelian sandpile model". It was first proved by Dhar in [18].

Proposition 2.3. *Given an unstable configuration c on a graph $G = (V, E, s)$, a stable configuration is always eventually reached through a sequence of topplings, and any sequence of topplings of unstable vertices will lead to the same stable configuration.*

Given a configuration c, if c' is obtained from c after a sequence of sand additions and topplings, we say that c' is *accessible* from c and we call c' a *successor* of c. We denote this by $c \rightsquigarrow c'$. Moreover, if c' is obtained from c by a sequence of topplings, then the *toppling vector* f associated to the stabilization $c \rightsquigarrow c'$ is the integer vector indexed by the non-sink vertices of G with $f(v)$ equal to the number of times vertex v appears in the vertex toppling sequence that sends c to c'. Finally, given a configuration c, the *unique stable* configuration obtained after a sequence of topplings is denoted by c° and is called the *stabilization* of c.

Let $G = (V, E, s)$ be a graph and a, b be two configurations on G. Then $a + b$ denotes the configuration obtained by adding the grains of sand vertex-wise, that is, $(a + b)(v) = a(v) + b(v)$ for each $v \in \widetilde{V}$. Note that even if a and b are stable, $a + b$ may not be. We denote the stabilization of $a + b$ by $a \oplus b$, that is, $a \oplus b = (a + b)^\circ$. The binary operator $\oplus$ is called *stable addition*.

Definition 2.4. Given a graph $G = (V, E, s)$, a stable configuration c is *recurrent* if there exists a configuration b such that $\max \oplus b = c$, where max denotes the maximal stable configuration on G defined by $\max(v) = d_v - 1$ for each v in $\widetilde{V}$.

Definition 2.4 immediately implies that max is recurrent. This definition also justifies the choice of starting the Markov chain with max in the simulation summarized in Figure 2. Definition 2.4 actually appears as Remark (*iii*) after Definition

1 in [15], where max is denoted by δ. In that paper, the authors use the standard definition of a recurrent configuration that usually appears in the statistical physics literature, namely, a configuration is *recurrent* if it can be reached infinitely often in the Markov chain on stable configurations, where the dynamics result from adding a grain at random and stabilizing.

As mentioned above, the set of recurrent configurations on a graph G under stable addition forms a finite abelian group denoted $\mathcal{S}(G)$, see [19]. Explicitly, the sandpile group of G is isomorphic to the cokernel of the *reduced Laplacian* matrix of G. Moreover, this matrix can be used to compute the stabilization of a configuration algebraically.

2.1 Graph Laplacians

Definition 2.5. Let G be a graph with n vertices $v_1, v_2, \ldots, v_n$. The *Laplacian* of G, denoted $L = L(G)$, is the $n \times n$ matrix defined by

$$L_{ij} = \begin{cases} -\operatorname{wt}(v_i, v_j) & \text{for } i \neq j, \\ d_{v_i} & \text{for } i = j. \end{cases}$$

The *reduced Laplacian* of a graph $G = (V, E, s)$, denoted, $\widetilde{L} = \widetilde{L}(G)$, is the matrix obtained by deleting the row and column corresponding to the sink s from the matrix L. Kirchhoff's Matrix-Tree Theorem implies that the number of recurrent configurations in G, that is, the determinant of reduced Laplacian $\widetilde{L}$ equals the number of spanning trees of G. From our definition we also have that if $c \rightsquigarrow c'$ by toppling vertex v, then $c' = c - \widetilde{L}1_v$, where 1_v denotes the (column) vector with $1_v(v) = 1$ and $1_v(w) = 0$, for all $w \neq v$ in $\widetilde{V}$. The previous observation leads to the following result.

Proposition 2.6. *Given a graph $G = (V, E, s)$, and a configuration c. If $c \rightsquigarrow c'$ by a sequence of topplings, then $c' = c - \widetilde{L}f$, where f is the (column) toppling vector associated to $c \rightsquigarrow c'$.*

The following result, known as Dhar's Burning Criterion, gives an alternative and useful way to characterize recurrent configurations in an undirected graph. Given $G = (V, E, s)$, let u denote the configuration given by $u_j = \operatorname{wt}(v_j, s)$ for each $v_j \in \widetilde{V}$. We will refer to the configuration u as the configuration obtained by 'toppling the sink'.

Proposition 2.7 ([15, Corollary 2.6])**.** *The configuration c is recurrent if and only if $u \oplus c = c$. Moveover, the toppling vector in the stabilization of $u \oplus c$ is* $(1, \ldots, 1)$.

Corollary 2.8. *Let $G = (V, E, s)$ be a graph and c a recurrent configuration on G. When one grain of sand is added to a vertex adjacent to the sink then every vertex can topple at most once.*

2.2 Avalanche Polynomials

The (univariate) avalanche polynomial was introduced in [13]. This polynomial enumerates the sizes of all principal avalanches.

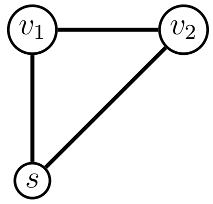
Definition 2.9. Let $G = (V, E, s)$ be a graph and let $v \in \widetilde{V}$. Let c be a recurrent configuration on G the *principal avalanche* of c at v is the sequence of vertex topplings resulting from the stabilization of the configuration $c + 1_v$.

Definition 2.10. The univariate avalanche polynomial for a graph G is defined as

$$\mathcal{A}_G^{(u)}(x) = \sum_{c \in \mathcal{S}(G)} \sum_{v \in \widetilde{V}} x^{|c \oplus 1_v|}$$

where $|c \oplus 1_v|$ is the number of vertices that topple in the stabilization of $c + 1_v$. Thus the coefficient of x^m is the number of principle avalanches of size m. We will denote the coefficient of x^m by λ_m.

Example 2.11. Let us consider the 3-cycle C_3. The recurrent configurations on C_3 are $(1, 0), (0, 1)$, and $(1, 1)$. The table in Figure 3 records the size of the avalanche for the corresponding recurrent and vertex. Therefore, $\mathcal{A}_{C_3}^{(u)}(x) = 2x^2 + 2x + 2$.



Recurrents	v_1	v_2
(1,0)	1	0
(0,1)	0	1
(1,1)	2	2

Figure 3: Principal avalanche sizes on C_3.

2.3 Multivariate Avalanche Polynomial

The univariate avalanche polynomial does not contain any information regarding which vertices topple in a given principal avalanche. Here we introduce the *multivatiate avalanche polynomial* that encodes this information.

Definition 2.12. Let $G = (V, E, s)$ be a graph on $n + 1$ vertices and let $\widetilde{V} = \{v_1, \ldots, v_n\}$. Given $k \in \{1, \ldots, n\}$ and a recurrent configuration c, the *avalanche monomial* of c at v_k is

$$\mu_G(c, v_k) = \boldsymbol{x}^{\nu(c, v_k)} = \prod_{i=1}^{n} x_i^{f_i}$$

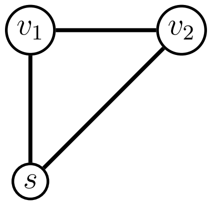
where $\nu(c, v_k) = (f_1, \ldots, f_n)$ is the toppling vector of the stabilization of $c + 1_{v_k}$.

Definition 2.13. Let $G = (V, E, s)$ be a graph. The *multivariate avalanche polynomial* of G is defined by

$$\mathcal{A}_G(x_1, \ldots, x_n) = \sum_{c \in \mathcal{S}(G)} \sum_{v \in \widetilde{V}} \mu_G(c, v).$$

Note that the multivariate avalanche polynomial is the sum of all possible avalanche monomials. In what follows the term "avalanche polynomial" will refer to the multivariate case.

Example 2.14. As in Example 2.11 we have recurrents $(1,0), (0,1)$, and $(1,1)$ in C_3. The table in Figure 4 records the toppling vector $\nu(c, v_i)$ for the principal avalanche of c at v_i.



Recurrents	v_1	v_2
(1,0)	(1,0)	(0,0)
(0,1)	(0,0)	(0,1)
(1,1)	(1,1)	(1,1)

Figure 4: Toppling vectors on C_3.

From this table we can easily read the avalanche monomials. For example, $\mu_{C_3}((1,0), v_1) = x_1^1 x_2^0 = x_1$. Adding all avalanche monomials gives the avalanche polynomial

$$\mathcal{A}_{C_3}(x_1, x_2) = x_1^1 x_2^0 + 2x_1^0 x_2^0 + x_1^0 x_2^1 + 2x_1^1 x_2^1 = 2x_1 x_2 + x_1 + x_2 + 2.$$

Note that the univariate avalanche polynomial for a graph G can be recovered from the multivariate avalanche polynomial by substituting each x_i by x, i.e.,

$$\mathcal{A}_G^{(u)}(x) = \mathcal{A}_G(x, \ldots, x).$$

It is a well-known fact that the sandpile group of an undirected graph is independent of the choice of sink [15, Proposition 1.1]. Nevertheless, the structure of the individual recurrent configurations may differ. This implies that the avalanche polynomial of a graph is dependent on the choice of sink. For example, for the path graph P_3 on 3 vertices we have that $\mathcal{A}_{P_3}^{(u)} = 2x$ or $\mathcal{A}_{P_3}^{(u)} = x^2 + x^3$ depending on whether the sink is the middle vertex or one of the end vertices, respectively. For this reason, we will always fix a sink before discussing the avalanche polynomial of a graph. Nevertheless, for certain graphs like cycles and complete graphs, the avalanche polynomial does not depend on the choice of sink. For other families such as wheel and fan graphs there is a natural choice of sink, namely the dominating vertex. For trees, our method computes the avalanche polynomial for any choice of sink.

2.4 Burst Size

There are other evaluations of the multivariate avalanche polynomial $\mathcal{A}_G(x_1, \ldots, x_n)$ that are relevant in the larger field of sandpile groups. In [24], Levine introduces the concept of *burst size* to prove a conjecture of Poghosyan, Poghosyan, Priezzhev, Ruelle [28] on the relationship between the threshold state of the fixed-energy sandpile and the stationary state of Dhar's abelian sandpile.

Definition 2.15. Let $G = (V, E, s)$ be a graph and c be a recurrent configuration on G. Given $v \in \widetilde{V}$, define the *burst size* of c at v as

$$\text{burst}(c, v) := |c'| - |c| + 1,$$

where c' is the unique recurrent configuration that satisfies $c' \oplus 1_v = c$ and $|c|$ denotes the number of grains of sand in c. Equivalently, $\text{burst}(c, v)$ is the number of grains that fall into the sink s during the stabilization of $c' + 1_v \rightsquigarrow c$.

As noted in [24], given a vertex $v \in \widetilde{V}$, the *addition operator* defined by $a_v(c) = c \oplus 1_v$ acts as a permutation on the set of recurrent configurations. This implies both the existence and uniqueness of the recurrent configuration c' in Definition 2.15.

Let $G = (V, E, s)$ be a *simple graph* and $\mathcal{A}_G(x_1, \ldots, x_n)$ be its multivariate avalanche polynomial. Now, let $x_i = 1$ for each vertex v_i that is not adjacent to the sink s. Also, for each vertex v_j adjacent to s, let $x_j = x$. The resulting univariate polynomial $\mathcal{B}(x) = \sum_k b_k x^k$ satisifes the condition that b_k is the number of principal avalanches with burst size k.

3 Avalanche Polynomials of Trees

Let T be a tree on $n+1$ vertices labelled $v_1, \ldots, v_n, s$. Assume further that T is rooted at the sink s. It is a basic observation that T has only one recurrent configuration, namely $\max_T$. So

$$\mathcal{A}_T(x_1, \ldots, x_n) = \sum_{v \in \widetilde{V}} \mu_T(\max_T, v).$$

As noted in [13] any tree can be constructed from a single vertex using two operations ϕ and $+$ defined below.

Definition 3.1. For two trees T and T' rooted at s and s' respectively, the operation $+$, called *tree addition*, identifies s and s'. For a tree T rooted at s, the operation ϕ, called *grafting* or *root extension*, refers to adding an edge from s to a new root s'.

The operations ϕ and $+$ are depicted in Figure 5. Theorem 3.2 explains what happens to the avalanche polynomial of a tree under these operations. This theorem extends the results presented in Section 3.1 of [13] regarding the univariate avalanche polynomial of trees.

Theorem 3.2. *Let $\mathcal{A}_T$, $\mathcal{A}_{T_1}$, and $\mathcal{A}_{T_2}$ be the avalanche polynomials of trees T, T_1, and T_2, respectively. Then*

1. $\mathcal{A}_{T_1+T_2} = \mathcal{A}_{T_1} + \mathcal{A}_{T_2}$,
2. $\mathcal{A}_{\phi(T)} = x_1 x_2 \cdots x_n(\mathcal{A}_T + 1)$, *where* $n = |\widetilde{V}(\phi(T))|$.

Proof. Under tree addition, trees T_1 and T_2 are only connected at the sink s. Since the sink never topples, a principal avalanche at a vertex in T_1 will never affect the vertices in T_2, and vice versa. Therefore, $\mathcal{A}_{T_1+T_2} = \mathcal{A}_{T_1} + \mathcal{A}_{T_2}$. For the second part,

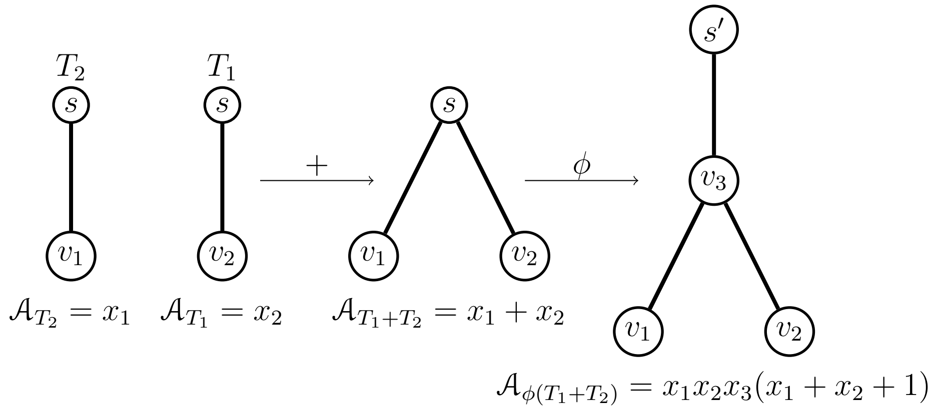


Figure 5: Effect of $+$ and ϕ operations on the avalanche polynomial

let T be a tree on n vertices with sink s. Let $\max_{\phi(T)}$ and $\max_T$ be the maximum stable configuration on $\phi(T)$ and T, respectively. First we consider $\max_{\phi(T)} \oplus 1_s$. By Proposition 2.7,

$$\mu_{\phi(T)}(\max_{\phi(T)}, s) = x_1x_2 \cdots x_n.$$

Now consider $\max_{\phi(T)} \oplus 1_{v_k}$ where $v_k \neq s$. Note that when we apply the toppling sequence $\nu_T(\max_T, v_k)$ to $\max_{\phi(T)} + 1_{v_k}$, we get the configuration $\max_{\phi(T)} + 1_s$. Thus, each vertex will now topple once more. So for each v_k with $v_k \neq s$,

$$\mu_{\phi(T)}(\max_{\phi(T)}, v_k) = (x_1x_2 \cdots x_n) \cdot \mu_T(\max_T, v_k).$$

Therefore,

$$\mathcal{A}_{\phi(T)} = (x_1x_2 \cdots x_n) + (x_1x_2 \cdots x_n) \cdot \mathcal{A}_T = x_1x_2 \cdots x_n(\mathcal{A}_T + 1).$$

□

Note that using Theorem 3.2 we can compute the multivariate avalanche polynomial of any tree. Furthermore, as noted in [13], it is possible for two non-isomorphic trees to have the same univariate avalanche polynomial. In contrast, the multivariate avalanche polynomial distinguishes between labeled trees.

Figure 6 gives an example, first presented in [13], of two non isomorphic trees with the same univariate avalanche polynomial. The vertices are labeled with the size of the principal avalanche starting at that vertex. One can clearly see that T_1 and T_2 have the same univariate avalanche polynomial. However, they have different multivariate avalanche polynomials. Let's examine the right subtrees of T_1 and T_2, denoted by R_1 and R_2, respectively.

The avalanche polynomial of R_1 is $x_6 + x_5x_7(x_5 + 1) + x_8 + x_9 + x_{10}$ and the avalanche polynomial of R_2 is $x_5 + x_6 + x_7 + x_8 + x_9 + x_{10}$. Note that the first polynomial has total degree 3 and the second polynomial has total degree 1. The avalanche polynomials of the left subtrees of T_1 and T_2 will be on a disjoint set of variables. Thus, the avalanche polynomials for T_1 and T_2 must be distinct.

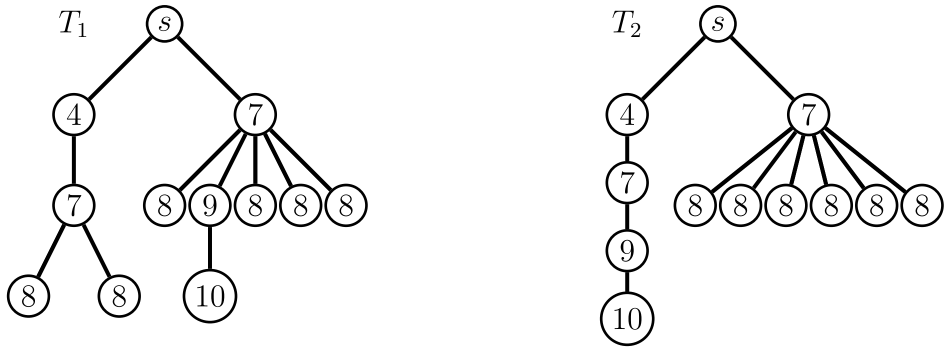


Figure 6: $\mathcal{A}^{(u)}_{T_1}(x) = \mathcal{A}^{(u)}_{T_2}(x) = x^{10} + x^9 + 6x^8 + 2x^7 + x^4$

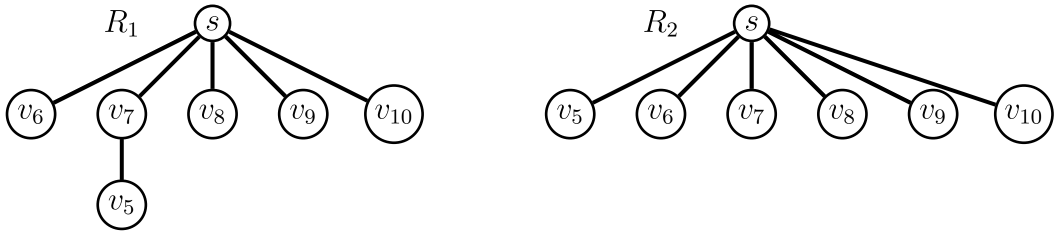


Figure 7: Labelled right subtrees of T_1 and T_2.

Corollary 3.3. *Let T and T' be two trees on $n+1$ vertices. Then, $\mathcal{A}_T(x_1, \dots, x_n) = \mathcal{A}_{T'}(x_1, \dots, x_n)$ if and only if $\mathcal{A}_{\phi(T)}(x_1, \dots, x_{n+1}) = \mathcal{A}_{\phi(T')}(x_1, \dots, x_{n+1})$.*

Proof. Suppose that $\mathcal{A}_T = \mathcal{A}_{T'}$. Theorem 3.2 implies

$$\mathcal{A}_{\phi(T)} = x_1 x_2 \cdots x_n(\mathcal{A}_T + 1) = x_1 x_2 \cdots x_n(\mathcal{A}_{T'} + 1) = \mathcal{A}_{\phi(T')}.$$

Now assume $\mathcal{A}_{\phi(T)} = \mathcal{A}_{\phi(T')}$. Then $x_1 x_2 \cdots x_n(\mathcal{A}_T + 1) = x_1 x_2 \cdots x_n(\mathcal{A}_{T'} + 1)$. This clearly implies $\mathcal{A}_T = \mathcal{A}_{T'}$. □

Theorem 3.4. *Let T be a tree on $n+1$ vertices. If $\mathcal{A}_T(x_1, \dots, x_n) = \mathcal{A}_{T'}(x_1, \dots, x_n)$ for some tree T', then $T = T'$.*

Proof. We use induction on the height of T. Recall that the *height* of a tree is the number of edges in the longest path between the root and a leaf. Suppose T has height 0, that is, T consists of one vertex. Then $\mathcal{A}_T = 0$. Clearly, if T' has two or more vertices then $\mathcal{A}_{T'} \neq 0$. Since $\mathcal{A}_T = \mathcal{A}_{T'}$, then T' must also consist of one vertex and $T = T'$. Now suppose that T has height $h > 0$. In this case, the sink s of T must have at least one child. Assume s has degree d. Deleting s creates d trees $T_1, T_2, \dots, T_d$. We have that $T = \phi(T_1) + \phi(T_2) + \cdots + \phi(T_d)$. So $\mathcal{A}_T$ is the sum of d multivariate polynomials with pairwise disjoint supports

$$\mathcal{A}_T = \mathcal{A}_{\phi(T_1)} + \mathcal{A}_{\phi(T_2)} + \cdots + \mathcal{A}_{\phi(T_d)}.$$

Since $\mathcal{A}_T = \mathcal{A}_{T'}$, then $\mathcal{A}_{T'}$ must also satisfy the same condition. We claim that this implies that the sink of T' must also have degree d. This follows from the fact that in a principle avalanche on a tree, all vertices on the branch of the tree containing the vertex where a grain is added will topple. Therefore, $T' = \phi(T'_1) + \phi(T'_2) + \cdots + \phi(T'_d)$ for some trees $T'_1, T'_2, \ldots, T'_d$. Since the supports are pairwise disjoint, we must also have that $\mathcal{A}_{\phi(T_1)} = \mathcal{A}_{\phi(T'_j)}$, for some j. Corollary 3.3 implies $\mathcal{A}_{T_1} = \mathcal{A}_{T'_j}$. But T_1 is a tree of height $h-1$. By induction, $T_1 = T'_j$. Therefore, after relabeling the subtrees in T', we must have $T_i = T'_i$ for all $1 \le i \le d$ and $T = T'$. □

4 Avalanche Polynomials of Cycles

Now, we will compute the avalanche polynomial of the cycle graph C_{n+1} on $n+1$ vertices. Unless otherwise stated, we will label the vertices $s, v_1, v_2, \ldots, v_n$ in a clockwise manner. As shown in Example 2.11, $\mathcal{A}_{C_3}(x_1, x_2) = 2x_1x_2 + x_1 + x_2 + 2$. We will denote by C_2 the graph with two vertices and two edges between these vertices. It is clear that $\mathcal{A}_{C_2}(x_1) = x_1 + 1$.

In this section, we will write configurations and toppling vectors as strings instead of vectors. For example, the string $1^{p-1}01^{n-p}$ denotes the configuration with no grains of sand at vertex v_p and 1 grain of sand at every other vertex. We will also make the convention that a bit raised to the 0 power does not appear in the string, e.g., $0^21^00^2 = 0^4$. In the previous section, we saw that a tree has exactly one recurrent configuration. The cycle graph C_{n+1} has exactly $n+1$ recurrent configurations, namely, $\max = 1^n$ and $b_p = 1^{p-1}01^{n-p}$ for $p = 1, 2, \ldots, n$, see [13].

4.1 Toppling Sequence for the Maximal Stable configuration

We first focus our attention on understanding the toppling sequences for $1^n + 1_{v_i}$ for $1 \le i \le n$.

Example 4.1. Figure 8 shows that $\mu_{C_6}(1^5, v_2) = x_1x_2^2x_3^2x_4^2x_5$.

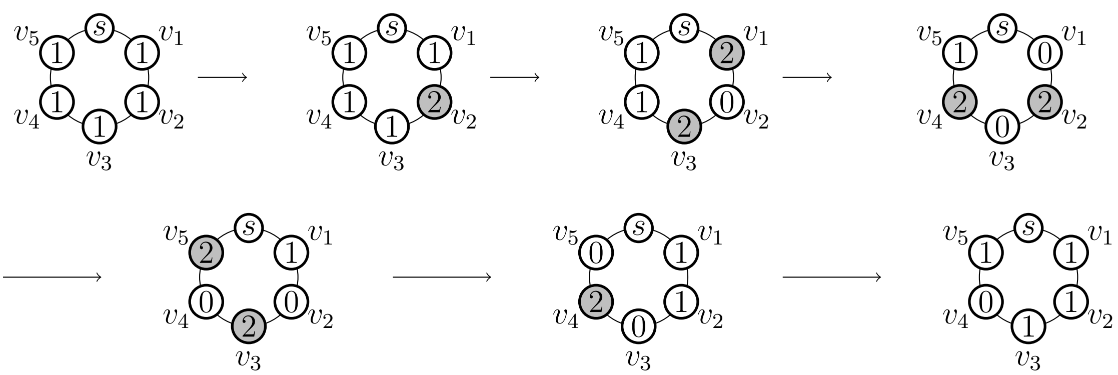


Figure 8: The principal avalanche created by adding a grain of sand to v_2.

Lemma 4.2. *Let $n \geq 1$. Then*

$$\mu_{C_{n+1}}(1^n, v_1) = \mu_{C_{n+1}}(1^n, v_n) = x_1 x_2 \cdots x_n.$$

Proof. Because v_1 and v_n are adjacent to the sink, each vertex topples at most once by Corollary 2.8. Now consider $\max + 1_{v_1}$. Toppling v_1 results in v_2 being unstable. Inductively, for $i \geq 2$, if v_i becomes unstable it will topple and v_{i+1} will become unstable. Thus each vertex topples exactly once. A similar argument works for $\mu_{C_{n+1}}(1^n, v_n)$. □

Observe that the reduced Laplacian of the cycle C_{n+1} is the $n \times n$ matrix

$$\widetilde{L} = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix},$$

with 2's on the diagonal, -1's on the off diagonals, and 0's elsewhere. We will use the reduced Laplacian in the next two proofs. We will abuse notation and write $\widetilde{L}b$ instead of $\widetilde{L}b^t$ to denote the product of $\widetilde{L}$ times the vector b.

Lemma 4.3. *Let $n \geq 3$, $2 \leq i \leq n-1$ and consider the configuration $1^n + 1_{v_i}$ on C_{n+1}. We can legally topple each vertex once and the resulting configuration is $01^{i-2}21^{n-i-1}0$.*

Proof. The sequence $v_i, v_{i-1}, v_{i-2}, \ldots, v_1, v_{i+1}, v_{i+2}, \ldots, v_n$ is a legal toppling sequence. By Proposition 2.6, $1^n + 1_{v_i}$ accesses the configuration

$$(1^n + 1_{v_i}) - \widetilde{L} \cdot 1^n = (1^n + 1_{v_i}) - 10^{n-2}1 = 01^{i-2}21^{n-i-1}0.$$

□

Lemma 4.3 states that $1^n + 1_{v_i} \rightsquigarrow 01^{i-2}21^{n-i-1}0$, another unstable configuration. Lemma 4.4 will describe the remaining toppling pattern.

Lemma 4.4. *Let $n \geq 3$, $2 \leq i \leq n-1$, and $1 \leq k \leq \min\{i-1, n-i\}$. Let c_k be the configuration*

$$c_k = 1^{k-1}01^{i-k-1}21^{n-i-k}01^{k-1}.$$

Then for each $1 \leq k \leq \min\{i-2, n-i-1\}$ the configuration c_k accesses the configuration c_{k+1} via toppling vertices $v_{k+1}, v_{k+2}, \ldots, v_{n-k}$.

Proof. The toppling sequence $v_i, v_{i-1}, \ldots, v_{k+1}, v_{i+1}, \ldots, v_{n-k}$ is a legal toppling sequence. Applying this toppling sequence we obtain

$$\widetilde{L} \cdot 0^k 1^{n-2k} 0^k = 0^{k-1}(-1)10^{n-2k-2}1(-1)0^{k-1}.$$

Thus, Proposition 2.6 implies c_k accesses the configuration

$$\begin{aligned} &1^{k-1}01^{i-k-1}21^{n-i-k}01^{k-1} - 0^{k-1}(-1)10^{n-2k-2}1(-1)0^{k-1} \\ &\qquad = 1^k01^{i-k-2}21^{n-i-k-1}01^k = c_{k+1}. \end{aligned}$$

□

Theorem 4.5. *For $n \geq 1$, $1 \leq i \leq n$, let $m = \min\{i, n-i+1\}$. Then*

$$\mu_{C_{n+1}}(1^n, v_i) = \prod_{j=1}^{m} x_j \cdots x_{n-j+1} = (x_1 \cdots x_n)(x_2 \cdots x_{n-1}) \cdots (x_m \cdots x_{n-m+1}).$$

Proof. The cases for $n = 1$ and $n = 2$ are discussed in the first paragraph of this section. Now assume $n \geq 3$. If $i = 1$ or $i = n$, the result follows from Lemma 4.2. Suppose $2 \leq i \leq n-1$, by Lemma 4.3 $\max + 1_{v_i} \rightsquigarrow c_1$ via the toppling of all vertices which gives the factor $x_1 x_2 \cdots x_n$. By Lemma 4.4, $c_1 \rightsquigarrow c_2 \rightsquigarrow \cdots \rightsquigarrow c_{m-1}$. Note that $c_k \rightsquigarrow c_{k+1}$ via the toppling of vertices $v_{k+1}, \ldots, v_{n-k}$. This produces the factor $x_{k+1} \cdots x_{n-k}$. Suppose that $m = i = \min\{i, n-i+1\}$. Then $c_{m-1} = c_{i-1} = 1^{i-2}021^{n-2i+1}01^{i-2}$. Only vertex v_i is unstable, but when v_i topples v_{i-1} does not become unstable. Vertex v_{i+1} does become unstable and will topple. In fact, vertices $v_i, v_{i+1}, \ldots, v_{n-i+1}$ will topple resulting in the stable configuration $1^{n-i}01^{i-1}$. This gives the last factor $x_m \cdots x_{n-m+1}$. So, if $m = i = \min\{i, n-i+1\}$,

$$\mu_{C_{n+1}}(1^n, v_i) = \prod_{j=1}^{m} x_j \cdots x_{n-j+1}.$$

The proof is similar for the case $m = n-i+1 = \min\{i, n-i+1\}$. □

4.2 The toppling sequence for recurrents $1^{p-1}01^{n-p}$

Theorem 4.5 gives us the avalanche monomials for the maximal stable configuration at all vertices. Now we find the avalanche monomials for recurrents of the form $1^{p-1}01^{n-p}$. We will see that these monomials are closely related to the avalanche monomials arising from 1^n.

Example 4.6. In Example 4.1, we saw $\mu_{C_6}(1^5, v_2) = x_1 x_2^2 x_3^2 x_4^2 x_5$. Figure 9 shows that $\mu_{C_{10}}(1^3 01^5, v_6) = x_5 x_6^2 x_7^2 x_8^2 x_9$. Notice that the structure of these monomials. The only difference is that there is a relabeling of the variables $x_i \to x_{i+4}$.

Based on what we have seen in Example 4.6, we may guess that the toppling monomials associated to $1^{p-1}01^{n-p}$ are related to the toppling monomials of C_p and C_{n-p+1}. More generally, there is one vertex of the recurrent configuration with no grain of sand. This vertex acts as a blocking vertex, or surrogate sink, because that vertex will never topple and all topplings will occur on one side of that vertex (the side where the grain is added). For this reason, we introduce the following notation which will be useful in the statement and proof of Theorem 4.8.

Definition 4.7. Let q be an integer. Then C_{n+1}^q will denote the cycle graph on $n+1$ vertices labeled $v_{q+1}, \ldots, v_{q+n}, s$.

Theorem 4.8. *Let $b_p = 1^{p-1}01^{n-p}$ be a recurrent on C_{n+1} such that $1 \leq p \leq n$.*

(1) If $1 \leq i \leq p-1$, then $\mu_{C_{n+1}}(b_p, v_i) = \mu_{C_p}(1^{p-1}, v_i)$,

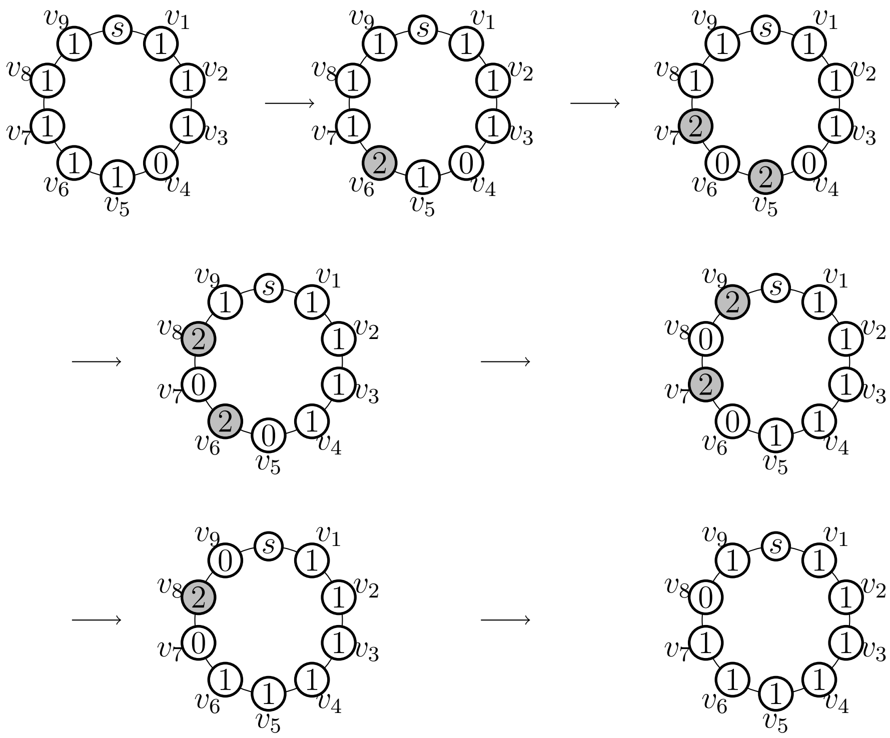


Figure 9: The principal avalanche of $1^3 01^5 + 1_{v_6}$.

(2) If $p+1 \leq i \leq n$*, then* $\mu_{C_{n+1}}(b_p, v_i) = \mu_{C^p_{n-p+1}}(1^{n-p}, v_i)$*,*

(3) $\mu_{C_{n+1}}(b_p, v_p) = 1$*.*

Proof. Let $1 \leq i \leq p-1$, this implies $p \geq 2$. If $p = 2$, then $i = 1$, and $b_p + 1_{v_1} = 201^{n-2} \rightsquigarrow 01^{n-1}$. So $\mu_{C_{n+1}}(b_p, v_1) = x_1 = \mu_{C_2}(1, v_1)$ and the result holds.

Assume $p \geq 3$. If $i = 1$, then $v_1, \ldots, v_{p-1}$ is a legal toppling sequence and $b_p + 1_{v_1} \rightsquigarrow 1^{p-2}01^{n-p+1}$. Lemma 4.2 implies $\mu_{C_{n+1}}(b_p, v_1) = x_1 x_2 \cdots x_{p-1} = \mu_{C_p}(1^{p-1}, v_1)$. Similarly, if $i = p-1$, then $v_{p-1}, v_{p-2}, \ldots, v_1$ is a legal toppling sequence and $b_p + 1_{v_{p-1}} \rightsquigarrow 01^{n-1}$. So again $\mu_{C_{n+1}}(b_p, v_{p-1}) = x_1 x_2 \cdots x_{p-1} = \mu_{C_p}(1^{p-1}, v_{p-1})$.

If $p \geq 3$ and $2 \leq i \leq p-2$, we have $b_p + 1_{v_i} = 1^{p-1}01^{n-p} + 1_{v_i} = 1^{i-1}21^{p-i-1}01^{n-p}$. Note that $v_i, v_{i-1}, \ldots, v_1, v_{i+1}, \ldots, v_{p-1}$ is a legal toppling sequence. So $b_p + 1_{v_i} \rightsquigarrow 01^{i-2}21^{p-i-2}01^{n-p+1}$. From this computation we can deduce two things. First, during the stabilization of $b_p + 1_{v_i}$, none of the vertices $v_p, \ldots, v_n$ will topple. Second, the configuration $01^{i-2}21^{p-i-2}01^{n-p+1}$ is in fact the configuration c_1 defined in Lemma 4.4 when $n+1 = p$ with the string 1^{n-p+1} concatenated at the end. Therefore, from these two observations we conclude that the principal avalanche resulting from $b_p + 1_{v_i}$ in C_{n+1} follows the pattern described in Lemma 4.4 for C_p. Hence $\mu_{C_{n+1}}(b_p, v_i) = \mu_{C_p}(1^{p-1}, v_i)$.

A similar argument works for the case $p+1 \leq i \leq n$ with the exception that now the vertices that topple are $v_{p+1}, \ldots, v_n$. So $\mu_{C_{n+1}}(b_p, v_i) = \mu_{C^p_{n-p+1}}(1^{n-p}, v_i)$. Finally, it is clear that $b_p + 1_{v_p}$ is stable. So $\mu_{c_{n+1}}(1^{p-1}01^{n-p}, v_p) = 1$. □

The following result follows immediately from Theorem 4.5 and Theorem 4.8. This corollary gives a complete description of the avalanche polynomial of a cycle graph. We further point out that it is possible to deduce Proposition 3.1 in [13] from our Corollary 4.9. This proposition provides a description of the univariate avalanche polynomial for cycles. However, we omit this argument since the direct proof presented in [13] is significantly easier.

Corollary 4.9. *The avalanche polynomial for* C_{n+1} *for* $n \geq 1$ *is*

$$\sum_{i=1}^{n} \mu_{C_{n+1}}(1^n, v_i) + \sum_{p=2}^{n}\sum_{i=1}^{p-1} \mu_{C_p}(1^{p-1}, v_i) + \sum_{p=1}^{n-1}\sum_{i=p+1}^{n} \mu_{C^p_{n-p+1}}(1^{n-p}, v_i) + n,$$

where $\mu_{C_{q+1}}(1^q, v_i) = \prod_{j=1}^{m} x_j \cdots x_{q-j+1}$, $1 \leq i \leq q$, *and* $m = \min\{i, q-i+1\}$.

5 Avalanche Polynomials of Complete Graphs

In this section we will compute the avalanche polynomial of the complete graph K_{n+1} on $n+1$ vertices $v_1, v_2, \ldots, v_n, s$. In Example 2.14, we showed that $\mathcal{A}_{K_3}(x_1, x_2) = 2x_1x_2 + x_1 + x_2 + 2$. Using SageMath [17], we can compute the avalanche polynomial of K_4:

$$\mathcal{A}_{K_4}(x_1, x_2, x_3) = 9x_1x_2x_3 + 2x_1x_2 + 2x_1x_3 + 2x_2x_3 + 3x_1 + 3x_2 + 3x_3 + 24.$$

Note that $\mathcal{A}^{(u)}_{K_4}(x) = 9x^3 + 6x^2 + 9x + 24$. So the set of principal avalanches of size m is partitioned into $\binom{3}{m}$ subsets of the same size for $0 \leq m \leq 3$. Moreover, $\mathcal{A}_{K_4}(x_1, x_2, x_3)$ is a linear combination of elementary symmetric polynomials. We will show that this characterizes the avalanche polynomial of K_{n+1}.

Definition 5.1. Let m be an integer such that $0 \leq m \leq n$. The *elementary symmetric polynomial* of degree m on variables $x_1, x_2, \ldots, x_n$ is

$$e_m(x_1, \ldots, x_n) = \sum_{\substack{A \subseteq [n] \\ |A| = m}} \prod_{i \in A} x_i.$$

Observe that $e_0(x_1, \ldots, x_n) = 1$ and the number of terms in $e_m(x_1, \ldots, x_n)$ is $\binom{n}{m}$.

As mentioned in Section 2, the number of recurrent configurations of a graph G equals the number of spanning trees of G. Cayley's formula implies that K_n has n^{n-2} recurrent configurations. In order to study the principal avalanches in this graph, we will use a beautiful result first proved in [15] that establishes a bijection between recurrent configurations in K_{n+1} and n-parking functions.

Definition 5.2. Given a function $p : \{0, 1, \ldots, n-1\} \to \{0, 1, \ldots, n-1\}$, let $a_0 \leq a_1 \leq \cdots \leq a_{n-1}$ be the non-decreasing rearrangement of $p(0), \ldots, p(n-1)$. We say that p is an *n-parking function* provided that $a_i \leq i$ for $0 \leq i \leq n-1$.

Note that the parking function p can be represented by the vector $(p(0), p(1), \ldots, p(n-1))$.

Proposition 5.3 ([15, Proposition 2.8]). *The configuration c is recurrent on K_{n+1} if and only if $\max_{K_{n+1}} - c$ is an n-parking function.*

It is clear from the definition that any permutation of a parking function is also a parking function. We can concatenate parking functions to obtain new parking functions.

Lemma 5.4. *Let $p = (p_0, p_1, \ldots, p_{m-1})$ and $q = (q_0, q_1, \ldots, q_{n-1})$ be two parking functions. Then $(p_0, p_1, \ldots, p_{m-1}, q_0 + m, q_1 + m, \ldots, q_{n-1} + m)$ is also a parking function.*

Proof. Let $a_0 \leq a_1 \leq \cdots \leq a_{m-1}$ and $b_0 \leq b_1 \leq \cdots \leq b_{n-1}$ be non-decreasing rearrangements of p and q, respectively. Note $b_0 + m \leq b_1 + m \leq \cdots \leq b_{n-1} + m$ and $b_i + m \leq i + m$ for each $i = 0, \ldots, n-1$, since q is a parking function. Moreover, $a_{m-1} \leq m - 1 < m = b_0 + m$. So

$$a_0 \leq a_1 \leq \cdots \leq a_{m-1} < b_0 + m \leq b_1 + m \leq \cdots \leq b_{n-1} + m$$

and each term is less than or equal to its index. $\square$

5.1 The Avalanche Polynomial of K_{n+1}

The following lemma gives a partial description of a configuration c given the size of a principal avalanche.

Lemma 5.5. *Let c be a recurrent configuration on K_{n+1}. Suppose that the principal avalanche resulting from stabilizing $c + 1_{v_k}$ has length $m \geq 1$. Let $w_0, w_1, \ldots, w_{m-1}$ be the associated toppling sequence and let $\{u_0, \ldots, u_{n-m-1}\}$ be the set of vertices that do not topple. Assume further, perhaps after relabeling, that $c(u_0) \geq c(u_1) \geq \cdots \geq c(u_{n-m-1})$. The following are true:*

1. *$c(w_0) = c(v_k) = n - 1$.*
2. *$n - i \leq c(w_i) \leq n - 1$, for $i = 1, \ldots, m-1$.*
3. *$n - m - i - 1 \leq c(u_i) \leq n - m - 1$, for $i = 0, \ldots, n-m-1$.*

Proof. First note that since c is stable then $c(v) \leq n - 1$. Since $m \geq 1$, then $w_0 = v_k$ must topple. Thus, $c(w_0) = n-1$. Corollary 2.8 implies that each w_j appears exactly once in the toppling sequence. Moreover, when a vertex topples, it adds one grain of sand to every other non-sink vertex of K_{n+1}. Thus, toppling vertices $w_0, \ldots, w_{i-1}$ adds i grains of sand to w_i. Since this vertex must topple next, then $c(w_i) + i \geq n$. So, $n - i \leq c(w_i) \leq n - 1$. On the other hand, since u_i does not topple, then $c(u_i) \leq n - m - 1$ for $0 \leq i \leq n - m - 1$. By Proposition 5.3, $p = \max_{K_{n+1}} - c$ is an n-parking function. Let p' be its non-decreasing rearrangement. Note that the first m entries in p' correspond to the m vertices that topple and the last $n - m$ entries correspond to the vertices that do not topple. So $p'(m + i) = n - 1 - c(u_i)$. Since $p'(m+i) \leq m+i$, then $c(u_i) \geq n-m-i-1$. So, $n-m-i-1 \leq c(u_i) \leq n-m-1$. $\square$

Proposition 5.6. *Let λ_m denote the number of principal avalanches of size m in K_{n+1}. Then*

$$\mathcal{A}_{K_{n+1}}(x_1, \ldots, x_n) = \sum_{m=0}^{n} \frac{\lambda_m}{\binom{n}{m}} e_m(x_1, \ldots, x_n).$$

Proof. First note that each vertex can topple at most once in any principal avalanche by Corollary 2.8. So every monomial in $\mathcal{A}_{K_{n+1}}$ is square-free and completely characterized by its support. Fix an integer m with $1 \leq m \leq n$. Let $A \subseteq [n]$ with $|A| = m$ and let $\mu_A = \prod_{i \in A} x_i$. Consider the configuration c defined by

$$c(v_i) = \begin{cases} n-1 & \text{if } i \in A, \\ n-1-m & \text{if } i \notin A. \end{cases}$$

Note that $\max - c(v_i) = 0$ if $i \in A$ and $\max - c(v_i) = m$ if $i \notin A$. The non-decreasing rearrangement of $\max - c$ is is $(\underbrace{0, 0, \ldots, 0}_{m}, \underbrace{m, m, \ldots, m}_{n-m})$ which is a parking function. Thus, by Proposition 5.3, the configuration c is recurrent. By Lemma 5.5, $\mu(c, v_i) = \mu_A$ for any vertex v_i with $i \in A$. This shows that every square-free monomial on $x_1, \ldots, x_n$ appears in the avalanche polynomial. Moreover, the symmetry of K_{n+1} directly implies that the coefficient α_A of the monomial μ_A depends only on the size m of A. Since $\sum_{A;|A|=m} \alpha_A = \lambda_m$ and there are $\binom{n}{m}$ such sets A, the result follows immediately. □

The coefficients λ_m in Proposition 5.6 were computed in [13, Propositions 4 and 5]. Explicitly, $\lambda_0 = n(n-1)(n+1)^{n-2}$ and

$$\lambda_m = \binom{n}{m} m^{m-1}(n-m+1)^{n-m-1}, \text{ for } 1 \leq m \leq n.$$

We include a proof of the latter result in order to correct a mistake in their original argument. Furthermore, we also want to point out that the coefficient λ_m is also the number of principal avalanches with burst size m since every non-sink vertex in K_{n+1} is adjacent to the sink.

Definition 5.7. Let $c \in \mathcal{S}(K_{n+1})$ and $v_i \in \widetilde{V}$ such that when a grain of sand is added to v_i, an avalanche of size $m \geq 1$ occurs. Define the function

$$\phi : \mathcal{S}(K_{n+1}) \times \widetilde{V} \longrightarrow \widetilde{V} \times \binom{\widetilde{V} \setminus \{v_i\}}{m-1} \times \mathcal{S}(K_m) \times \mathcal{S}(K_{n-m+1}),$$

such that $\phi(c, v_i) = (v_i, J, c_1, c_2)$, where $J = \{w_1, \ldots, w_{m-1}\}$ is the set of $m-1$ vertices that topple other than $w_0 = v_i$. The configuration c_1 in K_m is defined by

$$c_1 = (c(w_1) - (n-m+1), \ldots, c(w_{m-1}) - (n-m+1)),$$

and the configuration c_2 in K_{n-m+1} is defined by the values $c(v_k)$ for $v_k \notin J \cup \{v_i\}$.

Example 5.8. Consider the recurrent configuration $c = (8, 7, 8, 1, 0, 3, 7, 2, 4)$ on K_{10}. Note that adding a grain of sand at v_1 causes an avalanche of size $m = 4$. In this case $J = \{v_2, v_3, v_7\}$, $c_1 = (7-6, 8-6, 7-6) = (1, 2, 1)$ and $c_2 = (1, 0, 3, 2, 4)$. In [13], the authors define the configuration c_1 by substracting $m-2$ instead of $n-m+1$. In here, this would result in the configuration $(5, 6, 5)$ that is not even a stable configuration on K_4.

Lemma 5.9. *The map ϕ described in Definition 5.7 is a bijection.*

Proof. First we need to show that the map ϕ above is well-defined, that is, we need to show that c_1 and c_2 are, in fact, recurrent configurations on K_m and K_{n-m+1}, respectively. To show c_1 is recurrent, let $J = \{w_1, \ldots, w_{m-1}\}$ such that $c(w_i) \leq c(w_{i+1})$ for $1 \leq i \leq m-2$. By Lemma 5.5, for $1 \leq i \leq m-1$, we have that $n-i \leq c(w_i) \leq n-1$. So,

$$n-i-(n-m+1) \leq c(w_i)-(n-m+1) \leq n-1-(n-m+1)$$

and $m-i-1 \leq c_1(w_i) \leq m-2$. This implies c_1 is a stable configuration on K_m. Consider $p_1 = \max_{K_m} - c_1$. For $1 \leq i \leq m-1$, $0 \leq p_1(w_i) \leq i-1$, so p_1 is a parking function and c_1 is recurrent. To show c_2 is recurrent, let $\{u_0, \ldots, u_{n-m-1}\} = \widetilde{V} \setminus (J \cup \{v_i\})$ such that $c(u_i) \leq c(u_{i+1})$ for $0 \leq i \leq n-m-2$. By Lemma 5.5,

$$n-m-i-1 \leq c(u_i) \leq n-m-1.$$

Since $c_2(u_i) = c(u_i)$ then c_2 is stable in K_{n-m+1}. Consider $p_2 = \max_{K_{n-m+1}} - c_2$. For $0 \leq i \leq n-m-2$, we have $0 \leq p_2(i) \leq i$. Since p_2 is a parking function, c_2 is recurrent.

The fact that the map ϕ is injective follows immediately from the definition of c_1 and c_2. Finally, we will show that ϕ is onto. Given (v, J, c_1, c_2) we define c as follows. First, let $c(v) = n-1$. Now, for each $w_i \in J$, define $c(w_i)$ by adding $n-m+1$ to the ith entry in c_1. The remaining $n-m$ entries in c are filled with the entries in c_2. Since c_1 and c_2 are recurrent, Proposition 5.3 implies $p_1 = \max_{K_m} - c_1$ and $p_2 = \max_{K_{n-m+1}} - c_2$ are $(m-1)$ and $(n-m)$-parking functions, respectively. By Lemma 5.4, concatenating p_1 and $p_2 + \overline{m}$ defines an $(n-1)$-parking function p', where $\overline{m} = (m, \ldots, m)$. Furthermore, concatenating 0 and p' gives an n-parking function p. Moreover, $\max_{K_{n+1}} - p$ is a rearrangement of c, so c is a recurrent configuration on K_{n+1}. Clearly, $\phi(c, v) = (v, J, c_1, c_2)$ and this completes the proof. □

From the bijection ϕ we are able to compute the number λ_m of principal avalanches of size $m > 0$. Given $A \subseteq [n]$ with $|A| = m$, Proposition 5.6 states that $\lambda_m / \binom{n}{m}$ is the number of principal avalanches with avalanche monomial $\mu_A = \prod_{i \in A} x_i$. The bijection ϕ implies that this number equals the number of four-tuples (v_i, J, c_1, c_2) with $J \cup \{v_i\} = A$. There are m ways to pick v_i. Cayley's formula implies that the number of recurrents on K_m and K_{n-m+1} is m^{m-2} and $(n-m+1)^{n-m-1}$, respectively. Therefore,

$$\lambda_m = \binom{n}{m} \cdot m \cdot m^{m-2}(n-m+1)^{n-m-1} = \binom{n}{m} m^{m-1}(n-m+1)^{n-m-1}.$$

6 The Avalanche Polynomial of the Wheel

The wheel graph, denoted W_n, is a cycle on $n \geq 3$ vertices with an additional *dominating vertex*. Throughout, the vertices in the cycle will be labeled clockwise as $v_0, \ldots, v_{n-1}$, where the indices are taken modulo n. The dominating vertex, denoted s, will always be assumed to be the sink.

The sandpile group of W_n was first computed by Biggs in [9]:

$$\mathcal{S}(W_n) = \begin{cases} \mathbb{Z}_{l_n} \oplus \mathbb{Z}_{l_n} & \text{if } n \text{ is odd} \\ \mathbb{Z}_{f_n} \oplus \mathbb{Z}_{5f_n} & \text{if } n \text{ is even} \end{cases}$$

where $\{l_n\}$ is the Lucas sequence and $\{f_n\}$ is the Fibonacci sequence. These sequences are defined by initial conditions $l_0 = 2, l_1 = 1$ and $f_0 = 0, f_1 = 1$, respectively, and the recursion $x_n = x_{n-1} + x_{n-2}$. There are many relationships among these numbers. For example, $l_n = f_{n-1} + f_{n+1}$. Morever, the order of $\mathcal{S}(W_n)$ equals the number of spanning trees $\tau(W_n)$ in W_n. This number equals $\tau(W_n) = l_{2n} - 2$, see [20, 6].

We have already computed the avalanche polynomial of W_3 since $W_3 = K_4$. In this case,

$$\mathcal{A}_{W_3}(x_0, x_1, x_2) = 9x_0x_1x_2 + 2(x_0x_1 + x_1x_2 + x_2x_0) + 3(x_0 + x_1 + x_2) + 24.$$

Observe that the set of principal avalanches of size $0 < m < 3$ is partitioned into $n = 3$ subsets of the same size. Moreover, $\mathcal{A}_{W_3}(x_0, x_1, x_2)$ is a linear combination of *cyclic polynomials*. We will show that this characterizes $\mathcal{A}_{W_n}(x_0, \ldots, x_{n-1})$.

Definition 6.1. Let m be an integer such that $1 \leq m \leq n-1$. We will denote by $w_m(x_0, \ldots, x_{n-1})$ the *cyclic polynomial* of degree m on variables $x_0, \ldots, x_{n-1}$ defined as

$$w_m(x_0, \ldots, x_{n-1}) = \sum_{i=0}^{n-1} x_i x_{i+1} \cdots x_{i+m-1}$$

where the indices are taken modulo n.

First note that for each $1 \leq m \leq n-1$, w_m is the sum of n terms of degree m. For example, $w_1 = x_0 + \cdots + x_{n-1}$. For the case $m = n$, the above definition would give

$$w_n = \sum_{i=0}^{n-1} x_i x_{i+1} \cdots x_{i+n-1} = n x_0 \cdots x_{n-1}.$$

However, we will remove the coefficient n and define

$$w_n(x_0, \ldots, x_{n-1}) = x_0 \cdots x_{n-1}.$$

In [16], Dartois and Rossin gave exact results on the distribution of avalanches on W_n. Their approach consisted in showing that the recurrents on W_n can be seen as words of a *regular language*. They built an automaton associated to this language and used the concept of *transducers* to determine the exact distribution of avalanche lengths in this graph. Here we take a different approach focused solely on the structure of the recurrent configurations.

Note that the degree of every non-sink vertex in W_n is 3. So any stable configuration on this graph can be written as a word of length n in the alphabet $\{0, 1, 2\}$. Applying Dhar's Burning Criterion (Proposition 2.7), Cori and Rossin [15] showed that a configuration on W_n is recurrent if and only if there is at least one vertex with 2 grains of sand and between any two vertices with 0 grains, there is at least one vertex with 2 grains.

Definition 6.2. Let m be an integer with $1 \leq m \leq n-1$. A configuration c in W_n has a *maximal 2-string of length* m if there are vertices $v_i, v_{i+1}, \ldots, v_{i+m-1}$, such that $c(v_i) = \cdots = c(v_{i+m-1}) = 2$ and $c(v_{i-1}) \neq 2 \neq c(v_{i+m})$.

Note that $\max_{W_n} = 2^n$ is the unique recurrent with a maximal 2-string of length n. All other recurrent configurations will have a 0 or a 1 at the endpoints of a maximal 2-string. Similar to the comment made before Theorem 4.8, these vertices act as a blocking vertices, because they will never topple and all topplings will occur on the vertices forming the maximal 2-string. We formalize this idea in the following three results.

Lemma 6.3. *Let $c \in \mathcal{S}(W_n)$. The principal avalanche of c at a non-sink vertex v has size $1 \leq m \leq n-2$ if and only if v is part of a maximal 2-string of length m.*

Proof. Suppose a grain is added to a vertex v that is part of a maximal 2-string of length m. Since $m < n-1$, the two non-sink vertices adjacent to the ends of the 2-string are distinct. Thus, exactly the m vertices in the maximal 2-string will topple. On the other hand, if v is part of a longer or shorter maximal 2-string, the avalanche will not have size m. □

Lemma 6.3 implies that for each $1 \leq m \leq n-2$, the number λ_m of principal avalanches of size m equals m times the number of maximal 2-strings of length m over all recurrents. This is not the case for avalanches of size $n-1$ or n as the following simple lemma shows.

Lemma 6.4. *For any non-sink vertex v in W_n, $\mu(2^n, v) = x_0 \cdots x_{n-1}$. Also, let p be an integer with $0 \leq p \leq n-1$. For any non-sink vertex v with $v \neq v_p$ we have*

$$\mu_{W_n}(2^p 1 2^{n-p-1}, v) = x_0 \cdots x_{n-1},$$
$$\mu_{W_n}(2^p 0 2^{n-p-1}, v) = \frac{x_0 \cdots x_{n-1}}{x_p}.$$

This implies $\lambda_n = n^2$ and $\lambda_{n-1} = n(n-1)$.

Proof. Clearly the avalanche monomials for the given recurrents satisfy the above claims. Note also that the only avalanches of size n occur on recurrents of the form $2^p 1 2^{n-p-1}$ and 2^n. So there are $n(n-1)+n = n^2$ avalanches of size n. The avalanches of size $n-1$ occur on recurrents of the form $2^p 0 2^{n-p-1}$. Hence there are $n(n-1)$ avalanches of size $n-1$. □

For each $1 \leq m \leq n-2$, we will count the maximal 2-strings of length m by establishing a map from the set of recurrents on W_n with a given maximal 2-string of length m into the set of recurrents on the *fan graph* F_{n-m}. Let $k \geq 2$, the fan graph on $k+1$ vertices, denoted F_k, is a path on k vertices, plus an additional dominating vertex s, which is the sink.

Proposition 6.5. *For each $1 \leq m \leq n-2$, there is a bijection between the set of recurrents on W_n with a maximal 2-string of length m starting at v_0 and the set of recurrents on F_{n-m}.*

Proof. Let c be a recurrent configuration on F_{n-m}. Dhar's Burning Criterion (Proposition 2.7) implies that adding 1 grain of sand to each vertex must result in an avalanche where every vertex topples exactly once. This implies that at least one of the endpoint vertices has 1 grain of sand or both endpoints have 0 grains of sand and there is an internal vertex with 2 grains of sand. Moreover, if a vertex has 0 grains of sand, then its neighbors must topple before it, hence there are no consecutive vertices with 0 grains of sand. For the same reason, between any two vertices with 0 grains of sand there cannot be a sequence of 1's. In summary, c is a recurrent on F_{n-m} if and only if between any two vertices with 0 grains of sand there is a vertex with 2 grains of sand. Hence c is a recurrent configuration on F_{n-m} if and only if the configuration obtained by prepending a string of m 2's to c is recurrent on W_n. □

It is well-known that the number of spanning trees in the fan graph F_k is precisely the Fibonacci number f_{2k}, see [20]. So Proposition 6.5 implies that for each $1 \leq m \leq n-2$, there are $f_{2(n-m)}$ recurrent configurations that have a maximal 2-string of length m starting at v_0.

Theorem 6.6. *Given $n \geq 3$, the avalanche polynomial of the wheel graph W_n is*

$$\mathcal{A}_{W_n} = n^2 w_n(x_0, \ldots, x_{n-1}) + \sum_{m=1}^{n-1} m \cdot f_{2(n-m)} w_m(x_0, \ldots, x_{n-1}) + 2n\left(f_{2n-1} - 1\right).$$

Proof. In Lemma 6.4 we showed that $\lambda_n = n^2$. This lemma also shows that the avalanches of size $n-1$ are caused by adding a grain of sand at any vertex with 2 grains in any recurrent of the form $2^p 0 2^{n-p-1}$ with $0 \leq p \leq n-1$. Since $\mu_{W_n}(2^p 0 2^{n-p-1}, v) = x_0 \cdots x_{n-1}/x_p$, for any $v \neq v_p$. Then the degree $n-1$ part of $\mathcal{A}_{W_n}$ equals $(n-1) w_{n-1}(x_0, \ldots, x_{n-1})$. Note that when $m = n-1$ we have $f_{2(n-m)} = f_2 = 1$.

Now let $1 \leq m \leq n-2$. Proposition 6.5 implies that there are $f_{2(n-m)}$ recurrents on W_n with a maximal 2-string of length m starting at v_0. So by Lemma 6.3, there are $m f_{2(n-m)}$ principal avalanches with avalanche monomial $x_0 \cdots x_{m-1}$. This lemma also shows that any avalanche of size m must occur at a maximal 2-string of length m. So the only possible avalanche monomials of degree m are the monomials occuring in the cyclic polynomial w_m. Moreover, the cyclic symmetry of W_n implies that the number of principal avalanches that produce the toppling sequence $(v_0, v_1, \ldots, v_{m-1})$ equals the number of principal avalanches that produce the toppling sequence $(v_i, v_{i+1}, \ldots, v_{i+m-1})$ for any $0 \leq i \leq n-1$. Therefore, for any $1 \leq m \leq n-2$, the degree m part of $\mathcal{A}_{W_n}$ equals $m f_{2(n-m)} w_m(x_0, \ldots, x_{n-1})$.

Lastly, note that an avalanche of size 0 is produced by adding a grain of sand to a vertex with 0 or 1 grains of sand. So λ_0 equals the number of 0's and 1's in every recurrent. Since there are $l_{2n} - 2$ recurrents on W_n, then λ_0 equals $n(l_{2n} - 2)$ minus the total number of 2's in every recurrent. Recall that for $1 \leq m \leq n-2$, the number λ_m of principal avalanches of size m equals m times the number of maximal 2-strings of length m over all recurrents, that is, λ_m equals the total number of 2's in every maximal 2-string of length m. Moreover, $\lambda_{n-1} + \lambda_n = n^2 + n(n-1) = 2n^2 - n$ equals the number of principal avalanches of size $\geq n-1$. But this number also equals the number of 2's in every recurrent with a maximal 2-string of size $\geq n-1$. Therefore, $\lambda_1 + \cdots + \lambda_n$ equals the number of 2's in every recurrent. Hence

$$\begin{aligned}
\lambda_0 &= n(l_{2n} - 2) - (\lambda_1 + \cdots + \lambda_n) = n(l_{2n} - 2) - 2n^2 + n - \sum_{m=1}^{n-2} nmf_{2(n-m)} \\
&= n\left[l_{2n} - 2n - 1 - \sum_{m=1}^{n-2} mf_{2(n-m)}\right] = n\left[l_{2n} - 2n - 1 - \sum_{m=2}^{n-1} (n-m)f_{2m}\right] \\
&= n\left[l_{2n} - n - 2 - \sum_{m=1}^{n-1} (n-m)f_{2m}\right] = n\left[l_{2n} - n - 2 - \sum_{m=1}^{n-1}\sum_{k=1}^{m} f_{2k}\right] \\
&= n\left[l_{2n} - n - 2 - \sum_{m=1}^{n-1} (f_{2m+1} - 1)\right] = n\left[l_{2n} - 3 - \sum_{m=1}^{n-1} f_{2m+1}\right] \\
&= n(l_{2n} - 2 - f_{2n}) = n(f_{2n+1} + f_{2n-1} - f_{2n} - 2) \\
&= n(2f_{2n-1} - 2) = 2n(f_{2n-1} - 1).
\end{aligned}$$

□

In this case, λ_m is also the number of principal avalanches with burst size m since every non-sink vertex in W_n is adjacent to the sink. Note also that as $n \to \infty$, the proportion of avalanches of size 0 is

$$\lim_{n\to\infty} \frac{2n(f_{2n-1} - 1)}{n(l_{2n} - 2)} = 1 - \frac{1}{\sqrt{5}}.$$

Thus, recovering the last result in [16, Section 2].

Acknowledgements

The authors would like to thank the 2014 PURE Math program and the University of Hawai'i at Hilo for making this collaboration possible. We would especially like to thank David Perkinson for suggesting to study the multivariate version of the avalanche polynomial and for his invaluable guidance and support throughout this project. We would also like to acknowledge Andrew Fry, Lionel Levine, Christopher O'Neill, and Gautam Webb for their kind input and suggestions related to this work. We also want to thank the anonymous reviewers that helped improved our work with their careful reading and many suggestions.

References

[1] R. Bacher, P. de la Harpe and T. Nagnibeda, The lattice of integral flows and the lattice of integral cuts on a finite graph, *Bull. Soc. Math. France* 125(2) (1997),167–198.

[2] P. Bak, *How Nature Works: The Science of Self-organized Criticality*, Copernicus, New York, (1996).

[3] P Bak and M Paczuski, Complexity, contingency, and criticality, *Proc. Nat. Acad. Sciences* 92(15) (1995), 6689–6696.

[4] P. Bak, C. Tang and K. Wiesenfeld, Self-organized criticality, *Phys. Rev. A (3)* 38(1) (1988), 364–374.

[5] M. Baker and S. Norine, Riemann-Roch and Abel-Jacobi theory on a finite graph, *Adv. Math.* 215(2) (2007), 766–788.

[6] A. Benjamin and C. Yerger, Combinatorial interpretations of spanning tree identities, *Bull. Inst. Combin. Appl.* 47 (2006), 37–42.

[7] B. Benson, D. Chakrabarty and P. Tetali, G-parking functions, acyclic orientations and spanning trees, *Discrete Math.* 310(8) (2010), 1340–1353.

[8] K. Berman, Bicycles and spanning trees, *SIAM J. Algebraic Discrete Methods* 7(1) (1986), 1–12.

[9] N. Biggs, Chip-firing and the critical group of a graph, *J. Algebraic Comb.* 9 (1999), 25–45.

[10] A. Björner and L. Lovász, Chip-firing games on directed graphs, *J. Algebraic Combin.* 1(4) (1992), 305–328.

[11] A. Björner, L. Lovász and P. Shor, Chip-firing games on graphs, *European J. Combin.* 12(4) (1991), 283–291.

[12] K. Chen, P. Bak and S. Obukhov, Self-organized criticality in a crack-propagation model of earthquakes, *Phys. Rev. A* 43 (1991), 625–630.

[13] R. Cori, A. Dartois and D. Rossin, Avalanche polynomials of some families of graphs, In *Mathematics and computer science III*, Trends Math.(2004), 81–94.

[14] R. Cori, A. Micheli and D. Rossin, Avalanche polynomials, *ArXiv e-prints* (2009).

[15] R. Cori and D. Rossin, On the sandpile group of dual graphs, *European J. Combin.* 21(4) (2000), 447–459.

[16] A. Dartois and D. Rossin, Analysis of the distribution of the length of avalanches on the sandpile group of the (n, k)-wheel, In *Formal Power Series and Algebraic Combinatorics—15th+ Conference (FPSAC'03)*, (2003) (Eds. S. Linusson, K. Eriksson and A. Björner).

[17] The Sage Developers, *SageMath, the Sage Mathematics Software System (Version 7.1)* (2016), `http://www.sagemath.org`.

[18] D. Dhar, Self-organized critical state of sandpile automaton models, *Phys. Rev. Lett.* 64(14) (1990), 1613–1616.

[19] D. Dhar, P. Ruelle, S. Sen and D. Verma, Algebraic aspects of sandpile models, *J. Phys. A* (1995), 1995.

[20] A. J. W. Hilton, The number of spanning trees of labeled wheels, fans and baskets, In *Combinatorics (Proc. Conf. Combinatorial Math., Math. Inst., Oxford)* (1972), 203–206.

[21] A. Holroyd, L. Levine, K. Mészáros, Y. Peres, J. Propp and D. Wilson, Chip-firing and rotor-routing on directed graphs, In *In and out of equilibrium. 2, Progress in Probability* Vol. 60 (2008), 331–364.

[22] H. Jensen, *Self-organized criticality : emergent complex behavior in physical and biological systems*, Cambridge Lec. Notes in Physics, Cambridge University Press, Cambridge (1998).

[23] D. V. Ktitarev, S. Lübeck, P. Grassberger and V. B. Priezzhev, Scaling of waves in the bak-tang-wiesenfeld sandpile model, *Phys. Rev. E* 61 (200), 81–92.

[24] L. Levine, Threshold state and a conjecture of Poghosyan, Poghosyan, Priezzhev and Ruelle, *Comm. Math. Phys.* 335(2) (2015) 1003–1017.

[25] D. Lorenzini, Arithmetical graphs, *Math. Ann.* 285(3) (1989), 481–501.

[26] D. Marković and C. Gros, Power laws and self-organized criticality in theory and nature, *Physics Reports* 536(2) (2014), 41–74.

[27] C. Merino, The chip-firing game, *Discrete Math.* 302(1-3) (2005), 188–210.

[28] Su. S. Poghosyan, V. S. Poghosyan, V. B. Priezzhev and P. Ruelle, Numerical study of the correspondence between the dissipative and fixed-energy abelian sandpile models, *Phys. Rev. E* 84 (2011), 066119.

[29] G. Pruessner, *Self-organised criticality : theory, models, and characterisation*, Cambridge; New York: Cambridge University Press, (2012).

[30] A. Sornette and D. Sornette, Self-organized criticality and earthquakes, *EPL (Europhysics Lett.)* 9(3) (1989), 197.

(Received 17 June 2016; revised 21 Sep 2018)