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Minimum degree conditions for the existence of fractional factors in planar graphs

P. Katerinis

Department of Informatics,
Athens University of Economics and Business
Greece
pek@aueb.gr

Abstract

Let G be a simple planar graph, X, Y disjoint subsets of $E(G)$ and let $1 \leq \ell \leq 3$ be an integer. If one of the following four conditions holds then the graph $G - X$ contains a fractional ℓ-factor with indicator function h such that $h(e) = 1$ for every $e \in Y$.

(a) $\delta(G) \geq 4, \ell = 1, |X| \leq 2$ and $|Y| = 0$.

(b) $\delta(G) \geq 4, \ell = 1, |X| = 0$ and $|Y| \leq 1$.

(c) $\delta(G) \geq 4, \ell = 2$ and $|X| + |Y| \leq 2$.

(d) $\delta(G) \geq 5, \ell = 3, |X| + |Y| \leq 3$ and $|X| \leq 2$ if all the elements of X have a common end-vertex.

1 Introduction and Terminology

All graphs considered are assumed to be simple and finite. We refer the reader to [2] for standard graph theoretic terms not defined in this paper.

Let G be a graph. The degree $d_G(u)$ of a vertex u in G is the number of edges of G incident with u. The minimum degree of G is denoted by $\delta(G)$. If X and Y are subsets of $V(G)$, the set and the number of the edges of G joining X to Y are denoted by $E_G(X, Y)$ and $e_G(X, Y)$ respectively. For any set X of vertices in G, the subgraph induced by X is denoted by $G[X]$ and the neighbour set of X in G by $N_G(X)$. Similarly, for any set X of edges in G, the subgraph induced by X is denoted by $G[X]$. The number of connected components of G is denoted by $\omega(G)$. A cut edge of G is an edge e such that $\omega(G - \{e\}) > \omega(G)$.

A *bipartite graph* is one whose vertex set can be partitioned into two subsets X and Y, so that each edge has one end in X and one end in Y; such a partition (X, Y) is called a *bipartition* of the graph. The following result is a well-known characterization of bipartite graphs.

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Theorem 1.1 *A graph is bipartite if and only if it contains no odd cycle.*

Let g and f be two nonnegative integer-valued functions defined on $V(G)$ such that $g(x) \leq f(x)$ for each $x \in V(G)$ and let $h : E(G) \mapsto [0,1]$ be also a function such that $g(x) \leq d_G^h(x) \leq f(x)$ for every $x \in V(G)$, where $d_G^h(x) = \sum_{e \in E(x)} h(e)$ and $E(x)$ denotes the set of edges incident with vertex x. If we define $F_h = \{e \in E(G) : h(e) > 0\}$, then we call $G[F_h]$ a fractional (g,f)-factor of G with indicator function h. If $g(x) = a$ and $f(x) = b$ for all $x \in V(G)$, then we will call such a fractional (g,f)-factor, a fractional $[a,b]$-factor. A fractional (f,f)-factor is called simply a fractional f-factor. If $f(x) = \ell$ for every $x \in V(G)$, then a fractional f-factor is called a fractional ℓ-factor.

Furthermore if function h takes only integral values (0 and 1), then a fractional k-factor and fractional $[a,b]$-factor are called k-factor and $[a,b]$-factor respectively. Clearly a k-factor of G is a k-regular spanning subgraph of that graph.

The following necessary and sufficient condition for a graph to have a fractional (g,f)-factor was obtained by Anstee [1]. Liu and Zhang [5] later presented a simple proof.

Theorem 1.2 *Let G be a graph and let g, f be two nonnegative integer-valued functions defined on $V(G)$ such that $g(x) \leq f(x)$ for all $x \in V(G)$. Then, G has a fractional (g,f)-factor if and only if for any $S \subseteq V(G)$,*

$$\sum_{x \in T} (g(x) - d_{G-S}(x)) \leq \sum_{x \in S} f(x)$$

where $T = \{x : x \in V(G) - S, d_{G-S}(x) \leq g(x)\}$.

A graph is said to be planar or embeddable in the plane, if it can be drawn in the plane so that its edges intersect only at their ends. Such a drawing of a planar graph G is called a planar embedding of G. It can be regarded as a graph isomorphic to G and we sometimes refer to it as a plane graph. A planar embedding of planar graph divides the plane into a number of connected regions, called faces, each bounded by edges of the graph. We shall denote by $F(G)$ and $\phi(G)$ the set and the number respectively of faces of a plane graph G.

Each plane graph has exactly one unbounded face called the exterior face. For every plane graph G, we denote the boundary of a face f of G by $b(f)$. If G is connected, $b(f)$ can be regarded as a closed walk in which each cut edge of G in $b(f)$ is traversed twice. A face f is said to be incident with the vertices and edges in its boundary. If e is a cut edge in G, just one face is incident with e, otherwise there are two faces incident with e. The degree $d_G(f)$, of a face f of G is the number of edges with which it is incident (cut edges are counted twice).

The following proposition and three theorems related to planar graphs are well-known results.

Proposition 1.3 *If G is planar, then every subgraph of G is also planar.*

Theorem 1.4 (Euler's formula) *If G is a connected plane graph, then*

$$|V(G)| - |E(G)| + \phi(G) = 2.$$

Theorem 1.5 *If G is a plane graph, then*

$$\sum_{f \in F(G)} d_G(f) = 2|E(G)|.$$

Theorem 1.6 *For every planar graph G, $\delta(G) \leq 5$.*

The following result can be easily derived as a corollary of Theorems 1.4 and 1.5.

Corollary 1.7 *If G is a connected plane triangle-free graph with at least three vertices, then $|E(G)| \leq 2|V(G)| - 4$.*

2 The main result

The discussion concerning the existence of a (fractional) k-factor or a (fractional) $[a, b]$-factor in a planar graph is meaningful only for the cases when $k \leq 5$ and $a \leq 5$ respectively, by using Theorem 1.6. The existence of such $[a, b]$-factors in planar graphs were studied recently by the author [4] and related results for the existence of connected $[a, b]$-factors can also be found in [3].

The main purpose of this paper is to present the following sufficient minimum degree conditions for the existence of fractional k-factors in planar graphs, having indicator function which assigns to a prescribed number of edges integral values, 0 and 1.

Theorem 2.1 *Let G be a planar graph, X, Y disjoint subsets of $E(G)$ and let $1 \leq \ell \leq 3$ be an integer. If one of the following four conditions holds then the graph $G - X$ contains a fractional ℓ-factor with indicator function h such that $h(e) = 1$ for every $e \in Y$.*

(a) $\delta(G) \geq 4, \ell = 1, |X| \leq 2$ *and* $|Y| = 0$.

(b) $\delta(G) \geq 4, \ell = 1, |X| = 0$ *and* $|Y| \leq 1$.

(c) $\delta(G) \geq 4, \ell = 2$ *and* $|X| + |Y| \leq 2$.

(d) $\delta(G) \geq 5, \ell = 3, |X| + |Y| \leq 3$ *and* $|X| \leq 2$ *if all the elements of X have a common end-vertex.*

Proof. Let G^* be the graph obtained from $G-X$ by inserting to every edge belonging to Y, a vertex of degree 2. Clearly $G-X$ has a fractional ℓ-factor with indicator function h such that $h(e)=1$ for every $e \in Y$ if and only if G^* has a fractional f-factor satisfying $f(x)=2$ for $x \in V(G^*)-V(G)$ and $f(x)=\ell$ for $x \in V(G)$.

Suppose that the theorem does not hold. Then G^* does not contain a fractional f-factor having the properties described above. So by using Theorem 1.2, there exists $S \subseteq V(G^*)$ such that

$$\sum_{x\in T}(f(x)-d_{G^*-S}(x)) > \sum_{x\in S} f(x) \tag{2.1}$$

where $T=\{x \in V(G^*)-S : d_{G^*-S}(x) \leq f(x)\}$.

We suppose that S is minimal with respect to (2.1). We will first prove the following claim.

Claim 2.2 *For every $u \in S$,*

$$|N_{G^*}(u)\cap T'| \geq f(u)+1$$

where $T'=\{x \in T : d_{G^*-S}(x) < f(x)\}$.

Proof. Suppose that there exists $z \in S$ such that $|N_{G^*}(z)\cap T'| \leq f(z)$. If we define $S_o = S-\{z\}$ and $T_o=\{x \in V(G^*)-S_o : d_{G^*-S_o}(x) \leq f(x)\}$,

$$\begin{aligned}
\sum_{x\in T_o}(f(x)-d_{G^*-S_o}(x)) &\geq \sum_{x\in T}(f(x)-d_{G^*-S}(x)) - |N_{G^*}(z)\cap T'| \\
&\geq \sum_{x\in T}(f(x)-d_{G^*-S}(x)) - f(z) \\
&> \sum_{x\in S} f(x) - f(z) \qquad \text{by (2.1)} \\
&\geq \sum_{x\in S_o} f(x). \qquad (2.2)
\end{aligned}$$

But (2.2) contradicts the minimality of S with respect to (2.1). So Claim 2.2 holds. □

Now define $W=V(G^*)-V(G)$. For every $x \in W$, $f(x)=d_{G^*}(x)=2$. Thus if $x \in S\cap W$, then $|N_{G^*}(x)\cap T'| \leq f(x)$ contradicting Claim 2.2. So we may assume that $S\cap W=\emptyset$. Hence $W \subseteq V(G^*)-S$ and in fact

$$W \subseteq T \tag{2.3}$$

since for every $x \in W$, $d_{G^*-S}(x) \leq f(x)$ because as we mentioned earlier $f(x) = d_{G^*}(x)=2$. Thus using (2.3), (2.1) yields

$$\sum_{x\in W}(2-d_{G^*-S}(x)) + \sum_{x\in T-W}(\ell-d_{G^*-S}(x)) > \ell|S|. \tag{2.4}$$

Suppose that $S = \emptyset$. Then

$$\sum_{x \in W} (2 - d_{G^*-S}(x)) = 0 \tag{2.5}$$

since $d_{G^*-S}(x) = d_{G^*}(x) = 2$ for every $x \in W$. Moreover for every $x \in ((V(G^*) - S) - W) = V(G^*) - W = V(G)$, $d_{G^*-S}(x) = d_{G^*}(x) \geq d_G(x) - |X|$ and so this together with the minimum degree condition implies that $d_{G^*-S}(x) \geq 2$, when conditions (a),(b),(c) hold, and $d_{G^*-S}(x) \geq 3$, when condition (d) holds.

Thus $\ell - d_{G^*-S}(x) \leq 0$ and hence

$$\sum_{x \in T-W} (\ell - d_{G^*-S}(x)) = 0. \tag{2.6}$$

But if we use (2.5) and (2.6), then (2.4) yields a contradiction. So we may assume that

$$S \neq \emptyset. \tag{2.7}$$

We next suppose that $T' - W = \emptyset$. Then (2.4) implies

$$\sum_{x \in W} (2 - d_{G^*-S}(x)) > \ell|S|. \tag{2.8}$$

But $|W| = |Y| \leq \ell$ by the conditions of the theorem. Thus we have

$$\begin{aligned} \sum_{x \in W} (2 - d_{G^*-S}(x)) &\leq 2|W| \\ &\leq 2\ell \end{aligned}$$

and so (2.7) and (2.8) imply $|S| = 1$.

On the other hand if this is the case then

$$\begin{aligned} d_{G^*-S}(x) &\geq d_{G^*}(x) - |S| \\ &\geq 1 \end{aligned}$$

for every $x \in W$; and thus

$$\sum_{x \in W} (2 - d_{G^*-S}(x)) \leq |W| \leq \ell,$$

which contradicts (2.8). So we may also assume that

$$T' - W \neq \emptyset. \tag{2.9}$$

Let M_1 be the set of edges of $G - S$ belonging to X having exactly one end-vertex in $T' - W$ and let M_2 be the set of edges belonging to X having both end-vertices in $T' - W$. Define $|M_1| = m_1$ and $|M_2| = m_2$. Then

$$\sum_{x \in T'-W} d_{G^*-S}(x) \geq \sum_{x \in T'-W} d_{G-S}(x) - m_1 - 2m_2$$

since $d_{G^*-S}(x) \geq d_{(G-X)-S}(x)$ for every $x \in T' - W$. Thus (2.4) yields that

$$2|W| + \ell|T' - W| - \sum_{x\in T'-W} d_{G-S}(x) + m_1 + 2m_2 > \ell|S|. \tag{2.10}$$

At this point we consider the induced bipartite subgraph H of G having as bipartition the sets S and $T' - W$. As we saw earlier in (2.7) and (2.9), S and $T' - W$ are non-empty sets. Furthermore for every $x \in S$,

$$|N_{G^*}(x) \cap T'| \geq \ell + 1 \qquad \text{by Claim 2.2}$$

and so

$$\begin{aligned} d_H(x) &\geq \ell + 1 - |W| \\ &\geq \ell + 1 - |Y| \geq 1 \end{aligned} \tag{2.11}$$

since $|Y| \leq \ell$ by the conditions of the theorem.

On the other hand for every $x \in T'-W$, $d_{G^*-S}(x) \leq \ell-1$ and so $d_{G-S}(x) \leq \ell-1+|X|$. Thus

$$d_H(x) \geq \delta(G) - \ell + 1 - |X| \geq 1 \tag{2.12}$$

by using again the conditions of the theorem.

Hence (2.11) and (2.12) yield that every element of S is adjacent in H to at least one element of $T' - W$ and conversely every element of $T' - W$ is adjacent in H to at least one element of S.

Suppose now that there exists an element x of S, which is adjacent in H to exactly one element of $T' - W$, say y. Then (2.11) implies that $|Y| = \ell$. So by using the conditions of the theorem, either $\delta(G) \geq 4$, $1 \leq \ell \leq 2$ and $|X|$=0 or $\delta(G) \geq 5$, $\ell = 3$ and $|X|$=0.

Thus (2.12) yields

$$d_H(y) \geq \delta(G) - \ell + 1 - |X| \geq 3,$$

and hence we may assume that every component of H contains at least 3 vertices. Clearly the bipartite graph H is a planar graph by Proposition 1.3; and so

$$\begin{aligned} |E(H)| &\leq 2|V(H)| - 4 \\ &\leq 2(|S| + |T' - W|) - 4 \end{aligned} \tag{2.13}$$

by using Theorem 1.1 and Corollary 1.7.

But $|E(H)| \geq \delta(G)|T' - W| - \sum_{x\in T'-W} d_{G-S}(x)$ and thus

$$(\delta(G) - 2)|T' - W| - \sum_{x\in T'-W} d_{G-S}(x) \leq 2|S| - 4,$$

which implies

$$\frac{\ell(\delta(G) - 2)}{2}|T' - W| - \sum_{x\in T'-W} d_{G-S}(x) \leq \ell|S| - 2\ell + \frac{(\ell - 2)}{2} \sum_{x\in T'-W} d_{G-S}(x). \tag{2.14}$$

At this point, we consider the following cases:

Case 1: $\ell = 1, \delta(G) \geq 4$.
Then (2.14) implies

$$|T' - W| - \sum_{x \in T'-W} d_{G-S}(x) \leq |S| - 2 - \frac{1}{2} \sum_{x \in T'-W} d_{G-S}(x). \tag{2.15}$$

On the other hand (2.10) yields

$$|T' - W| - \sum_{x \in T'-W} d_{G-S}(x) > |S| - m_1 - 2m_2 - 2|W|. \tag{2.16}$$

We consider the following subcases:

Case 1a: $|X| \leq 2$ and $|Y| = 0$.
Then we have from (2.15),

$$|T' - W| - \sum_{x \in T'-W} d_{G-S}(x) \leq |S| - 2 - \frac{1}{2}(m_1 + 2m_2)$$

since $\sum_{x \in T'-W} d_{G-S}(x) \geq m_1 + 2m_2$. But $m_1 + m_2 \leq |X| \leq 2$, so it follows

$$|T' - W| - \sum_{x \in T'-W} d_{G-S}(x) \leq |S| - \frac{3}{2}m_1 - 2m_2,$$

which contradicts (2.16) since $|W| = |Y| = 0$.

Case 1b: $|X| = 0$ and $|Y| \leq 1$.
Then (2.16) implies,

$$|T' - W| - \sum_{x \in T'-W} d_{G-S}(x) > |S| - 2 \tag{2.17}$$

since $m_1 + m_2 = 0$ and $|W| = |Y| \leq 1$. But (2.17) contradicts (2.15).

Case 2: $\ell = 2$, $\delta(G) \geq 4$, $|X| + |Y| \leq 2$.
Then (2.14) yields

$$2|T' - W| - \sum_{x \in T'-W} d_{G-S}(x) \leq 2|S| - 4. \tag{2.18}$$

On the other hand (2.10) implies

$$\begin{aligned} 2|T' - W| - \sum_{x \in T'-W} d_{G-S}(x) &> 2|S| - m_1 - 2m_2 - 2|W| \\ &\geq 2|S| - 2(|W| + m_1 + m_2) \\ &\geq 2|S| - 2(|X| + |Y|) \\ &\geq 2|S| - 4 \qquad \text{since } |X| + |Y| \leq 2, \end{aligned}$$

which contradicts (2.18).

Case 3: $\ell = 3$, $\delta(G) \geq 5$, $|X| + |Y| \leq 3$ and $|X| \leq 2$ if all the elements of X have a common end-vertex.

We consider the following subcases.

Case 3a: $|X| = 3$.

It is obvious in this case that $|W| = |Y| = 0$ since $|X| + |Y| \leq 3$, so (2.10) implies

$$3|T'| - \sum_{x \in T'} d_{G-S}(x) + m_1 + 2m_2 > 3|S|. \tag{2.19}$$

But

$$\begin{aligned}\sum_{x \in T'} d_{G-S}(x) &\geq \delta(G)|T'| - e_G(S, T') \\ &\geq 5|T'| - |E(H)| \\ &\geq 5|T'| - (2(|S| + |T'|) - 4) && \text{by (2.13)} \\ &\geq 3|T'| - 2|S| + 4.\end{aligned}$$

So (2.19) yields

$$m_1 + 2m_2 - 5 \geq |S|. \tag{2.20}$$

But as we saw earlier $S \neq \emptyset$ and $m_1 + 2m_2 \leq 2|X| \leq 6$. So we obtain from (2.20), $|S| = 1$ and $m_1 + 2m_2 = 6$. On the other hand,

$$\begin{aligned}\sum_{x \in T'} d_{G-S}(x) &\geq \sum_{x \in T'} (d_G(x) - |S|) \\ &\geq 4|T'|, && \text{since } |S| = 1 \text{ and } \delta(G) \geq 5.\end{aligned}$$

Thus (2.19) implies

$$m_1 + 2m_2 - 3|S| > |T'|$$

or

$$|T'| \leq 2$$

since $|S| = 1$ and $m_1 + 2m_2 = 6$. But if $|T'| \leq 2$ then $m_2 \leq 1$ and so $m_1 \geq 4$ since $m_1 + 2m_2 = 6$; which contradicts the fact that $m_1 \leq |X| \leq 3$.

Case 3b: $|X| \leq 2$.

Then (2.14) implies

$$3|T' - W| + \frac{3}{2}|T' - W| - \sum_{x \in T' - W} d_{G-S}(x) \leq 3|S| - 6 + \frac{1}{2} \sum_{x \in T' - W} d_{G-S}(x).$$

But

$$\sum_{x\in T'-W} d_{G-S}(x) \le \sum_{x\in T'-W} d_{G^*-S}(x) + m_1 + 2m_2$$
$$\le 2|T'-W| + m_1 + 2m_2$$

since $d_{G^*-S}(x) \le 2$, for every $x \in T'$.

Thus

$$3|T'-W| + \frac{|T'-W|}{2} - \sum_{x\in T'-W} d_{G-S}(x) \le 3|S| - 6 + \frac{m_1+2m_2}{2}.$$

But if $m_2 = 1$ then $|T'-W| \ge 2$ and if $m_2 = 2$ then $|T'-W| \ge 3$. So by using the fact that $m_1 + m_2 \le |X| \le 2$, we have

$$|T'-W| - m_1 - 2m_2 \ge -1$$

and thus

$$3|T'-W| - \frac{1}{2} - \sum_{x\in T'-W} d_{G-S}(x) \le 3|S| - 6.$$

Hence

$$3|T'-W| - \sum_{x\in T'-W} d_{G-S}(x) \le 3|S| - 6. \tag{2.21}$$

On the other hand since $|W| + m_1 + m_2 \le |X| + |Y| \le 3$, (2.10) implies

$$3|T'-W| - \sum_{x\in T'-W} d_{G-S}(x) > 3|S| - 6$$

which contradicts (2.21).

The contradictions that we get in all cases complete the proof of the theorem. □

3 Remarks on the sharpness of the result

We will show in this section that the conditions of the hypothesis of Theorem 2.1 are in some sense best possible. We will first prove this for the minimum degree condition. We will describe initially a family of graphs having slightly lower minimum degree and not possessing a fractional ℓ-factor for $1 \le \ell \le 2$. We start from a cycle $C = u_1u_2u_3\dots u_{2k}u_1$, where $k \ge 6$ and vertices v_1, v_2, w. We join w to all vertices of C, v_1 to vertices u_1, u_3, u_5 and v_2 to vertices u_7, u_9, u_{11}. The resulting family of graphs G are clearly planar and $\delta(G) = 3$. Furthermore graphs G do not possess a fractional ℓ-factor for $\ell = 1, 2$ by Theorem 1.2, because if we let $S = \{u_1, u_3, u_5, u_7, \dots, u_{2k-3}, u_{2k-1}\} \cup \{w\}$ and define $T = \{x \in V(G) - S : d_{G-S}(x) \le \ell\}$,

$$\sum_{x\in T} (\ell - d_{G-S}(x)) > \ell|S|$$

since $|S| = k+1$, $\sum_{x\in T}(\ell - d_{G-S}(x)) = \ell|T|$ and $|T| = k+2$.

We will also show that the degree condition is best possible for $\ell = 3$. We consider a cycle $C = u_1u_2u_3\ldots u_ku_1$, where $k \geq 7$ and vertices v_1, v_2 which are joined to all vertices of C. The resulting family of graphs G are clearly planar and $\delta(G) = 4$. Furthermore graphs G do not possess a fractional 3-factor because if we let $S = \{v_1, v_2\}$ and define $T = \{x \in V(G) - S : d_{G-S}(x) \leq 3\}$,

$$\sum_{x\in T}(3 - d_{G-S}(x) > 3|S|$$

since $|S| = 2$, $\sum_{x\in T}(3 - d_{G-S}(x)) = |T|$ and $|T| = k \geq 7$.

We will next show that the elements of X and Y cannot be increased. We will show in other words that the number of edges to which the indicator function h assigns integral values, either 1 or 0, cannot be increased. For this purpose we will first describe a family of graphs G which constitutes counterexamples to an opposite claim for cases (a), (b), and (c).

Let H be a simple plane graph such that $\delta(H) \geq 4$ and let z be a vertex belonging to the exterior face of H. We also consider a cycle $C = u_1e_1u_2e_2u_3e_3u_4e_4u_5e_5u_1$ and vertex w. The family of graphs under consideration are obtained by joining w, z to all vertices of C. Clearly G are planar and $\delta(G) \geq 4$.

Claim 3.1 *The elements of X in case (a) of Theorem 2.1 cannot be increased.*

Proof. Let $X = \{e_1, e_4, e_5\}$ and $G^* = G - X$. The family of graphs G^* does not possess a fractional 1-factor because if we let $S = \{u_3, z, w\}$ and define $T = \{x \in V(G^*) - S : d_{G^*-S}(x) \leq 1\}$, then

$$\sum_{x\in T}(1 - d_{G^*-S}(x)) > |S|$$

since $\sum_{x\in T}(1 - d_{G^*-S}(x)) = 4$ and $|S| = 3$. □

Claim 3.2 *The number of elements of Y in case (b) of Theorem 2.1 cannot be increased.*

Proof. If $|Y| \geq 2$ and all the elements of Y have a common end-vertex then G clearly does not possess a fractional 1-factor with indicator function h such that $h(e) = 1$ for every $e \in Y$. We will show that the number of edges of Y cannot be increased, even if the elements of Y are independent edges in G. Let $Y = \{e_1, e_6\}$ where e_6 is the edge of G having as end-vertices w and u_4. Define G^* to be the graph obtained from G by inserting a vertex of degree 2 to every element of Y. The family of graphs G^* does not possess a fractional f-factor such that $f(x) = 2$ for $x \in V(G^*) - V(G)$ and $f(x) = 1$ for $x \in V(G)$, because if we let $S = \{u_1, u_2, u_4, w, z\}$ and define $T = \{x \in V(G^*) - S : d_{G^*-S}(x) \leq f(x)\}$, then

$$\sum_{x\in T}(f(x) - d_{G^*-S}(x)) > |S|$$

since $\sum_{x\in T}(f(x) - d_{G^*-S}(x)) = 6$ and $|S| = 5$. □

Claim 3.3 *The sets X and Y in cases (a) and (b) of Theorem 2.1 cannot be both non-empty.*

Proof. Let $X = \{e_5\}$ and $Y = \{e_6\}$, where e_6 is the edge of G having end-vertices w and u_4, as we mentioned earlier. Define G^* to be the graph obtained from $G - X$ by inserting a vertex of degree 2 to edge e_6. The family of graphs G^* does not possess a fractional f-factor such that $f(x) = 2$ for $x \in V(G^*) - V(G)$ and $f(x) = 1$ for $x \in V(G)$, because if we let $S = \{z, w, u_2, u_4\}$, then

$$\sum_{x \in T}(f(x) - d_{G^*-S}(x)) > |S|$$

since $\sum_{x\in T}(f(x) - d_{G^*-S}(x)) = 5$ and $|S| = 4$. □

Claim 3.4 *The sum $|X| + |Y|$ cannot be increased in case (c).*

Proof. Let $X = \{e_2, e_5\}$, $Y = \{e_6\}$, where e_6 is the edge of G having again end-vertices w, u_4 and let G^* be the graph obtained from $G - X$ by inserting to edge e_6 a vertex of degree 2. The family of graphs G^* does not possess a fractional 2-factor because if we let $S = \{z, w, u_4\}$ and define $T = \{x \in V(G^*) - S : d_{G^*-S}(x) \le 2\}$, then

$$\sum_{x \in T}(2 - d_{G^*-S}(x)) > 2|S|$$

since $\sum_{x\in T}(2 - d_{G^*-S}(x)) = 8$ and $|S| = 3$. □

We will also describe a family of graphs which shows that the elements of X and Y cannot be increased in case (d). For the construction of such graphs G, we first consider a simple plane graph H such that $\delta(H) \ge 5$, whose exterior face is incident with at least 4 vertices. We take 7 copies of H and for every such copy H_i, we choose vertices $u_{1,i}, u_{2,i}, u_{3,i}, u_{4,i}$ belonging to the exterior face of H_i, where $i = 1, 2, \ldots, 7$. We also consider 6 copies of K_2 and for every such copy F_i, let $V(F_i) = \{v_{1,i}, v_{2,i}\}$, where $i = 1, 2, \ldots, 6$. In addition we consider vertices w_1 and w_2. The family of graphs G mentioned above is constructed as follows: For all $i = 1, 2, \ldots, 6$ we join $u_{3,i}, u_{4,i}$ to $v_{1,i}$ and for all $i = 2, \ldots, 7$ we join $u_{1,i}, u_{2,i}$ to $v_{2,i-1}$. Finally we join w_1, w_2 to all the elements of $V(F_1) \cup V(F_2) \cup \cdots \cup V(F_6)$ and w_1 to w_2.

Claim 3.5 *The sum $|X| + |Y|$ in case (d) cannot be increased.*

Proof. Let e_1, e_2 be the edges of G having as end-vertices the elements of the sets $\{w_1, w_2\}$ and $\{w_1, v_{1,1}\}$ respectively and define $Y = \{e_1, e_2\}$, $X = E(F_2) \cup E(F_3)$. Let also G^* be the family of graphs obtained from $G - X$ by inserting to every element of Y a vertex of degree 2. The family of graphs G^* does not possess a fractional f-factor such that $f(x) = 2$ for $x \in V(G^*) - V(G)$ and $f(x) = 3$ for $x \in V(G)$, because if we let $S = \{w_1, w_2\}$ and define $T = \{x \in V(G^*) - S : d_{G^*-S}(x) \le f(x)\}$, then

$$\sum_{x \in T}(f(x) - d_{G^*-S}(x)) > 3|S|$$

since $\sum_{x\in T}(f(x) - d_{G^*-S}(x)) = 7$ and $|S| = 2$. □

Claim 3.6 *The number of elements of* X *in case* (d)*, if all of them have a common end-vertex, cannot be increased.*

Proof. Let e'_1, e'_2, e'_3 be the edges having $v_{1,1}$ as a common end-vertex and having $u_{3,1}, u_{4,1}, v_{2,1}$ as the other end-vertex respectively. Let also $X = \{e'_1, e'_2, e'_3\}$ and define $G^* = G - X$. The family of graphs G^* clearly does not possess a 3-factor because if we let $S = \emptyset$ and define $T = \{x \in V(G^*) - S : d_{G^*-S}(x) \leq 3\}$, then

$$\sum_{x \in T}(3 - d_{G^*-S}(x)) > 3|S|$$

since $\sum_{x \in T}(3 - d_{G^*-S}(x)) = 1$ and $|S| = 0$. □

Finally a natural question that may arise is whether minimum degree of the highest value in a planar graph can guarantee the existence of a fractional 4-factor or of a fractional 5-factor. We will answer the above question by proving the following claim.

Claim 3.7 *Planar graphs having minimum degree of the highest value do not necessarily contain a fractional* 4*-factor or a fractional* 5*-factor.*

Proof. We will prove the claim by using again the family of graphs, also used for the proofs of claims 3.5 and 3.6.

Let $S = \{w_1, w_2\}, \ell = 4$ or $\ell = 5$ and define $T = \{x \in V(G) - S : d_{G-S}(x) \leq \ell\}$. Then

$$\sum_{x \in T}(\ell - d_{G-S}(x)) > \ell|S|$$

since $\sum_{x \in T}(\ell - d_{G-S}(x)) = (\ell - 3)12$ and $|S| = 2$. Hence although G satisfies $\delta(G) = 5$, the family of graphs G does not possess a fractional ℓ-factor when $\ell = 4$ or $\ell = 5$. □

Therefore, all the above claims yield that the conditions of Theorem 2.1 are in some sense best possible.

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