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About the structure of the blocker of the hyperplanes of a matroid*

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Abstract

A clutter on a ground set E is a collection of incomparable (inclusionwise) subsets of E. A map on clutters is a function from the class of all clutters on E to itself. Examples of maps on clutters are the blocking and the complementary maps. Let M be a matroid; the complementary and blocker maps give some relations between the clutters of M; however, $b(\mathcal{H})$, the blocker of the hyperplanes of M, has not been studied. An axiomatic characterization of $b(\mathcal{H})$ seems difficult to find, so in this paper we give some structural properties of such a clutter. In particular, necessary conditions about the dual rank of the members of $b(\mathcal{H})$ are given. We show how $b(\mathcal{H})$ can be decomposed in the blockers of the hyperplanes of submatroids of M. A characterization of the clutter of hyperplanes of a matroid and its blocker through forbidden minors is also given.

1 Introduction

It is assumed that the reader is familiar with the basic facts of matroid theory and graph theory. A standard reference to these are the books of Oxley [7] and Bondy and Murty [1], respectively. Clutters are also known in the literature as Sperner families; the references included here contain almost all of the concepts that we use. Two reference books about clutters are Cornuéjols [9] and Engel [10].

A *clutter* is an ordered pair $(E, \mathcal{K})$ consisting of a finite ground set E and a collection $\mathcal{K}$ of incomparable inclusionwise subsets of E. The *hereditary closure* of a clutter $(E, \mathcal{K})$ is the collection $\mathcal{K}{\downarrow}= \{A \subseteq K : K \in \mathcal{K}\}$; similarly, the *upward closure* of a clutter $(E, \mathcal{K})$ is the collection $\mathcal{K}{\uparrow}= \{A \supseteq K : K \in \mathcal{K}\}$. A set $K \subseteq E$ is a

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transversal of the clutter $(E, \mathcal{K})$ if K intersects all the members of the collection $\mathcal{K}$. A *map on clutters* is a function defined from the class of all clutters on E to itself. The *complementary* map c is a map on clutters that sends the clutter $(E, \mathcal{K})$ to the clutter $c((E, \mathcal{K})) := (E, \{E - K : K \in \mathcal{K}\})$, called the complementary clutter of $(E, \mathcal{K})$. The *blocker map* b is a map on clutters that sends the clutter $(E, \mathcal{K})$ to the blocker clutter $b((E, \mathcal{K})) := (E, \min\{D \subseteq E : D \cap K \neq \emptyset, \text{for all } K \in \mathcal{K}\})$, where min is an operator on a family of sets that renders its minimal elements; note that $\{D \subseteq E : D \cap K \neq \emptyset, \text{for all } K \in \mathcal{K}\}$ is the collection of transversals of $(E, \mathcal{K})$, so the blocker map sends a clutter $(E, \mathcal{K})$ to the clutter of its minimal transversals. We recall that the complementary and the blocker maps are involutions, and a proof of this fact can be found in the paper of Edmonds and Fulkerson [3].

While finding the complementary clutter of a given clutter $(E, \mathcal{K})$ is straightforward, finding the blocker clutter may be extremely difficult. An extensive research in that direction was performed by Khachiyan, Boros and Gurvich [4].

Through this document, b and c will denote the blocker map and the complementary map, respectively. When no confusion is possible we may use $\mathcal{K}$ instead of $(E, \mathcal{K})$, $b(\mathcal{K})$ instead of $b((E, \mathcal{K}))$ and $c(\mathcal{K})$ instead of $c((E, \mathcal{K}))$. Also, we will denote by $M = (E, \mathcal{I})$ (or only M) a matroid over a set E, by M^* its dual, and by $\mathcal{B}, \mathcal{B}^*, \mathcal{C}, \mathcal{C}^*, \mathcal{H}, \mathcal{H}^*$, respectively, the collections of bases, cobases, circuits, cocircuits, hyperplanes and cohyperplanes of M. We also follow the usual notation for the rank, r, and the corank, r^*, functions of M; and recall that these two functions are related by $r^*(X) = r(E - X) + |X| - r(M)$ for all $X \subseteq E$. The following are well-known relations among some important clutters of a matroid:

i. $c(\mathcal{B}) = \mathcal{B}^*$ and $c(\mathcal{B}^*) = \mathcal{B}$;

ii. $c(\mathcal{C}) = \mathcal{H}^*$ and $c(\mathcal{H}) = \mathcal{C}^*$;

iii. $b(\mathcal{B}) = \mathcal{C}^*$ and $b(\mathcal{B}^*) = \mathcal{C}$.

Observe that $b(\mathcal{H})$ does not appear in these relations, and it is easy to find examples (like the Vámos matroid) where $b(\mathcal{H})$ is none of the clutters involved in the above list, so this clutter is a different object.

On the other hand, the property of involution of the blocker map implies that $b(\mathcal{H})$ determines the matroid, because $b(b(\mathcal{H})) = \mathcal{H}$; therefore, we can construct a matroid if we know the blocker of its hyperplanes. So we think that it is worthwhile to study $b(\mathcal{H})$ and in fact all the clutters obtained by the alternate application of maps b and c. Here we restrict our attention to $b(\mathcal{H})$.

In this paper, we show some properties of $b(\mathcal{H})$ and we describe a way to construct $b(\mathcal{H})$ using the blockers of the hyperplanes of the simple matroids associated with the restrictions of the matroid to each of its connected components. Finally, we show the effect on the blocker clutter of the hyperplanes when we perform the relaxation of a circuit-hyperplane. We are planning to publish a second part of this paper, where we will give some characterizations of the blocker of the set of hyperplanes of some classes of matroids.

2 A characterization of $b(\mathcal{H})$

We start this section with a simple but remarkable fact about clutters.

Lemma 2.1. *For any clutter $\mathcal{K}$ on E, the pair $(b(\mathcal{K})\!\uparrow, c(\mathcal{K})\!\downarrow)$ is a partition of 2^E, the power set of E.*

Proof: $b(\mathcal{K})\!\uparrow$ is the set of transversals of $\mathcal{K}$ while every $A \in c(\mathcal{K})$ avoids at least one element of $\mathcal{K}$, namely $E \setminus A$. So A is not a transversal of $\mathcal{K}$, and neither are their subsets. So $b(\mathcal{K})\!\uparrow \cap\, c(\mathcal{K})\!\downarrow = \emptyset$. Now consider $S \notin c(\mathcal{K})\!\downarrow$. For every $K \in \mathcal{K}$, $S \nsubseteq E \setminus K$. So $S \cap K \neq \emptyset$ and so $S \in b(\mathcal{K})\!\uparrow$. □

Corollary 2.2. *For any clutter $\mathcal{K}$ on E,*

K1) $b(\mathcal{K}) = \min\{2^E - c(\mathcal{K})\!\downarrow\}$*; and*

K2) $c(\mathcal{K}) = \max\{2^E - b(\mathcal{K})\!\uparrow\}$.

where min *and* max *render the minimal and the maximal elements of a family of sets, respectively.* □

We call $\mathcal{P}(\mathcal{K}) := (b(\mathcal{K})\!\uparrow, c(\mathcal{K})\!\downarrow)$ a *partition structure* on E. Consider $\mathcal{P}(b(\mathcal{K})) = (b(b(\mathcal{K}))\!\uparrow, c(b(\mathcal{K}))\!\downarrow) = (\mathcal{K}\!\uparrow, c(b(\mathcal{K}))\!\downarrow)$; observe that $\mathcal{P}(\mathcal{K})$ and $\mathcal{P}(b(\mathcal{K}))$ are related by the extended complementary map, $c : 2^E \to 2^E$ where $c(S) = E \setminus S$. In fact, for any clutter $\mathcal{J}$ on E, it is true that $c((\mathcal{J}\!\uparrow)) = (c(\mathcal{J}))\!\downarrow$, so $c(b(\mathcal{K})\!\uparrow) = c(b(\mathcal{K}))\!\downarrow$ and $c((\mathcal{K}\!\uparrow)) = (c(\mathcal{K}))\!\downarrow$. Nicoletti [5], and Nicoletti and White [6] observed that the complementary map not only defines a matroid duality but also a duality of statements in matroid theory. In fact, this is true in the more general context of hypergraphs. Here we make the point that when the blocking map is considered, a nicer picture (Figure 2.3) arises.

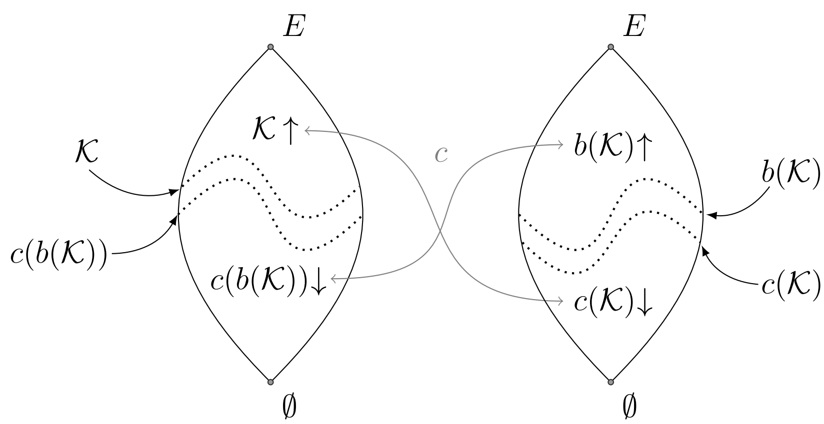

Figure 2.3. *The lattice of subsets of E with two partition structures related by the extended complementary map c.*

In fact c is an automorphism on 2^E that associates each partition structure $\mathcal{P}(\mathcal{K})$ with $\mathcal{P}(b(\mathcal{K}))$. So every true statement on $\mathcal{P}(\mathcal{K})$ renders a true statement on $\mathcal{P}(b(\mathcal{K}))$ much as duality in projective geometry.

A beautiful example is obtained by considering the clutter $\mathcal{B}$ of bases of a matroid as a starting point. From $\mathcal{B}$ we obtain the partition structure $\mathcal{P}(\mathcal{B}) = (b(\mathcal{B})\!\uparrow, c(\mathcal{B})\!\downarrow) = (\mathcal{C}^*\!\uparrow, \mathcal{B}^*\!\downarrow)$ and its complementary structure $\mathcal{P}(b(\mathcal{B})) = (\mathcal{B}\!\uparrow, \mathcal{H}\!\downarrow)$. So the well-known statement: "For any cobasis $B^* \in \mathcal{B}^*$ and any $x \notin B^*$, $B^* \cup \{x\}$ contains exactly one cocircuit C^*" translates into: "For any basis B and any $x \in B$, $B - \{x\}$ is contained in exactly one hyperplane $H = cl(B - \{x\})$" where cl is the closure operator of matroids.

If instead of $\mathcal{B}$ we take $\mathcal{H}$ as a starting point, we obtain another pair of dual partition structures $\mathcal{P}(\mathcal{H}) = (b(\mathcal{H})\uparrow, c(\mathcal{H})\downarrow) = (b(\mathcal{H})\uparrow, \mathcal{C}^*\downarrow)$ and $\mathcal{P}(b(\mathcal{H})) = (\mathcal{H}\!\uparrow, c(b(\mathcal{H}))\!\downarrow)$. This is the pair of structures we are interested in. In this section we will give some results along these lines.

Although the characterization that the members of the blocker of the hyperplanes are the minimal sets not contained in cocircuits follows from the expression $b(\mathcal{K}) = \min\{2^E - (c(\mathcal{K}))\!\downarrow\}$, there exist special necessary conditions for a clutter to be the blocker of the hyperplanes of a matroid.

Theorem 2.4. *Let $M = (E, \mathcal{I})$ be a matroid. Then D is a member of $b(\mathcal{H})$ if and only if D satisfies exactly one of the following properties.*

i) *For all $x \in D$, $D - x \in \mathcal{C}^*$.*

ii) *There exists a unique $x \in D$ such that $D - x \in \mathcal{C}^*$, and for all $y \in D - x$, $D - y \in \mathcal{C}^*\!\downarrow - \mathcal{C}^*$.*

iii) *$D \notin \mathcal{C}^*\!\downarrow$, and for all $x \in D$, we have $D - x \in \mathcal{C}^*\!\downarrow - \mathcal{C}^*$.* □

The following example illustrates the three cases above.

We recall that in the cycle matroid $M(G)$ of a graph G, the cocircuits of the matroid correspond to the bonds of G and the hyperplanes correspond to complements of bonds.

Example 2.5. Consider the cycle matroid of the graph G in Figure 2.6.

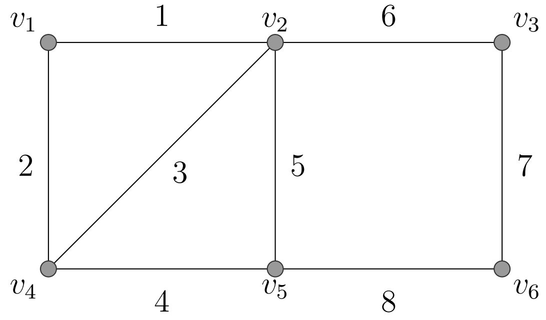

Figure 2.6. *A graph G.*

The sets $\{6,7,8\}$, $\{1,2,3\}$ and $\{1,4,7\}$ are members of the blocker of the hyperplanes of $M(G)$ and they correspond, respectively, to cases *i), ii),* and *iii)* of Theorem 2.4. $\square$

The proof of Theorem 2.4 follows by duality from Theorem 2.7. In Example 2.5, the complements of the sets $\{6,7,8\}$, $\{1,2,3\}$ and $\{1,4,7\}$ correspond, respectively, to cases *1)*, *2)*, and *3)* of Theorem 2.7.

Theorem 2.7. *Let $M=(E,\mathcal{I})$ be a matroid. Then A is a member of $c(b(\mathcal{H}))$ if and only if A satisfies exactly one of the following properties:*

1) For all $x \in E-A$, $A \cup \{x\} \in \mathcal{H}$.

2) There exists a unique element $x \in E-A$ such that $A \cup \{x\} \in \mathcal{H}$, and for all $y \in (E-A)-x$, we have $A \cup \{y\} \in \mathcal{H}\!\uparrow - \mathcal{H}$.

3) $A \notin \mathcal{H}\!\uparrow$, and for all $x \in E-A$, we have $A \cup \{x\} \in \mathcal{H}\!\uparrow - \mathcal{H}$.

Proof of Theorem 2.7: If a set $A \subseteq E$ satisfies one of the conditions enumerated above, then A is a maximal set which does not contains hyperplanes, and therefore $A \in c(b(\mathcal{H}))$.

To prove the converse implication, let $A \in c(b(\mathcal{H}))$, because the members of $c(b(\mathcal{H}))$ are maximal sets not containing hyperplanes; in particular, the members of $c(b(\mathcal{H}))$ cannot be hyperplanes, so they are not closed sets of M or their rank is not $r(M)-1$.

On the other hand, since for $x \in E-A$, $A \cup \{x\}$ contains a hyperplane, it follows that $r(A \cup \{x\}) \geq r(M)-1$. So $r(A) \geq r(M)-2$. It then follows that the rank of A is between $r(M)$ and $r(M)-2$. For $A \in c(b(\mathcal{H}))$, each possibility for $r(A)$ gives rise to one of the following three cases:

1) $r(A)=r(M)-2$. In this case, considering that A is a maximal set not containing hyperplanes, we have for all $x \in E-A$, $A \cup \{x\}$ contains a hyperplane, and so $r(A \cup \{x\}) \geq r(M)-1$.

   But since $r(A)=r(M)-2$, we have $r(A \cup \{x\})=r(M)-1$. Therefore $A \cup \{x\}$ is a hyperplane, because it is a set of rank $r(M)-1$ which contains a hyperplane, and the hyperplanes are maximal sets with rank $r(M)-1$.

2) $r(A)=r(M)-1$. In this case A is not a closed set since otherwise A would be a hyperplane.

   So there exists $x \in cl(A)-A$. Moreover, since $A \cup \{x\}$ contains a hyperplane, $cl(A)-A = cl(A \cup \{x\})-A = (A \cup \{x\})-A = \{x\}$. Therefore there exists a unique $x \in cl(A)-A$, that is, there exists a unique $x \in E-A$ such that $A \cup \{x\}$ is a hyperplane. Any other $y \in E-A$, such that $y \neq x$, is not in $cl(A)$ and so $A \cup \{y\}$ has rank $r(M)$ and properly contains a hyperplane.

3) $r(A) = r(M)$ so $A \notin \mathcal{H}$. On the other hand, $A \in c(b(\mathcal{H}))$ and so A is a maximal set not containing hyperplanes; that is, for all $x \in E - A$, we have $A \cup \{x\} \in \mathcal{H}\uparrow - \mathcal{H}$.

The cases 1), 2) and 3) of this proof correspond, respectively, to the cases *1), 2)*, and *3)* in the statement of the theorem. □

As a corollary of Theorem 2.7 we have

Corollary 2.8. *Let $M = (E, \mathcal{I})$ be a matroid. If D is a member of $b(\mathcal{H})$, then $|D| - 2 \leq r^*(D) \leq |D|$.*

Proof: This follows from Theorem 2.7 and the relation $r^*(X) = r(E - X) + |X| - r(M)$ for all $X \subseteq E$. □

This corollary suggests that the blocker clutter of the hyperplanes of a matroid M is an object of the dual matroid of M, in the same way as $b(\mathcal{B})$ are the circuits of M^*. So usually we will denote $b(\mathcal{H})$ by $\mathcal{D}^*$.

3 Decomposition of the Blocker of the Set of Hyperplanes

3.1 Decomposition of $b(\mathcal{H})$ through $si(M)$

Given a matroid M, the *simple matroid associated with* M is the matroid obtained from M by deleting all the loops, and for each parallel class $\{a_1, a_2, \ldots, a_m\}$ with cardinality greater than one, identifying the parallel class with a distinguished element $\alpha \in \{a_1, a_2, \ldots, a_m\}$; we denote this matroid by $si(M)$. In this section we give a way to construct the blocker of the hyperplanes of M using the blocker of the hyperplanes of $si(M)$.

Let $M = (E, \mathcal{I})$ be a matroid and let $(E, \mathcal{H})$ be its clutter of hyperplanes. For $S \subset E$ we define the collection $\mathcal{H}\backslash\backslash S = \{H - S : H \in \mathcal{H}\}$.

Theorem 3.1. *Let $M = (E, \mathcal{I})$ be a matroid. If $S = \{e_1, e_2, \ldots, e_m\}$ is the set of loops of M, then*

$$b(\mathcal{H}) = \{\{e_1\}, \{e_2\}, \ldots, \{e_m\}\} \cup b(\mathcal{H}\backslash\backslash S).$$

Proof: Let $D \in b(\mathcal{H})$. We have two cases.

Case 1: the set D contains a loop: In this case D is precisely a loop, because the loops are contained in all the hyperplanes, and D is a minimal transversal because of its cardinality.

Case 2: the set D does not contain a loop. In this case, the set D necessarily intersects all the hyperplanes in elements which are not contained in S. So $D \in b(\mathcal{H}\backslash\backslash S)$.

To prove the reverse inclusion, let $D \in \{\{e_1\}, \{e_2\}, \ldots, \{e_m\}\} \cup b(\mathcal{H}\backslash\backslash S)$. If D is a loop then clearly $D \in b(\mathcal{H})$. If $D \in b(\mathcal{H}\backslash\backslash S)$, then $D \cap S = \emptyset$ and D intersects all the hyperplanes of M. The set D is a minimal transversal of $\mathcal{H}$ because for all

$e \in D$, there exists a hyperplane H such that $(D-e) \cap (H-S) = \emptyset$ and this implies $(D-e) \cap H = \emptyset$. Therefore $D \in b(\mathcal{H})$. □

As a consequence of the last result, if a clutter $(E, b(\mathcal{H})) \neq (E, \{\emptyset\})$ is a blocker of the hyperplanes of a matroid $M = (E, \mathcal{I})$, we can add a new singleton $\{\alpha\}$, $\alpha \notin E$ and obtain a blocker of the hyperplanes of another new matroid $M' = (E \cup \alpha, \mathcal{I})$.

Now we will focus our attention on the treatment of parallel elements. Let $\mathfrak{S} = \{S_1, S_2, \ldots, S_k\}$ be a collection of disjoint sets. Consider the set $S = \{s_1, s_2, \ldots, s_k\}$, where $s_i \in S_i$ for $i = 1, 2, \ldots, k$. For a set $A \subseteq S$, we define the *expansion of A by $\mathfrak{S}$* as the collection

$$A \cdot \mathfrak{S} := \{A' : A' \subseteq (\bigcup_{i=1}^{k} S_i), |A' \cap S_i| = 1 \Leftrightarrow s_i \in A, |A' \cap S_i| = 0 \Leftrightarrow s_i \notin A\}.$$

If $(\bigcup_{i=1}^{k} S_i, \mathcal{K})$ is a clutter, we define the *expansion of $\mathcal{K}$ by $\mathfrak{S}$* as the collection $\mathcal{K} \cdot \mathfrak{S} = \cup\{A \cdot \mathfrak{S} : A \in \mathcal{K}\}$. It is not difficult to see that $(\bigcup_{i=1}^{k} S_i, \mathcal{K} \cdot \mathfrak{S})$ is a clutter.

Let $\mathcal{E} = \{E_1, E_2, \ldots, E_k\}$ be the collection of parallel classes of a matroid $M = (E, \mathcal{I})$ without loops. For each parallel class E_i we distinguish a unique element $e_i \in E_i$ to build $si(M)$; this is $\mathcal{E}' := \{e_1, e_2, \ldots, e_k\}$, the ground set of $si(M)$.

We will consider this particular construction of $si(M)$ in the following theorem.

Theorem 3.2. *Let $M = (E, \mathcal{I})$ be a matroid without loops, and let $\mathcal{E} = \{E_1, E_2, \ldots, E_k\}$ be its set of parallel classes. Let $(\mathcal{E}', si(\mathcal{H}))$ be the clutter of hyperplanes of $si(M)$. Then $b(\mathcal{H}) = b(si(\mathcal{H})) \cdot \mathcal{E}$.*

Proof: In this proof E_i will denote the parallel class that contains the identified element e_i that we used to build $si(M)$; similarly, if we are talking about an element e_i, we mean the element identified in the parallel class E_i of M to build $si(M)$. We use this convention to avoid the use of maps between matroids just for relabelling elements, and in this way, the element e_i is an element of M and $si(M)$.

Before getting into the details of the proof, let us make the observation that the elements of M which belong to the same parallel class are contained in a unique atom of the lattice of closed sets of M, so if a hyperplane H of M contains an element e_i, then H must contain all the elements of the parallel class E_i; as a consequence of this fact, we have that H is a hyperplane of $si(M)$ if and only if $H' = \cup\{E_i : e_i \in H\}$ is a hyperplane of M.

Let $D \in b(\mathcal{H})$. We claim that $A = \{e_i \in \mathcal{E}' : D \cap E_i \neq \emptyset\}$ is an element of $b(si(H))$. Suppose, to the contrary, that the set A does not intersect a hyperplane H_j of $si(M)$. Then the set D does not intersect the hyperplane $H'_j = \cup\{E_i : e_i \in H_j\}$ of M and this contradicts our assumption that $D \in b(\mathcal{H})$. Similarly, if A is a transversal set of $si(H)$ and it is not minimal, then there exists an element e_i of A such that $A - e_i$ is a transversal of $si(H)$, but this implies that there exists an element $e \in D$ in the same parallel class as e_i and such that $D - e$ is a transversal set of $\mathcal{H}$. This contradicts the minimality of D. Hence we have $A \in b(si(\mathcal{H}))$, and therefore $D \in A \cdot \mathcal{E} \subseteq b(si(\mathcal{H}))$.

Now, we will prove containment the other way around: if A is a member of $b(\mathcal{E}', si(\mathcal{H}))$, then A has at most one member of each E_i. If $e_i \in A$, the set $A - e_i$

does not intersect all the members of $si(\mathcal{H})$, because A is a minimal transversal of the hyperplanes of $si(\mathcal{H})$; hence the members of $(A - e_i) \cdot \mathcal{E}$ are not transversals of the hyperplanes of M (they do not intersect the hyperplanes which contain E_i), but if we choose an element of $e_i' \in E_i$, then the members of $((A - e_i) \cup e_i') \cdot \mathcal{E}$ intersect all the hyperplanes of M and they are minimal, so the sets in $A \cdot \mathcal{E}$ are members of $(E, b(\mathcal{H}))$. Applying the same arguments with the remaining elements of $b(si(\mathcal{H}))$, we conclude that $b(\mathcal{H}) = b(si(\mathcal{H})) \cdot \mathcal{E}$. □

3.2 Decomposition through Connected Components

In this section we show how to construct the blocker of the hyperplanes of a non connected matroid using the blocker of the hyperplanes of the restrictions of the matroid to each one of the connected components.

Theorem 3.3. *If $M = (E, \mathcal{I})$ is a simple matroid, let $T_1, \ldots, T_k$ be its connected components, and let $\mathcal{H}_i$ be the clutter of hyperplanes of $M|T_i$. If for $1 \leq i \leq k$, $\mathcal{H}_i \neq \{\emptyset\}$, then*

$$b(\mathcal{H}) = \bigcup_{i=1}^{k} \{b(\mathcal{H}_i) : \mathcal{H}_i \neq \{\emptyset\}\} \cup \{\{a, b\} : a \in T_i, b \in T_j, i \neq j\}.$$

Proof: $M = (M|T_1) \oplus \cdots \oplus (M|T_k)$ and its clutter of hyperplanes is the collection $\mathcal{H} = \{H_i \cup (E - T_i) : i \in \{1, \ldots, k\}\}$.

If $D \neq \emptyset$ is contained in a component T_i and is a member of the blocker of the hyperplanes of $M|T_i$, then D intersects all the hyperplanes of M because all the hyperplanes of M contain T_i or they are of the form $H_i \cup (E - T_i)$ with H_i hyperplane of $M|T_i$. On the other hand, if $D \subseteq T_i, D \neq \emptyset$ is not a member of $b(\mathcal{H})$, then D does not intersect a hyperplane H_i of $M|T_i$, and D does not intersect the hyperplane $H_i \cup (E - T_i)$ of M. So the sets contained in exactly one component T_i are blockers of M if and only if they are blockers of the hyperplanes of $M|T_i$. Now, $\mathcal{H}_i = \{\emptyset\}$ if and only if $E - T_i$ is a hyperplane of M, and this happens if and only if there are no members of $b(\mathcal{H})$ contained in T_i.

If we take an element a in a component T_i and an element b in a component T_j, such that $T_j \neq T_i$, every hyperplane either contains T_i or T_j, and so $\{a, b\}$ is a transversal of the clutter $\mathcal{H}$, and it is minimal because M is simple. □

In the following example we ilustrate the theorems of this section about the decomposition of the blocker of the hyperplanes.

Example 3.4. Here we will use the notation $(a_1 a_2 \ldots a_k)$ to denote a set $\{a_1, a_2, \ldots, a_k\}$ so that the notation does not become overwhelming. Let M be the matroid of cycles of the graph G' represented in Figure 3.5.

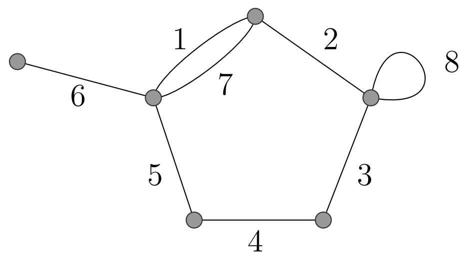

Figure 3.5. *A graph G'.*

The clutter of hyperplanes of M is the collection of sets $\mathcal{H} = \{(7123458), (34568), (24568), (23568), (23468), (714568), (713568), (713468), (712568), (712468), (712368)\}$. We can construct $b(\mathcal{H})$ through $si(M)$, the simple matroid asociated with M. In this case, $si(M)$ is isomorphic to the cycle matroid of the graph G'' in Figure 3.6.

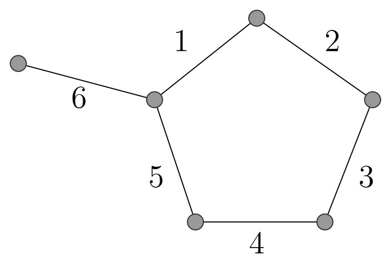

Figure 3.6. *A graph G''.*

The simple matroid $si(M)$ has two connected components $E_1 = (12345)$ and $E_2 = (6)$. The hyperplanes of $si(M)|E_1$ are $\mathcal{H}_1 = \{A \subseteq E_1 : |A| = 3\}$; and the hyperplanes of $si(M)|E_2$ are $\mathcal{H}_2 = (\emptyset)$. The blockers of these clutters are $b(\mathcal{H}_1) = \{A \subseteq E_1 : |A| = 2\}$ and $b(\mathcal{H}_2) = \emptyset$, respectively.

Now, by Theorem 3.3, the blocker of the hyperplanes of $si(M)$ is the clutter $\{(16), (26), (36), (46), (56), (123), (124), (125), (134), (135), (145), (234), (235), (245), (345)\}$, and by Theorems 3.1 and 3.2 the blocker of the hyperplanes of $M(G)$ is the clutter $\{(16), (26), (36), (46), (56), (123), (124), (125), (134), (135), (145), (234), (235), (245), (345)\} \cup \{(8), (76), (723), (724), (725), (734), (735), (745)\}$. □

4 Relaxation of Hyperplanes

The next proposition appears in Oxley's book [7].

Proposition 4.1. *Let $M = (E, \mathcal{I})$ be a matroid and let $X \subseteq E$ be a circuit-hyperplane of M. Then $\mathcal{B}' = \mathcal{B} \cup \{X\}$ is the set of bases of a matroid M' on the same ground set E of M.*

The matroid M' is called a *relaxation* of M. In particular, we say that it is the result of relaxing the circuit-hyperplane X.

Let $M = (E, \mathcal{I})$ be a matroid of rank r on a set E of cardinality n. In this section $\mathcal{H}_r = \{H_1, H_2, \ldots, H_s\}$ will denote the hyperplanes of M with cardinality greater or equal than r and $\mathcal{H}_{r-1} = \{\hat{H}_1, \hat{H}_2, \ldots, \hat{H}_t\}$ the hyperplanes of M with cardinality equal to $r-1$. Let $b(\mathcal{H})$ be the blocker clutter of the hyperplanes of M. We have the following theorem.

Theorem 4.2. *Let $M = (E, \mathcal{I})$ be a rank r matroid. If we relax a circuit-hyperplane $H_i \in \mathcal{H}_r$, then $b(\mathcal{H}^-)$, the blocker of the hyperplanes of the matroid M^- obtained after the relaxation of H_i is given by:*

$b(\mathcal{H}^-) = \min\{\{D \in b(\mathcal{H}) : |D \cap H_i| > 1\} \cup \{D \cup x : |D \cap H_i| = 1, x \in H_i - D, D \in b(\mathcal{H})\}\}$.

To prove the above theorem we need the following lemma.

Lemma 4.3. *Every transversal T of the hyperplanes of the matroid M^- obtained relaxing a circuit-hyperplane H_i either contains a member of $\{D \in b(\mathcal{H}) : |D \cap H_i| > 1\}$ or, there exists $D \in \{\hat{D} \in b(\mathcal{H}) : |\hat{D} \cap H_i| = 1\}$ and there exists $x \in H_i$ such that $D \cup \{x\} \subseteq T$.*

Proof: Let T be a transversal of the hyperplanes of M^-. Then it is clear that T is a member of the blocker of the hyperplanes of M^- or contains properly a member of the blocker of the hyperplanes of M^-.

Case 1: If T contains a member of $\{D \in b(\mathcal{H}) : |D \cap H_i| > 1\}$ then there is nothing to prove.

Case 2: If T only contains members of $b(\mathcal{H})$ such that $|D \cap H_i| = 1$, let D be one of them, that is, $D \subseteq T$. If $D \cap H_i = \{e\}$, then $e \in T \cap H_i$ and $H_i - e$ is a hyperplane of M^-. But then $D \cap (H_i - e) = \emptyset$, so D is a proper subset of T because T is a transversal of the hyperplanes of M^-; in particular, $T \cap (H_i - e) \neq \emptyset$. Let $x \in T \cap (H_i - e)$; clearly $x \notin D$ because $D \cap (H_i - e) = \emptyset$. It follows that $D \cup x \subseteq T$. $\square$

Proof of Theorem 4.2: Note that the sets of hyperplanes of M^- with cardinalities greater or equal than r and $r-1$, are respectively $\mathcal{H}_r^- = \mathcal{H}_r - \{H_i\}$ and $\mathcal{H}_{r-1}^- = \mathcal{H}_{r-1} \cup \{H_i - x : x \in H_i\}$.

First, we prove that the members of $\{D \in b(\mathcal{H}) : |D \cap H_i| > 1\}$ are transversals of the hyperplanes of M^-. If $D \in b(\mathcal{H})$ and $|D \cap H_i| > 1$, then for all $x \in H_i$, $|D \cap (H_i - x)| > 0$, this is, D intersects the sets of cardinality $r-1$ that are hyperplanes of the matroid M^- but are not hyperplanes of M. On the other hand, D is in $b(\mathcal{H})$, so D intersects every member of $(\mathcal{H}_r \cup \mathcal{H}_{r-1}) - \{H_i\}$ because these sets are hyperplanes of M. Therefore, we have proved the members of $\{D \in b(\mathcal{H}) : |D \cap H_i| > 1\}$ are transversals of the hyperplanes of M^-.

Now, if $D \in b(\mathcal{H})$ and $|D \cap H_i| = 1$, then there exists a unique $e \in E$ such that $D \cap H_i = \{e\}$, so $D \cap (H_i - \{e\}) = \emptyset$; the set $H_i - \{e\}$ is a hyperplane of M^-, it follows that D is not transversal of the hyperplanes of the matroid M^-. However, if we choose $y \in H_i - \{e\}$, then $(D \cup \{y\}) \cap (H_i - \{e\}) = \{y\}$. Now $D \cup \{y\}$ intersects all the members of $(\mathcal{H}_r \cup \mathcal{H}_{r-1}) - \{H_i\}$ because D is a transversal of the hyperplanes of M; even more, $D \cup \{y\}$ intersects all the members of $\{H_i - \{x\} : x \in H_i - \{e\}\}$ because all of them contain the element y, therefore, $D \cup \{y\}$ is a transversal of the hyperplanes of M^-.

Now, by Lemma 4.3 the members of $\{D \cup x : |D \cap H_i| = 1, x \in H_i - D, D \in b(\mathcal{H})\}$ and $\{D \in b(\mathcal{H}) : |D \cap H_i| > 1\}$ are transversals of the hyperplanes of M^- and every transversal of the hyperplanes of M^- contains a member of these collections, in particular, the members of the blocker of the hyperplanes of M^- contain members of these collections and by minimality, they are members of these collections. $\square$

Example 4.4. Consider the Fano matroid F_7 with the geometric representation of Figure 4.5.

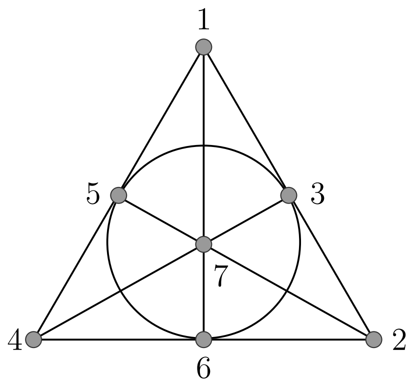

Figure 4.5. *F_7, the Fano matroid*

The collection of hyperplanes of F_7 is $\mathcal{H} = \{\{1, 2, 3\}, \{1, 4, 5\}, \{1, 6, 7\}, \{2, 4, 6\}, \{2, 5, 7\}, \{3, 4, 7\}, \{3, 5, 6\}\}$; in this case $b(\mathcal{H}) = \mathcal{H}$. If we relax the circuit-hyperplane $\{3, 5, 6\}$ we obtain the matroid F_7^-. We observe that the unique set in $b(\mathcal{H})$ that intersects $\{3, 5, 6\}$ in more than one element is $\{3, 5, 6\}$ itself, so by Theorem 4.2 we have that the blocker of the hyperplanes of F_7^- is the clutter

$$\{\{3,5,6\},\{1,2,3,5\},\{1,2,3,6\},\{1,3,4,5\},\{1,3,6,7\},\{1,4,5,6\},$$
$$\{1,5,6,7\},\{2,3,4,6\},\{2,3,5,7\},\{2,4,5,6\},\{2,5,6,7\},\{3,4,5,7\},\{3,4,6,7\}\}.$$

$\square$

5 A Characterization of the Clutter of Hyperplanes of a Matroid by Excluded Minors

In the paper "Clutters and matroids", Cordovil et al. [2] define two pairs of operations to obtain minors of clutters and give an excluded minor characterizations of bases and circuits of a matroid using the complementary and the blocker maps. We follow their approach and define two new operations on clutters (suggested by the sets of hyperplanes of the minors of a matroid) and we use them to give forbidden minors for a clutter to be the collection of hyperplanes of a matroid. Also we give another characterization of the clutter of hyperplanes using the blocker, the complementary, and another map that sends the hyperplanes of a matroid M into hyperplanes of the dual matroid of M. We conclude that a similar process can be performed to obtain new operations and forbidden minors for a clutter to be the blocker (respectively the complementary clutter of the blocker) clutter of the hyperplanes of a matroid.

5.1 Minors of clutters

The *minor* of a matroid is another matroid obtained from the original by deleting and/or contracting elements of the ground set of the matroid. These operations can be translated into operations on the main clutters of a matroid in such a way that a minor of the collection of circuits(respectively bases, hyperplanes) is itself a clutter of circuits (respectively bases, hyperplanes) of the minor of the matroid. Cordovil et al. defined such operations for circuits and bases. Here we define them for hyperplanes:

Definition 5.1. *Let $(E, \mathcal{L})$ be a clutter and e an element of E. The clutter obtained by "deletion" of e, denoted by $\mathcal{L}\backslash\backslash e$, is the clutter:*

$$\mathcal{L}\backslash\backslash e := (E - e, \max\{L - e \subsetneq E - e : L \in \mathcal{L}\}),$$

and the clutter obtained by "contraction" of e, denoted by $\mathcal{L}//e$ is:

$$\mathcal{L}//e := (E - e, \{L - e \subsetneq E - e : e \in L \in \mathcal{L}\}). \qquad \square$$

We now define the pair of operations used in [2]; if $(E, \mathcal{K})$ is the clutter of circuits of a matroid M we obtain, respectively, the clutters of circuits of the minors of M obtained by deletion and contraction of an element $e \in E$.

Definition 5.2. *Let $(E, \mathcal{K})$ be a clutter and e and element of E. The clutter obtained by deletion of e, denoted by $\mathcal{K}\backslash e$ is the clutter:*

$$\mathcal{K}\backslash e := (E - e, \{K : e \notin K \in \mathcal{K}\}),$$

and the clutter obtained by contraction of e, denoted by $\mathcal{K}/e$ is:

$$\mathcal{K}/e := (E - e, \min\{K - e : K \in \mathcal{K}, K - e \neq \emptyset\}). \qquad \square$$

Proposition 5.3. *If $(E, \mathcal{K})$ is a clutter, and $(E, \mathcal{L})$ is the complementary clutter of $(E, \mathcal{K})$. For every $e \in E$ the diagrams in Figure 5.4 and Figure 5.5 are commutative:*

$$\begin{tikzcd}
(E, \mathcal{K}) \arrow[r, leftrightarrow, "c"] \arrow[d, "/"'] \arrow[dr, phantom, "\circlearrowleft"] & (E, \mathcal{L}) \arrow[d, "\backslash\backslash"] \\
(E - e, \mathcal{K}/e) \arrow[r, leftrightarrow, "c"] & (E - e, \mathcal{L}\backslash\backslash e)
\end{tikzcd}$$

Figure 5.4. *Commutativity induced by the operators / and \\.*

$$\begin{tikzcd}
(E,\mathcal{K}) \arrow[r, leftrightarrow, "c"] \arrow[d, "\backslash"'] \arrow[dr, phantom, "\circlearrowright"] & (E,\mathcal{L}) \arrow[d, "//"] \\
(E-e,\mathcal{K}/e) \arrow[r, leftrightarrow, "c"] & (E-e,\mathcal{L}\backslash\backslash e)
\end{tikzcd}$$

Figure 5.5. *Commutativity induced by the operators* $\backslash$ *and* $//$.

Proof: For the first diagram, it is sufficient to prove that $c(E-e,\mathcal{K}/e) = (E-e,\mathcal{L}\backslash\backslash e)$.

First, we will show that $c(E-e,\mathcal{K}/e) \subseteq (E-e,\mathcal{L}\backslash\backslash e)$. Let $\hat{K} \in (E-e,\mathcal{K}/e)$ by definition, $\hat{K} = K_1 - e$ for some $K_1 \in \mathcal{K}$, moreover, $E - K_1 = L_1 \in (E,\mathcal{L})$ so $(E-e)-\hat{K} = (E-e)-(K_1-e) = (E-K_1)-e = L_1-e$. Observe that $L_1-e \neq E-e$ because $\hat{K} \neq \emptyset$.

Let us now prove that $L_1 - e$ is maximal in $\{L-e \subsetneq E-e : L \in \mathcal{L}\}$. Suppose $L_1 - e$ is not maximal, then there exists $L_2 \in \mathcal{L}$ such that $L_1 - e \subsetneq L_2 - e$. We have to examine four cases:

1. If $e \notin L_1$ and $e \notin L_2$, then $L_1 \subsetneq L_2$.
2. If $e \notin L_1$ and $e \in L_2$, then $L_1 \subset L_2 - e \subsetneq L_2$.
3. If $e \in L_1$ and $e \in L_2$, then $L_1 - e \subsetneq L_2 - e$, this implies $L_2 \subsetneq L_1$.
4. $e \in L_1$ and $e \notin L_2$.

The first three cases are not possible, because $\mathcal{L}$ is a clutter. So we must have $e \in L_1 - L_2$. Let $K_2 = E - L_2 \in \mathcal{K}$, then $e \in K_2 - K_1$ and $L_1 - e \subsetneq L_2 - e = L_2$, so, taking complements in E, $K_2 \subsetneq K_1 \cup e$, but $e \in K_2 - K_1$, so $K_2 - e \subsetneq K_1 = K_1 - e = \hat{K}$ which contradicts the minimality of $\hat{K}$. Therefore $L_1 - e$ is maximal in $\{L-e \subsetneq E-e : L \in \mathcal{L}\}$ and we have proved the first inclusion.

Now, we will prove the other direction of the statement, that is $\mathcal{L}\backslash\backslash e \subseteq c(\mathcal{K}/e)$. Let $\hat{L} \in \mathcal{L}\backslash\backslash e = (E-e, \max\{L-e \subsetneq E-e : L \in \mathcal{L}\})$, we have to prove that $\hat{L} = (E-e)-\hat{K} \in \mathcal{K}$ for some $\hat{K} \in \mathcal{K}/e$. Since $\hat{L} = L_1 - e$ for some $L_1 \in \mathcal{L}$ we obtain that $E - L_1 = K_1 \in \mathcal{K}$. So we have $\hat{L} = L_1 - e = (E-K_1)-e = (E-e)-(K_1-e)$. We note that $K_1 - e \neq \emptyset$ because $\hat{L}$ cannot be $(E-e)$ (members of $\mathcal{L}\backslash\backslash e$ are proper subsets of $E-e$).

So, it only remains to be proved that $K_1 - e$ is minimal in $\{K-e : K \in \mathcal{K}, K-e \neq \emptyset\}$. Suppose $K_1 - e$ is not minimal in $\{K-e : K \in \mathcal{K}, K-e \neq \emptyset\}$, then, there exists $K_2 \in \mathcal{K}$ such that $K_2 - e \subsetneq K_1 - e$. We have the next four cases:

1. If $e \notin K_1$ and $e \notin K_2$, then $K_2 \subsetneq K_1$.
2. If $e \in K_1$ and $e \notin K_2$, then $K_2 = K_2 - e \subsetneq K_1 - e \subsetneq K_1$.
3. If $e \in K_1$ and $e \in K_2$, then $K_2 - e \subsetneq K_1 - e$ implies $K_2 \subsetneq K_1$.

4. $e \notin K_1$ and $e \in K_2$.

The first three cases are not possible because K_1, K_2 are members of a clutter. So, it must be that $e \in K_2 - K_1$. Let $L_2 = E - K_2 \in \mathcal{L}$ then $e \in L_1 - L_2$. Because $e \notin K_1$ we have $K_2 - e \subsetneq K_1 - e = K_1$, and taking complements in E, we have $L_2 \cup \{e\} \supsetneq L_1$, and then, because $e \notin L_2$, $\hat{L} = L_1 - e \subsetneq L_2 = L_2 - e$, but this contradicts the maximality of $\hat{L} = L_1 - e$, therefore $K_1 - e$ is minimal in $\{K - e : K \in \mathcal{K}, K - e \neq \emptyset\}$ and we have proved that the first diagram is commutative.

For the second part, we have to prove that $\mathcal{L}//e = c(\mathcal{K}\backslash e)$. We have these equivalences:

The collection $\hat{L} \in \mathcal{L}//e = (E - e, \{L - e \subsetneq E - e : e \in L \in \mathcal{L}\})$ if and only if $\hat{L} = L - e$ and $e \in L \in \mathcal{L}$; this is, if and only if $\hat{L} = (E - K) - e$ and $e \notin K \in \mathcal{K}$ (taking $L = E - K$), but this happens if and only if $\hat{L} = (E - e) - K$ and $e \notin K \in \mathcal{K}$; this is, if and only if $\hat{L} = (E - e) - K$ and $K \in \mathcal{K}\backslash e$ or, equivalently $c(E - e, \{K : e \notin K \in \mathcal{K}\}) = c(\mathcal{K}\backslash e) \ni L$

Therefore $\mathcal{L}//e = c(\mathcal{K}\backslash e)$ and the second diagram commutes. This ends the proof. □

5.2 Characterizations of hyperplanes and its blocker clutter

The following theorem was proved by Cordovil et al. in [2]. Here we use it to prove Theorem 5.7.

Theorem 5.6. *Let $(E, \mathcal{K})$ be a clutter. Then $(E, \mathcal{K})$ is not the sets of circuits of a matroid if and only if the following conditions hold:*

i) $(E, \mathcal{K})$ *is the clutter* $(E, \{\emptyset\})$

ii) Using the operations $\backslash$, $/$; $(E, \mathcal{K})$ *has a minor isomorphic to* $(\{1, 2, 3\}, \{\{1, 2\}, \{1, 3\}\})$. □

Theorem 5.7. *Let $(E, \mathcal{L})$ be a clutter. Then $(E, \mathcal{L})$ is the set of hyperplanes of a matroid if and only if the following conditions hold:*

i) $(E, \mathcal{L})$ *is not the clutter* $(E, \{E\})$

ii) Using the operations $\backslash\backslash$, $//$; $(E, \mathcal{L})$ *does not have a minor isomorphic to* $(\{1, 2, 3\}, \{\{2\}, \{3\}\})$.

Proof: The clutter $(E, \{E\})$ cannot be the set of hyperplanes of a matroid because the hyperplanes of a matroid must be proper subsets of the total set where the matroid is defined.

Also, the clutter $(\{1, 2, 3\}, \{\{2\}, \{3\}\})$ cannot be a minor of $(E, \mathcal{L})$, because if we apply the operations $\backslash\backslash$, $//$ on a clutter that is the set of hyperplanes of a matroid, the minors are the sets of hyperplanes of the minors of the matroid; and $(\{1,2,3\},\{\{2\},\{3\}\})$ is not a set of hyperplanes because $1 \in \{1,2,3\} - \{\{2\} \cup \{3\}\}$ but there is not a set $L_3 \in \{\{2\},\{3\}\}$ such that $L_3 \supset (\{\{2\} \cap \{3\}\}) \cup \{1\} = \{1\}$ (i.e. the clutter $(\{1,2,3\},\{\{2\},\{3\}\})$ does not satisfy the hyperplanes axioms).

To prove that if $i)$ and $ii)$ do not happen, then $\mathcal{L}$ is the set of hyperplanes of a matroid we consider the contrapositve proposition, that is, we will prove that if $\mathcal{L}$ is not the set of hyperplanes of a matroid, then either $(E,\mathcal{L}) = (E,\{E\})$ or $(E,\mathcal{L})$ has a minor isomorphic to the clutter $(\{1,2,3\},\{\{2\},\{3\}\})$.

If $\mathcal{L} = (E,\{E\})$ there is nothing to prove.

So, let $(E,\mathcal{L})$ be a clutter that is not the set of hyperplanes of a matroid and such that $(E,\mathcal{L}) \neq (E,\{E\})$.

Suppose to the contrary $(E,\mathcal{L})$ has no minor isomorphic to $(\{1,2,3\},\{\{2\},\{3\}\})$. Then $c(E,\mathcal{L}) = (E,\mathcal{K})$ (the clutter complementary of $(E,\mathcal{L})$) does not have a minor isomorphic to $(\{1,2,3\},\{\{1,2\},\{1,3\}\})$ because otherwise the succession of operations $\backslash, /$ performed to the clutter $(E,\mathcal{K})$ to obtain $(\{1,2,3\}, \{\{1,2\},\{1,3\}\})$ can be applied to the clutter $(E,\mathcal{L})$ replacing $\backslash$ with $//$, and $/$ with $\backslash\backslash$ and then, by the commutative diagrams of previous section, $(E,\mathcal{L})$ must have a minor isomorphic to $(\{1,2,3\},\{\{2\},\{3\}\})$ contradicting our assumption.

Now, by Theorem 5.6, we have that $(E,\mathcal{K})$ is the set of circuits of a matroid, and then $c((E,\mathcal{K}))$ is the set of hyperplanes of a matroid; this contradicts our hypothesis. So if $(E,\mathcal{L})$ is not the set of hyperplanes of a matroid on a set E and is not $(E,\{E\})$, $(E,\mathcal{L})$ must have a minor isomorphic to $(\{1,2,3\},\{\{2\},\{3\}\})$.

By the two implications, the theorem is true. □

The next theorem appears in the Section 5 of [2]. This theorem was first proved by Vaderlind [8]; here we use it to prove Theorem 5.9.

Theorem 5.8. *A clutter $(E,\mathcal{K})$ is the set of circuits of a matroid if and only if $\mathcal{K}$ is a fixed point of the map $b \circ c \circ b \circ *$ where b is the blocker map, c is the complementary map, and $*$ is the map $*((E,\mathcal{K})) = (E, \min\{X : X \subseteq E, X \neq \emptyset$ and $|X \cap K| \neq 1,$ for all $K \in \mathcal{K}\})$.* □

Theorem 5.9. *Let $(E,\mathcal{L})$ be a clutter, c the complementary map, $*$ the map defined above, and $\hat{*}$ the map $\hat{*}((E,\mathcal{L})) = (E, \max\{X : X \subseteq E, X \neq E$ and $|E-(X \cup \hat{L})| \neq 1,$ for all $\hat{L} \in \mathcal{L}\}$. Then $\hat{*}((E,\mathcal{L})) = c \circ * \circ c((E,\mathcal{L}))$.*

Proof: Let $\mathcal{L}$ be a clutter on a set E. We have $L \in c \circ * \circ c(\mathcal{L})$ if and only if $E - L \in *(c(\mathcal{L}))$. That is, if and only if:

$$E - L \in \min\{X : X \subseteq E, X \neq \emptyset \text{ and } |X \cap K| \neq 1, \forall K \in c(\mathcal{L})\}$$

or, equivalently, $(\emptyset \neq E - L \subseteq E), (|(E-L) \cap K| \neq 1,$ for all $K \in c(\mathcal{L}))$ and, if $(\emptyset \neq E - L' \subseteq E), (|(E-L') \cap K| \neq 1,$ for all $K \in c(\mathcal{L}))$ and $E - L' \subseteq E - L$, then $E - L' = E - L$. But this happens if and only if:

$(E \neq L \subseteq E), (|(E-L) \cap (E-\hat{L}| \neq 1,$ for all $E - \hat{L} \in c(\mathcal{L}))$ and, if $(E \neq L' \subseteq E)$, $(|(E - L') \cap (E - \hat{L})| \neq 1,$ for all $\hat{L} \in \mathcal{L})$ and $L' \supseteq L$, then $L' = L$. In other words, we have that:

$$L \in \max\{X : X \subseteq E, X \neq E \text{ and } |E - (X \cup \hat{L})| \neq 1, \text{ for all } \hat{L} \in \mathcal{L}\}$$

or, equivalently, $L \in \hat{*}(\mathcal{L})$. □

We now are able to prove the following theorem.

Theorem 5.10. *A clutter $\mathcal{L}$ on a set E is the set of hyperplanes of a matroid if and only if $\mathcal{L}$ is a fixed point of the map $c \circ b \circ c \circ b \circ c \circ \hat{*}$.*

Proof: The collection $\mathcal{L}$ is the set of hyperplanes of a matroid if and only if $c(\mathcal{L})$ is the set of circuits of a matroid, this is, if and only if $b \circ c \circ b \circ *(c(\mathcal{L})) = c(\mathcal{L})$ (by Theorem 5.8), but this happens if and only if $*(c(\mathcal{L})) = b \circ c \circ b(c(\mathcal{L}))$.

This is $c \circ * \circ c(\mathcal{L}) = \hat{*}(\mathcal{L}) = c \circ b \circ c \circ b \circ c(\mathcal{L})$ (by Theorem 5.9) or, equivalently $c \circ b \circ c \circ b \circ c \circ \hat{*}(\mathcal{L}) = (\mathcal{L})$. □

Using similar ideas we can extend these results to the blocker of the hyperplanes of a matroid and to the complementary clutter of the blocker of the hyperplanes of a matroid.

Corollary 5.11. *Let b be the blocker map, $\backslash\backslash$, $//$ the operations of Definition 1. We define $\backslash^d := b \circ \backslash\backslash \circ b$, and $/^d := b \circ // \circ b$. A clutter $(E, \mathcal{D}^*)$ is the blocker clutter of the hyperplanes of a matroid if and only if the following conditions hold:*

i) $(E, \mathcal{D}^*)$ *is not the clutter* $(E, \{\{e\} : e \in E\})$

ii) *using the operations* $\backslash^d$, $/^d$; $(E, \mathcal{D}^*)$ *has no minor isomorphic to* $(\{1, 2, 3\}, \{\{2, 3\}\})$. □

Corollary 5.12. *Let c be the complementary map, b the blocker map, $\backslash\backslash$, $//$ the operations of Definition 1. We define $\backslash^\delta := c \circ b \circ \backslash\backslash \circ b \circ c$, and $/^\delta := c \circ b \circ // \circ b \circ c$. A clutter (E, Δ) is the complementary clutter of the blocker clutter of the hyperplanes of a matroid if and only if the following conditions hold:*

i) (E, Δ) *is not the clutter* $(E, \{A : A \subseteq E, |A| = n - 1\})$

ii) *using the operations* $\backslash^\delta$, $/^\delta$; (E, Δ) *has no minor isomorphic to* $(\{1, 2, 3\}, \{\{1\}\})$. □

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