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Decompositions of complete tripartite graphs into cycles of lengths 3 and 6

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Abstract

A decomposition of a graph G into r copies of the cycle C_{m_1} and s copies of the cycle C_{m_2} is denoted by a $\{C^r_{m_1},\, C^s_{m_2}\}$-decomposition of G. In this paper, a necessary condition for the existence of a $\{C^r_3,\, C^s_6\}$-decomposition of the complete tripartite graph $K_{a,\,b,\,c}$, $a \le b \le c$, is obtained. Further, a sufficient condition for the existence of a $\{C^r_3,\, C^s_6\}$-decomposition of $K_{a,\,b,\,c}$, $a \le b \le c$, is given. As a corollary, the graph $K_{m,\,m,\,m}$ is shown to have a $\{C^r_3,\, C^s_6\}$-decomposition.

1 Introduction

Let C_m denote the cycle on m vertices. If $H_1, H_2, \ldots, H_k$ are edge-disjoint subgraphs of G such that $E(G) = E(H_1) \cup E(H_2) \cup \cdots \cup E(H_k)$, then we say that $H_1, H_2, \ldots, H_k$ *decompose* G and we write this as $G = H_1 \oplus H_2 \oplus \cdots \oplus H_k$, where $\oplus$ denotes edge disjoint union of graphs. If each $H_i \simeq H$, $1 \le i \le k$, then we say that H *decomposes* G and we denote this by $H \,|\, G$. If each $H_i \simeq C_m$, the cycle of length m, then we write $C_m \,|\, G$ and in this case we say that G has a *C_m-decomposition* or an *m-cycle decomposition*. A decomposition of G into r copies of C_{m_1} and s copies of C_{m_2} is denoted by a $\{C^r_{m_1},\, C^s_{m_2}\}$-decomposition of G. For a graph G, $G(\lambda)$ denotes the graph obtained from G by replacing each edge of G by λ edges. The complete graph on n vertices is denoted by K_n and the complete multipartite graph with partite sets having sizes $a_1, a_2, \ldots, a_k$ is denoted by $K_{a_1,a_2,\ldots,a_k}$. In particular, the complete tripartite graph with partite sets having sizes a, b, c with $a \le b \le c$ is denoted by $K_{a,b,c}$. The complete m-partite graph with each of its partite sets having size n is called a *complete equipartite* graph and it is denoted by $K_{m(n)}$. Throughout this paper, the partite sets of the complete tripartite graph $K_{a,b,c}$, $a \le b \le c$, are assumed to be $\{x_1, x_2, x_3, \ldots, x_a\}$, $\{y_1, y_2, y_3, \ldots, y_b\}$ and $\{z_1, z_2, z_3, \ldots, z_c\}$.

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A *latin square* of order k is a $k \times k$ array, each cell of which contains exactly one of the symbols in $\{1, 2, \ldots, k\}$, such that each row and each column of the array contains each of the symbols in $\{1, 2, \ldots, k\}$ exactly once. A latin square of order k is said to be *idempotent* if the cell (s, s) contains the symbol s, $1 \leq s \leq k$. A latin square of order k is said to be *cyclic* if the 1^{st} row entries are $a_1, a_2, a_3, \ldots, a_k$, and the s^{th} row entries are $a_s, a_{s+1}, a_{s+2}, \ldots, a_{s-1}$, in order. As in [9], a cell (i, j) is termed "empty" if it contains no entry and "filled" otherwise. For our convenience, when we represent a *partial latin square* we avoid drawing empty cells. Definitions which are not given here can be found in [5, 21].

Decompositions of complete graphs and complete multipartite graphs into cycles of fixed length are well-studied. Decomposition of the complete graph K_n (respectively $K_n - I$, where I is a perfect matching of K_n) when n is odd (respectively, even) into cycles has been considered by various authors: see [2, 18, 28] and [11]. Billington et al. considered a C_5-decomposition of a λ-fold complete equipartite graph: see [6]. Further, Manikandan and Paulraja proved that $C_p \,|\, K_{m(n)}$, $p \geq 5$ a prime, whenever the obvious necessary conditions are satisfied: see [23, 24, 25]. Moreover, in [29, 30, 31], Smith studied the existence of a k-cycle decomposition for $k \in \{2p, 3p, p^2\}$, of $K_{m(n)}$, where $p \geq 3$ is a prime. Further, existence of a $2k$-cycle decomposition of a λ-fold complete equipartite graph was obtained by Muthusamy and Shanmuga Vadivu: see [27]. Very recently, the authors of [12] actually solved the existence problem for a C_k-decomposition of $K_{m(n)}(\lambda)$ whose cycle-set can be partitioned into 2-regular graphs containing all the vertices except those belonging to one part. In [20], Jordon and Morris studied the cyclic Hamiltonian cycle decomposition of $K_{2n} - I$, where I is a perfect matching. In [26], Merola et al. obtained a necessary and sufficient condition for the existence of a cyclic and symmetric Hamiltonian cycle decomposition of $K_{m(n)}$ for any even m.

Chou et al. [15] obtained a necessary and sufficient condition for the existence of a decomposition of $K_{a,b}$ (respectively, $K_{m,m} - I$, where $m \geq 3$ is odd and I denotes a perfect matching) into cycles of lengths 4, 6 and 8. In [16], Chou and Fu considered a $\{C_4^r, C_{2t}^s\}$-decomposition of $K_{a,b}$ and $K_{m,m} - I$, where m is odd and I denotes a perfect matching. Later, Fu et al. [17] proved that the necessary conditions for the existence of a decomposition of $K_{m,m}$ (respectively, $K_{m,m} - I$) into cycles of distinct lengths are sufficient whenever m is even (respectively, odd) except when $m = 4$. Recently, Asplund et al. [3] established necessary and sufficient conditions for the existence of a decomposition of $K_{a,b}(\lambda)$ into cycles of arbitrary lengths. Existence of a $\{C_4^r, C_5^s\}$-decomposition of $K_{m(n)}$ was proved by Huang and Fu [19]. Moreover, Bahmanian and Šajna [4] showed that if $K_m(\lambda n)$ has a decomposition into cycles of lengths $k_1, k_2, \ldots, k_t$ (plus a perfect matching if $\lambda n(m-1)$ is odd), then $K_{m(n)}(\lambda)$ has a decomposition into cycles of lengths $k_1 n, k_2 n, \ldots, k_t n$ (plus a perfect matching if $\lambda n(m-1)$ is odd).

But not many results have been obtained in the study of decomposition of complete multipartite graphs when the partite sets have different sizes. Mahmoodian and Mirzakhani proved the existence of a C_5-decomposition of $K_{a,b,c}$ whenever the necessary conditions are satisfied and two of the partite sets have equal size, except when $a = b \equiv 0 \pmod 5$ and $c \not\equiv 0 \pmod 5$; see [22]. The authors of [1, 10, 13, 14] also studied this problem; but the problem remains open when the partite sets have different sizes and are odd. In [7], Billington obtained a necessary and sufficient condition for the existence of a $\{C_3^r, C_4^s\}$-decomposition of the graph $K_{a,b,c}$. Further, Billington et al. [8] obtained a necessary and sufficient condition for the existence of a $2k$-cycle decomposition of complete multipartite graphs for $k \in \{2, 3, 4\}$.

In this paper we give the necessary conditions for the existence of a $\{C_3^r, C_6^s\}$-decomposition of the complete tripartite graph $K_{a,b,c}$, $a \leq b \leq c$. Also, we give a sufficient condition for the existence of a $\{C_3^r, C_6^s\}$-decomposition of $K_{a,b,c}$, $a \leq b \leq c$. Using this, we prove that the graph $K_{m,m,m}$ admits a $\{C_3^r, C_6^s\}$-decomposition.

Often we recall the following remark.

Remark 1.1. Let the partite sets of the graph $K_{a,a,a}$, $a \geq 1$, be $\{x_1, x_2, \ldots, x_a\}$, $\{y_1, y_2, \ldots, y_a\}$ and $\{z_1, z_2, z_3, \ldots, z_a\}$. A C_3-decomposition of $K_{a,a,a}$ can be achieved from a latin square L of order a as follows: an entry s in the cell (i, j) of L, $1 \leq i, j, s \leq a$, corresponds to the 3-cycle (x_i, y_j, z_s) of $K_{a,a,a}$. All the cells of the latin square give a C_3-decomposition of $K_{a,a,a}$; see [7].

In this paper we prove the following main theorem.

Theorem 1.2. *Let $K_{a,b,c}$ be the complete tripartite graph with $a \leq b \leq c$ and let $K_{a,b,c} \neq K_{1,1,c}$, when $c \equiv 1 \pmod 6$ and $c > 1$. If $a \equiv b \equiv c \pmod 6$, then $K_{a,b,c}$ admits a $\{C_3^r, C_6^s\}$-decomposition for any $r \equiv a \pmod 2$, with $0 \leq r \leq ab$.*

Corollary 1.3. *The complete tripartite graph $K_{m,m,m}$ admits a $\{C_3^r, C_6^s\}$-decomposition.*

2 Necessary conditions

In this section we prove the necessary conditions for the existence of a $\{C_3^r, C_6^s\}$-decomposition of $K_{a,b,c}$.

Theorem 2.1. *Let a, b, c be positive integers with $a \leq b \leq c$. If the graph $K_{a,b,c} \neq K_{1,1,c}$, when $c \equiv 1 \pmod 6$ and $c > 1$, admits a $\{C_3^r, C_6^s\}$-decomposition, then*

(i) $a \equiv b \equiv c \pmod 2$;

(ii) $ab + ac + bc \equiv 0 \pmod 3$;

(iii) *either* $a \equiv b \equiv c \pmod 3$ *or two of them are multiples of three;*

(iv) $r \equiv a \pmod 2$ *with* $0 \leq r \leq ab$.

Proof. The conditions (i) and (ii) are obvious. For (iii), let $a = 3A + A'$, $b = 3B + B'$ and $c = 3C + C'$, where $0 \leq A', B', C' \leq 2$ and $A, B, C \geq 0$. Then

$$\begin{aligned} ab + ac + bc &= (3A + A')(3B + B') + (3A + A')(3C + C') + (3B + B')(3C + C') \\ &= 9(AB + AC + BC) + 3(AB' + BA' + AC' + CA' + BC' + CB') \\ &\quad + A'B' + A'C' + B'C'. \end{aligned}$$

From (ii), $3 \mid (A'B'+A'C'+B'C')$, and from this we conclude that either $A' = B' = C'$, or two of them must be zero.

Next we prove (iv). If there exists a $\{C_3^r, C_6^s\}$-decomposition in $K_{a,b,c}$, then $3r + 6s = ab + ac + bc$. Suppose by way of contradiction that r is odd (respectively, even) and a, b and c are even (respectively, odd); then $ab + ac + bc - 3r$ is odd but $6s = ab + ac + bc - 3r$ is even, by (i), a contradiction. Hence a, b, c and r have the same parity. In a tripartite graph each C_3 meets all the three partite sets and hence $r \leq ab$. This proves (iv). □

3 Some useful lemmas

We prove some useful lemmas before giving a proof of the main theorem.

Lemma 3.1. *The graph $K_{3,3,3}$ has a $\{C_3^r, C_6^s\}$-decomposition.*

Proof. Let the partite sets of $K_{3,3,3}$ be $\{x_1, x_2, x_3\}$, $\{y_1, y_2, y_3\}$ and $\{z_1, z_2, z_3\}$. Using the idempotent latin square L of order 3 given below, we exhibit a $\{C_3^r, C_6^s\}$-decomposition of $K_{3,3,3}$. Since a is odd, by Theorem 2.1, also r is odd, with $0 \leq r \leq 9$. Moreover, $3r + 6s = 27$, so we have to consider the following cases:

$L =$

1	3	2
3	2	1
2	1	3

,

(1) $r = 9$ and $s = 0$.

Then the required decomposition follows by Remark 1.1.

(2) $r = 7$ and $s = 1$.

The three C_3 of $K_{3,3,3}$ corresponding to the three cells $(2,1)$, $(2,2)$ and $(3,1)$ of L give one 6-cycle and one 3-cycle, namely, $(x_2, y_2, z_2, x_3, y_1, z_3)$ and (x_2, y_1, z_2). The remaining cells of L correspond to six 3-cycles, by Remark 1.1.

(3) $r = 5$ and $s = 2$.

The edges of the four C_3 of $K_{3,3,3}$ corresponding to the cells $(1,2)$, $(1,3)$, $(2,1)$ and $(3,1)$ of L can be partitioned into two 6-cycles, namely, $(x_1, z_3, x_2, y_1, x_3, z_2)$ and $(x_1, y_2, z_3, y_1, z_2, y_3)$, and the remaining cells yield five 3-cycles, by Remark 1.1.

(4) $r = 3$ and $s = 3$.

The diagonal cells of L correspond to three 3-cycles of $K_{3,3,3}$ and the edges not on these three 3-cycles can be partitioned into three 6-cycles, namely, $(x_1, y_2, x_3, y_1, x_2, y_3)$, $(y_1, z_2, y_3, z_1, y_2, z_3)$ and $(x_1, z_2, x_3, z_1, x_2, z_3)$.

(5) $r = 1$ and $s = 4$.

The cells of L, except the cell $(1,1)$, correspond to four 6-cycles, $(x_1, z_3, x_2, y_1, x_3, z_2)$, $(x_1, y_2, z_3, y_1, z_2, y_3)$, $(x_2, y_2, x_3, z_3, y_3, z_1)$ and $(x_2, y_3, x_3, z_1, y_2, z_2)$. The C_3 corresponding to the cell $(1,1)$ is (x_1, y_1, z_1). □

Lemma 3.2. *The graph $K_{5,5,5}$ has a $\{C_3^r, C_6^s\}$-decomposition.*

Proof. Let the partite sets of $K_{5,5,5}$ be $\{x_1, x_2, x_3, x_4, x_5\}$, $\{y_1, y_2, y_3, y_4, y_5\}$ and $\{z_1, z_2, z_3, z_4, z_5\}$. Consider the idempotent latin square L of order 5 given below:

$L =$

1	4	2	5	3
4	2	5	3	1
2	5	3	1	4
5	3	1	4	2
3	1	4	2	5

.

From L above, we obtain five cell-disjoint partial latin squares L_1, L_2, L_3, L_4 and L_5, respectively, as shown below, where c_i and r_j denote the i^{th} column and j^{th} row of L, respectively.

	c_1	c_2	c_3
r_1	1	4	2
r_2	4	2	5
r_3	2	5	3

L_1

	c_4	c_5
r_2	3	1
r_3	1	4

L_2

	c_2	c_3
r_4	3	1
r_5	1	4

L_3

	c_4	c_5
r_4	4	2
r_5	2	5

L_4

	c_1	c_4	c_5
r_1		5	3
r_4	5		
r_5	3		

L_5

From the cells of the partial latin square L_i, $2 \leq i \leq 5$, we obtain four 3-cycles, by Remark 1.1, and the edges of these four C_3 can be partitioned into two 6-cycles; they are listed below:

(i) 6-cycles corresponding to L_2 are $(x_2, y_4, x_3, z_4, y_5, z_1)$, $(x_2, z_3, y_4, z_1, x_3, y_5)$.

(ii) 6-cycles corresponding to L_3 are $(x_4, y_2, x_5, z_4, y_3, z_1)$, $(x_4, z_3, y_2, z_1, x_5, y_3)$.

(iii) 6-cycles corresponding to L_4 are $(x_4, y_4, x_5, z_5, y_5, z_2)$, $(x_4, z_4, y_4, z_2, x_5, y_5)$.

(iv) 6-cycles corresponding to L_5 are $(x_1, z_3, x_5, y_1, x_4, z_5)$, $(x_1, y_4, z_5, y_1, z_3, y_5)$.

Now we consider the partial latin square L_1. The cells of L_1 correspond to one 3-cycle and four 6-cycles, or three 3-cycles and three 6-cycles, or seven 3-cycles and one 6-cycle, or nine 3-cycles as shown below:

(1) (x_1, y_1, z_1), $(x_1, z_2, x_3, y_1, x_2, z_4)$, $(x_1, y_2, z_4, y_1, z_2, y_3)$, $(x_2, y_2, x_3, z_3, y_3, z_5)$, $(x_2, y_3, x_3, z_5, y_2, z_2)$.

(2) (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), $(x_1, y_2, x_3, y_1, x_2, y_3)$, $(y_1, z_2, y_3, z_5, y_2, z_4)$, $(x_1, z_2, x_3, z_5, x_2, z_4)$.

(3) (x_1, y_1, z_1), (x_1, y_3, z_2), (x_2, y_3, z_5), (x_3, y_1, z_2), (x_3, y_2, z_5), (x_3, y_3, z_3), (x_2, y_2, z_4), $(x_1, y_2, z_2, x_2, y_1, z_4)$.

(4) nine 3-cycles by Remark 1.1.

Depending on r and s, we choose the 3-cycles and 6-cycles from the above list to obtain a $\{C_3^r, C_6^s\}$-decomposition of $K_{5,5,5}$. This completes the proof. $\square$

We quote the following theorem for our future reference.

Theorem 3.3. [32] *For positive integers a, b and k, $C_k \mid K_{a,b}$ if and only if a, b and k are all even with $a \geq \frac{k}{2}$, $b \geq \frac{k}{2}$ and $k \mid ab$.*

Lemma 3.4. *If* $b \equiv 1 \pmod 6$ *and* $3r + 6s = 2b + b^2$, $1 \le r \le b$, *then* $K_{1,b,b}$ *has a* $\{C_3^r, C_6^s\}$*-decomposition.*

Proof. Let $b = 6b' + 1$, where $b' \ge 0$. Let the partite sets of $K_{1,b,b}$ be $\{x_0\}$, $\{y_0, y_1, y_2, \dots, y_{6b'}\}$ and $\{z_0, z_1, z_2, \dots, z_{6b'}\}$. Delete the edges of the 3-cycle $C = \{x_0, y_0, z_0\}$ from $K_{1,b,b}$; the resulting subgraph can be decomposed into b' copies of the graph isomorphic to $K_{1,7,7} - E(C)$ and $b'(b' - 1)$ copies of $K_{6,6}$. Since $C_6 \mid K_{6,6}$, by Theorem 3.3, it is enough to obtain a $\{C_3^{r_1}, C_6^{s_1}\}$-decomposition of $K_{1,7,7} - E(C)$ for suitable r_1 and s_1. We exhibit a $\{C_3^{r_1}, C_6^{s_1}\}$-decomposition of $K_{1,7,7} - E(C)$ as follows, where we assume that the partite sets of $K_{1,7,7} - E(C)$ are $\{x_0\}$, $\{y_0, y_1, y_2, y_3, y_4, y_5, y_6\}$ and $\{z_0, z_1, z_2, z_3, z_4, z_5, z_6\}$.

(1) If $r_1 = 0$ and $s_1 = 10$, then the edge disjoint cycles are

$(x_0, y_2, z_1, y_1, z_0, y_4)$, $(x_0, y_3, z_2, y_2, z_0, y_5)$, $(x_0, y_1, z_3, y_3, z_0, y_6)$,
$(x_0, z_3, y_0, z_1, y_3, z_6)$, $(x_0, z_1, y_4, z_2, y_0, z_4)$, $(x_0, z_2, y_5, z_6, y_0, z_5)$,
$(y_5, z_1, y_6, z_2, y_1, z_5)$, $(y_6, z_4, y_1, z_6, y_2, z_5)$, $(y_2, z_4, y_3, z_5, y_4, z_3)$,
$(y_4, z_4, y_5, z_3, y_6, z_6)$.

(2) If $r_1 = 2$ and $s_1 = 9$, then the required set of edge disjoint 3-cycles and 6-cycles are C', C'', C^1, C^2, C^3, C^4, C^5, C^6, C^7, D^1, D^2, where

$C' = (x_0, y_1, z_1)$, $C'' = (x_0, y_2, z_2)$, $C^1 = (y_0, z_1, y_2, z_0, y_1, z_2)$,
$C^2 = (x_0, y_5, z_5, y_0, z_6, y_6)$, $C^3 = (x_0, z_5, y_6, z_0, y_5, z_6)$, $C^4 = (y_1, z_3, y_5, z_1, y_6, z_4)$,
$C^5 = (y_2, z_3, y_6, z_2, y_5, z_4)$, $C^6 = (y_1, z_5, y_3, z_1, y_4, z_6)$, $C^7 = (y_2, z_5, y_4, z_2, y_3, z_6)$,
$D^1 = (x_0, y_3, z_3, y_0, z_4, y_4)$, $D^2 = (x_0, z_3, y_4, z_0, y_3, z_4)$.

(3) If $r_1 = 4$ and $s_1 = 8$, from the above decomposition for the case $r_1 = 2$ and $s_1 = 9$, the union of the edges of D^1 and D^2 can be partitioned into two copies of C_3 and a copy of C_6, namely, $C''' = (x_0, y_3, z_3)$, $C'''' = (x_0, y_4, z_4)$ and $C^8 = (y_0, z_3, y_4, z_0, y_3, z_4)$. Hence the required decomposition is given by C', C'', C''', C'''', C^1, C^2, C^3, C^4, C^5, C^6, C^7 and C^8.

(4) If $r_1 = 6$ and $s_1 = 7$, then the cycles are

C', C'', C''', C'''', (x_0, y_5, z_5), (x_0, y_6, z_6), $(y_0, z_1, y_3, z_0, y_1, z_3)$,
$(y_1, z_2, y_4, z_1, y_2, z_4)$, $(y_2, z_3, y_5, z_2, y_3, z_5)$, $(y_3, z_4, y_6, z_3, y_4, z_6)$,
$(y_4, z_5, y_0, z_4, y_5, z_0)$, $(y_5, z_6, y_1, z_5, y_6, z_1)$, $(y_6, z_0, y_2, z_6, y_0, z_2)$,

where C', C'', C''' and C'''' are as in the case $(r_1, s_1) = (4, 8)$. □

For our convenience we use the following definition given in [7].

If a latin square L contains a subsquare of the type

α	$\alpha + 1$
$\alpha + 1$	α

,

then we call it a '*subsquare of the form* (α)'.

The following lemma is in [7]; as we extensively use it in our proof, we give a proof of it here.

Lemma 3.5. [7] *For any $k \geq 3$, there exists a latin square of order $2k+1$ containing $k(k-1)$ 2×2 cell-disjoint subsquares of the form (α).*

Proof. Consider an idempotent latin square L' of order k, on the set $\{1, 2, \ldots, k\}$. From L', we obtain a new latin square, L'' of order $2k$ by replacing each entry l in L' with

$2l-1$	$2l$
$2l$	$2l-1$

From L'', we obtain the required latin square L of order $2k+1$ on the set $\{0, 1, 2, \ldots, 2k\}$, by adjoining a new top row and new left-hand column to L'', and appropriately replacing the 2×2 squares on the diagonal of L'' as follows:

Let (r_i, c_j) denote the cell in the i^{th} row and j^{th} column of a latin square. Since L' is an idempotent latin square, the 2×2 subsquares on the "diagonal" of L'' are the following:

1	2
2	1

,

3	4
4	3

, ... ,

$2k-1$	$2k$
$2k$	$2k-1$

.

The required latin square L is obtained by replacing the diagonal 2×2 subsquares of L'' of the form $(2l)$, that is,

$2l-1$	$2l$
$2l$	$2l-1$

by

$2l-1$	0
0	$2l$

,

place 0 in the cell $(0, 0)$ and place $2l$ (respectively, $2l-1$) in the cells $(0, 2l-1), (2l-1, 0)$ (respectively, $(0, 2l), (2l, 0)$); see Example 3.6. The remaining 2×2 subsquares of L'' in L are unchanged. The resulting latin square is the required latin square, since the 2×2 subsquares corresponding to the non-diagonal cells of L' become 2×2 subsquares of type (α); see Example 3.6. ☐

Example 3.6. For $k = 3$, let

$L' =$

1	3	2
3	2	1
2	1	3

. Then $L'' =$

1	2	5	6	3	4
2	1	6	5	4	3
5	6	3	4	1	2
6	5	4	3	2	1
3	4	1	2	5	6
4	3	2	1	6	5

and

$L =$ (0^{th} *row* is the top row; 0^{th} *col* is the left column)

0	**2**	**1**	**4**	**3**	**6**	**5**
2	**1**	**0**	5	6	3	4
1	**0**	**2**	6	5	4	3
4	5	6	**3**	**0**	1	2
3	6	5	**0**	**4**	2	1
6	3	4	1	2	**5**	**0**
5	4	3	2	1	**0**	**6**

.

To prove the next theorem, we need a particular idempotent latin square, I_k, which is defined here; see [21]. For an odd integer $k \geq 3$, consider the cyclic latin square, C, of order k, on the set $\{1, 2, 3, \ldots, k\}$ with the i^{th} row $i, i+1, \ldots, i-1$, in order. Let $k = 2k' + 1$, for some $k' \geq 1$. Now we rename the entry i in C by the rule $i \to 1 + (i-1)n \pmod{k}$, where $n = k' + 1$; see the example below. The resulting latin square I_k is idempotent and the entries of the cells in $T = \{(1,2), (2,3), \ldots, (k-1,k), (k,1)\}$ of I_k is a transversal of I_k. Now applying the technique of stripping the transversal T (see [21]), an idempotent latin square of even order $k+1$ is obtained. Thus, for all $k \geq 3$, we have an idempotent latin square, which we denote by I_k. For example, when $k = 7$, the latin squares C, I_7 and I_8, respectively, are given below. This I_k is extensively used throughout the paper.

$C =$

1	2	3	4	5	6	7
2	3	4	5	6	7	1
3	4	5	6	7	1	2
4	5	6	7	1	2	3
5	6	7	1	2	3	4
6	7	1	2	3	4	5
7	1	2	3	4	5	6

$I_7 =$

1	**5**	2	6	3	7	4
5	2	**6**	3	7	4	1
2	6	3	**7**	4	1	5
6	3	7	4	**1**	5	2
3	7	4	1	5	**2**	6
7	4	1	5	2	6	**3**
4	1	5	2	6	3	7

Bold letters form a transversal T for I_7

$I_8 =$

1	**8**	2	6	3	7	4	**5**
5	2	**8**	3	7	4	1	**6**
2	6	3	**8**	4	1	5	**7**
6	3	7	4	**8**	5	2	**1**
3	7	4	1	5	**8**	6	**2**
7	4	1	5	2	6	**8**	**3**
8	1	5	2	6	3	7	**4**
4	**5**	**6**	**7**	**1**	**2**	**3**	**8**

I_8 is obtained from I_7 by the technique of stripping the transversal T.

Remark 3.7. Here we list some useful observations about I_k for our future reference.

Observation 1. For odd $k = 2k' + 1$, by our construction of I_k, the entries of the first row of I_k are

	c_1	c_2	c_3	c_4	$\ldots$	c_{k-1}	c_k
r_1	1	$k'+2$	2	$k'+3$	$\ldots$	k	$k'+1$

and the entries in the $(i+1)^{\text{st}}$ row of I_k, $1 \leq i \leq k-1$, are of the following form:

	c_1	c_2	c_3	c_4	$\ldots$	c_{k-1}	c_k
r_{i+1}	m	$m+n$	$m+2n$	$m+3n$	$\ldots$	$m-2n$	$m-n$

,

where $n = k' + 1$, $m = 1 + i \cdot n$.

Observation 2. As I_k, $k = 2k' + 1$, is cyclic, any three consecutive rows of I_k are of the form

	c_1	c_2	c_3	c_4	$\dots$	c_{k-1}	c_k
r_{i+1}	m	$m+n$	$m+2n$	$m+3n$	$\dots$	$m-2n$	$m-n$
r_{i+2}	$m+n$	$m+2n$	$m+3n$	$m+4n$	$\dots$	$m-n$	m
r_{i+3}	$m+2n$	$m+3n$	$m+4n$	$m+5n$	$\dots$	m	$m+n$

,

where $n = k' + 1$ and $m = 1 + i \cdot n$.

Observation 3. Since I_{k+1}, $k+1 = 2k''$, is obtained from I_k, any three consecutive rows of I_{k+1}, except its last three rows, are as shown below, where $n = k''$, $m = 1+i\cdot n$ and the entries are taken modulo k, except the entry $k+1$ in each of the cells $(i+1, i+2)$, $(i+2, i+3)$ and $(i+3, i+4)$, which is shown in bold face letters in the partial latin square below; these $(k+1)$'s arise out of the stripping of a transversal.

	c_1	c_2	c_3	$\dots$	c_{i+1}	c_{i+2}	c_{i+3}	c_{i+4}	$\dots$	c_k	c_{k+1}
r_{i+1}	m	$m+n$	$m+2n$	$\dots$	$i+1$	$\boldsymbol{k+1}$	$i+2$	$m+(i+3)n$	$\dots$	$m-n$	$m+(i+1)n$
r_{i+2}	$m+n$	$m+2n$	$m+3n$	$\dots$	$m+(i+1)n$	$i+2$	$\boldsymbol{k+1}$	$m+(i+4)n$	$\dots$	m	$m+(i+3)n$
r_{i+3}	$m+2n$	$m+3n$	$m+4n$	$\dots$	$i+2$	$m+(i+3)n$	$i+3$	$\boldsymbol{k+1}$	$\dots$	$m+n$	$m+(i+5)n$

.

Observation 4. The last three rows of I_{k+1}, $k+1 = 2k''$, are given below:

	c_1	c_2	c_3	c_4	c_5	c_6	$\dots$	c_{k-1}	c_k	c_{k+1}
r_{k-1}	k	k''	$2k''$	$3k''$	$4k''$	$5k''$	$\dots$	$k-1$	$\boldsymbol{k+1}$	$k''-1$
r_k	$\boldsymbol{k+1}$	$2k''$	$3k''$	$4k''$	$5k''$	$6k''$	$\dots$	$k''-1$	k	k''
r_{k+1}	k''	$k''+1$	$k''+2$	$k''+3$	$k''+4$	$k''+5$	$\dots$	$k''-2$	$k''-1$	$\boldsymbol{k+1}$

,

where the entries are taken modulo k, except the entries in the cells $(k-1, k)$, $(k, 1)$ and $(k+1, k+1)$ which arise out of the stripping of a transversal.

Theorem 3.8. *Let a and b be positive integers with $1 \le a \le b$. If $a \equiv b \pmod 6$, then $K_{a,b,b}$ admits a $\{C_3^r, C_6^s\}$-decomposition for any $r \equiv a \pmod 2$, with $0 \le r \le ab$.*

Proof. We split the proof into two cases.

Case 1. a is even.

Let $a = 2a'$ and $b = 2b'$, $1 \le a' \le b'$. Let C be a cyclic latin square of order b' with the first row entries $1, 2, \dots, b'$, in order. From C, we obtain a new latin square C' of order b on the set $\{1, 2, \dots, b\}$ by replacing the $(i, j)^{\text{th}}$ entry $i + j - 1 = \ell$ of C into a 2×2 subsquare of the form $(2\ell - 1)$, where $1 \le i,\ j,\ \ell \le b'$; that is, we replace the entry ℓ of C by

$2\ell - 1$	2ℓ
2ℓ	$2\ell - 1$

(1)

The first a rows of C' contain exactly $a'b'$ 2×2 cell-disjoint subsquares of the form (α). Each of these 2×2 subsquares corresponds to four 3-cycles, or two 3-cycles and one 6-cycle, or two 6-cycles of $K_{a,b,b}$ which are listed below; the cycles described below are based on the subsquare of the form $(2\ell - 1)$ in (1), where $\ell = i + j - 1$. The entries $2l - 1$ and $2l$ correspond to the cells (r_{2i-1}, c_{2j-1}), (r_{2i}, c_{2j}) and (r_{2i-1}, c_{2j}), (r_{2i}, c_{2j-1}), respectively, of C'.

$$\left.\begin{array}{rl} (i) & (x_{2i-1}, y_{2j-1}, z_{2\ell-1}),\ (x_{2i-1}, y_{2j}, z_{2\ell}),\ (x_{2i}, y_{2j-1}, z_{2\ell}),\ (x_{2i}, y_{2j}, z_{2\ell-1}). \\ (ii) & (x_{2i-1}, y_{2j-1}, z_{2\ell}),\ (x_{2i}, y_{2j}, z_{2\ell-1}),\ (x_{2i-1}, y_{2j}, z_{2\ell}, x_{2i}, y_{2j-1}, z_{2\ell-1}). \\ (iii) & (x_{2i-1}, y_{2j-1}, z_{2\ell-1}, x_{2i}, z_{2\ell}, y_{2j}),\ (x_{2i}, y_{2j}, z_{2\ell-1}, x_{2i-1}, z_{2\ell}, y_{2j-1}). \end{array}\right\} \quad (2)$$

The maximum number of 3-cycles in $K_{a,b,b}$ cannot exceed ab. To obtain r copies of C_3, choose $\lceil \frac{r}{4} \rceil$, 2×2 subsquares of the form (α) in the first a rows of C'. These subsquares give the required r copies of C_3, as the 12 edges of $K_{a,b,b}$ corresponding to each of these subsquares of the form (α) can be partitioned into either four C_3 or two C_3 and one C_6 by (2). Since the 12 edges corresponding to any 2×2 subsquare of the form (α) can be decomposed into two C_6 by (2), the remaining $a'b' - \lceil \frac{r}{4} \rceil$ subsquares of the form (α) within the first $a = 2a'$ rows of C', give $s_1 = 2(a'b' - \lceil \frac{r}{4} \rceil)$ cycles of length six. If $a = b$, then the above decomposition is the required decomposition. So we assume that $a < b$.

Observe that all the edges incident with the partite set of size a are on the triangles corresponding to the entries of the cells in the first a rows of C'. Consequently, after the deletion of the edges of r C_3 and s_1 C_6 from $K_{a,b,b}$, corresponding to the cells in the first a rows of C', the resulting edge induced subgraph is a bipartite subgraph, say, H, of $K_{b,b}$ contained in $K_{a,b,b}$. We now decompose this bipartite graph H into cycles of length six. Observe that if the $(a + i, j)^{\text{th}}$ entry of C' is l, then this entry now denotes only the edge $y_j z_l$ of H, because all the edges incident with the partite set of size a have been used by rC_3 and $s_1 C_6$ obtained above.

The edges of $K_{a,b,b}$ corresponding to the cells of the remaining $b - a$ rows of C' can be decomposed into 6-cycles as follows: since $b - a \equiv 0 \pmod 6$, we partition the $b - a$ rows of C' into six consecutive rows each, namely, C'_i, $1 \le i \le \frac{b-a}{6}$, beginning from the $(a + 1)^{\text{th}}$ row. A partial latin square, C'_i of C', consisting of six rows is of the following form:

	c_1	c_2	c_3	c_4	$\dots$	c_{b-1}	c_b
r_t	t	$t+1$	$t+2$	$t+3$	$\dots$	$t-2$	$t-1$
r_{t+1}	$t+1$	t	$t+3$	$t+2$	$\dots$	$t-1$	$t-2$
r_{t+2}	$t+2$	$t+3$	$t+4$	$t+5$	$\dots$	t	$t+1$
r_{t+3}	$t+3$	$t+2$	$t+5$	$t+4$	$\dots$	$t+1$	t
r_{t+4}	$t+4$	$t+5$	$t+6$	$t+7$	$\dots$	$t+2$	$t+3$
r_{t+5}	$t+5$	$t+4$	$t+7$	$t+6$	$\dots$	$t+3$	$t+2$

,

where $t = a+6i-5$, $1 \le i \le \frac{b-a}{6}$. Now partition C'_i into 6×4 subsquares, consisting of four consecutive columns of C'_i, beginning from the first column if $b \equiv 0 \pmod 4$, or into 6×4 subsquares except the last subsquare which is a 6×6 subsquare if $b \equiv 2 \pmod 4$.

Let the 6×4 subsquare of C'_i be C'_{ij}, $1 \le j \le \frac{b}{4}$, if $b \equiv 0 \pmod 4$; let C'_{ij}, $1 \le j \le \frac{b-6}{4}$, and $C'_{i\infty}$ be the 6×4 and 6×6 subsquares, respectively, of C'_i if $b \equiv 2 \pmod 4$. The entries of C'_{ij} and $C'_{i\infty}$ are shown below.

$C'_{ij} =$

	c_{4j-3}	c_{4j-2}	c_{4j-1}	c_{4j}
r_t	$t+4j-4$	$t+4j-3$	$t+4j-2$	$t+4j-1$
r_{t+1}	$t+4j-3$	$t+4j-4$	$t+4j-1$	$t+4j-2$
r_{t+2}	$t+4j-2$	$t+4j-1$	$t+4j$	$t+4j+1$
r_{t+3}	$t+4j-1$	$t+4j-2$	$t+4j+1$	$t+4j$
r_{t+4}	$t+4j$	$t+4j+1$	$t+4j+2$	$t+4j+3$
r_{t+5}	$t+4j+1$	$t+4j$	$t+4j+3$	$t+4j+2$

and

$C'_{i\infty} =$

	c_{b-5}	c_{b-4}	c_{b-3}	c_{b-2}	c_{b-1}	c_b
r_t	$t-6$	$t-5$	$t-4$	$t-3$	$t-2$	$t-1$
r_{t+1}	$t-5$	$t-6$	$t-3$	$t-4$	$t-1$	$t-2$
r_{t+2}	$t-4$	$t-3$	$t-2$	$t-1$	t	$t+1$
r_{t+3}	$t-3$	$t-4$	$t-1$	$t-2$	$t+1$	t
r_{t+4}	$t-2$	$t-1$	t	$t+1$	$t+2$	$t+3$
r_{t+5}	$t-1$	$t-2$	$t+1$	t	$t+3$	$t+2$

.

As each cell of C'_{ij} or $C'_{i\infty}$ corresponds to exactly one edge of H, all the entries of C'_{ij} and $C'_{i\infty}$ correspond to 24 and 36 edges of H, respectively; see Figure 1. If the $(p,q)^{\text{th}}$ entry of C'_{ij} (respectively, $C'_{i\infty}$) is ℓ, then that entry represents the edge $y_q z_\ell$ of H. We now partition the edges corresponding to C'_{ij} and $C'_{i\infty}$ into four 6-cycles and six 6-cycles, respectively, as follows:

A set of four 6-cycles of H corresponding to the cells of C'_{ij} is

$(y_{4j-3}, z_{t+4j-3}, y_{4j-2}, z_{t+4j-2}, y_{4j}, z_{t+4j-1})$, $(y_{4j-3}, z_{t+4j-4}, y_{4j-2}, z_{t+4j-1}, y_{4j-1}, z_{t+4j-2})$, $(y_{4j-3}, z_{t+4j}, y_{4j-1}, z_{t+4j+3}, y_{4j}, z_{t+4j+1})$, $(y_{4j-2}, z_{t+4j+1}, y_{4j-1}, z_{t+4j+2}, y_{4j}, z_{t+4j})$; see Figure 1(a).

A set of six 6-cycles of H corresponding to the cells of $C'_{i\infty}$ is

$(y_{b-5}, z_{t-4}, y_{b-3}, z_{t-1}, y_{b-2}, z_{t-3})$, $(y_{b-4}, z_{t-3}, y_{b-3}, z_{t-2}, y_{b-2}, z_{t-4})$,

$(y_{b-3}, z_t, y_{b-1}, z_{t+3}, y_b, z_{t+1})$, $(y_{b-2}, z_{t+1}, y_{b-1}, z_{t+2}, y_b, z_t)$,

$(y_{b-5}, z_{t-6}, y_{b-4}, z_{t-1}, y_{b-1}, z_{t-2})$, $(y_{b-5}, z_{t-5}, y_{b-4}, z_{t-2}, y_b, z_{t-1})$;

see Figure 1(b).

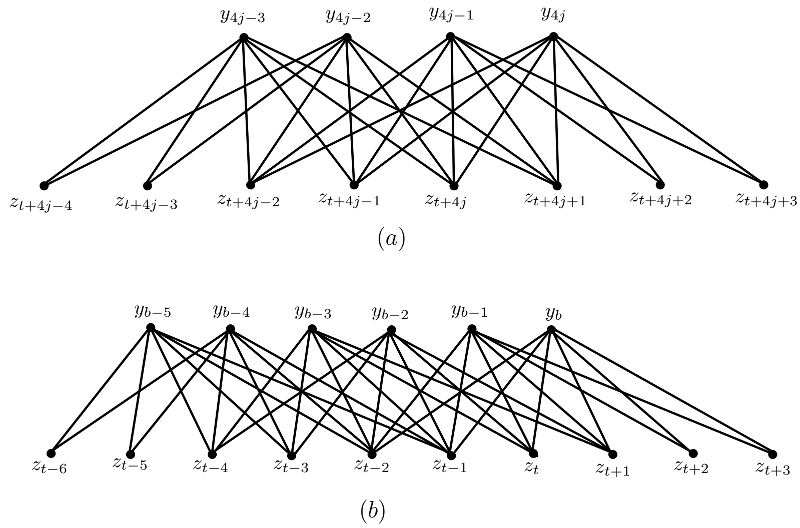

Figure 1: The subgraph of H corresponding to the cells of C'_{ij} (respectively $C'_{i\infty}$) is shown in (a) (respectively (b)).

Let s_2 be the number of six cycles of H corresponding to the cells of the last $b-a$ rows of C'. Thus we have obtained r 3-cycles and s_1 6-cycles corresponding to the cells of the first a rows of C' and s_2 6-cycles corresponding to the cells of the remaining $b-a$ rows of C'; and $(3r+6s_1)+6s_2 = 3ab+(b-a)b = 2ab+b^2$, which is the number of edges of $K_{a,b,b}$. This completes the proof of this case.

Case 2. a is odd.

Because a and b have same parity, let $a = 2a'+1$ and let $b = 2b'+1$, for some $b' \geq a'$. The graph $K_{1,1,1}$ can be decomposed into one C_3 and no C_6. Since the case $a = 1$ with $b \equiv 1 \pmod 6$, and the cases $a = b = 3$ and $a = b = 5$ are dealt with in Lemmas 3.4, 3.1 and 3.2, respectively, we do not consider them here.

Consider an idempotent latin square $I_{b'}$ of order b', on the set $\{1, 2, \ldots, b'\}$, as described in Remark 3.7. From $I_{b'}$, we obtain a latin square L of order b, using Lemma 3.5, on the set $\{0, 1, 2, \ldots, 2b'\}$. Part of the entries of L, obtained from $I_{b'}$, are given in Figure 2; the 2×2 subsquares, in order, without entries, in Figure 2, are subsquares of the form (α).

Let L_a, L_b and L_c be three partial latin squares of L, see Figure 3; note that if $b \neq a$, then the partial latin squares L_b and L_c of L exist.

A sketch of the rest of the proof of this case is described here. Our aim is to partition the cells of L into subsets L_a, L_b and L_c and decompose the subgraphs of $K_{a,b,b}$ corresponding to these subsets of cells according to our requirement. Using the cells of L_a (respectively, L_b) we obtain r' (respectively, r'') copies of 3-cycles and s_1 (respectively, s_2) copies of 6-cycles; s_1 (respectively, s_2) may be zero. These $r = r'+r''$ 3-cycles and $s' = s_1+s_2$ 6-cycles contain all the edges of $K_{a,b,b}$ incident with the partite set of size a. Edges not on these cycles induce a subgraph $H \subset K_{b,b} \subset K_{a,b,b}$. Each cell in L_c now represents an edge of H. We partition the edges corresponding to the cells of L_c into cycles of length six.

$L =$

	c_0	c_1	c_2	c_3	c_4	$\cdots$	$c_{2a'-1}$	$c_{2a'}$	$\cdots$	$c_{2b'-1}$	$c_{2b'}$
r_0	0	2	1	4	3		$2a'$	$2a'-1$		$2b'$	$2b'-1$
r_1	2	1	0								
r_2	1	0	2								
r_3	4			3	0						
r_4	3			0	4						
$\vdots$											
$r_{2a'-1}$	$2a'$						$2a'-1$	0			
$r_{2a'}$	$2a'-1$						0	$2a'$			
$\vdots$											
$r_{2b'-1}$	$2b'$									$2b'-1$	0
$r_{2b'}$	$2b'-1$									0	$2b'$

Figure 2: The latin square L. In the partial latin square obtained from L by deleting its 0^{th} row and 0^{th} column, all 2×2 subsquares are of the form (α), except the "diagonal" 2×2 cells which are of the form

$2i-1$	0
0	$2i$

.

We now proceed to the proof of the theorem.

Initially we partition the edges of $K_{a,b,b}$ corresponding to the cells of L_a of L into r' 3-cycles and s_1 (possibly zero) 6-cycles.

We fix the 3-cycle $C = (x_0, y_0, z_0)$ of $K_{a,b,b}$ corresponding to the entry 0 in the cell $(0, 0)$ of L_a. Clearly, L_a without its 0^{th} row and 0^{th} column contains 2×2 subsquares of the form

	c_{2i-1}	c_{2i}
r_{2i-1}	$2i-1$	0
r_{2i}	0	$2i$

, $1 \leq i \leq a'$, along the "diagonal"; see Figure 2. This subsquare together with four other cells of L_a, namely, two of the cells $(0, 2i-1)$ and $(0, 2i)$, for each i, in the 0^{th} row and two cells $(2i-1, 0)$ and $(2i, 0)$ in the 0^{th} column give the partial latin square L_{ai} of L_a, where $L_{ai} =$

	c_0	c_{2i-1}	c_{2i}
r_0		$2i$	$2i-1$
r_{2i-1}	$2i$	$2i-1$	0
r_{2i}	$2i-1$	0	$2i$

.

Each L_{ai}, $1 \leq i \leq a'$, with 8 entries, as shown above, is equivalent to 24 edges of $K_{a,b,b}$ and a $\{C_3^{r_1}, C_6^{s_1}\}$-decomposition of these 24 edges is listed below:

(1) If $r_1 = 8$ and $s_1 = 0$, it is clear as each cell corresponds to a C_3.

(2) If $r_1 = 6$ and $s_1 = 1$, then a required set of cycles is
(x_0, y_{2i-1}, z_{2i}), (x_0, y_{2i}, z_{2i-1}), (x_{2i-1}, y_{2i}, z_0), (x_{2i}, y_{2i-1}, z_0), (x_{2i}, y_{2i}, z_{2i}), $(x_{2i-1}, y_0, z_{2i-1})$, $(x_{2i}, y_0, z_{2i}, x_{2i-1}, y_{2i-1}, z_{2i-1})$.

(3) If $r_1 = 4$ and $s_1 = 2$, then a required set of cycles is
$(x_{2i-1}, y_{2i-1}, z_{2i-1})$, (x_{2i-1}, y_{2i}, z_0), (x_{2i}, y_{2i-1}, z_0), (x_{2i}, y_{2i}, z_{2i}), $C' = (x_0, y_{2i}, z_{2i-1}, x_{2i}, y_0, z_{2i})$, $C'' = (x_0, y_{2i-1}, z_{2i}, x_{2i-1}, y_0, z_{2i-1})$.

(4) If $r_1 = 2$ and $s_1 = 3$, then a required set of cycles is
$(x_{2i-1}, y_{2i-1}, z_0)$, (x_{2i}, y_{2i}, z_{2i}), C', C'', $(x_{2i}, y_{2i-1}, z_{2i-1}, x_{2i-1}, y_{2i}, z_0)$,

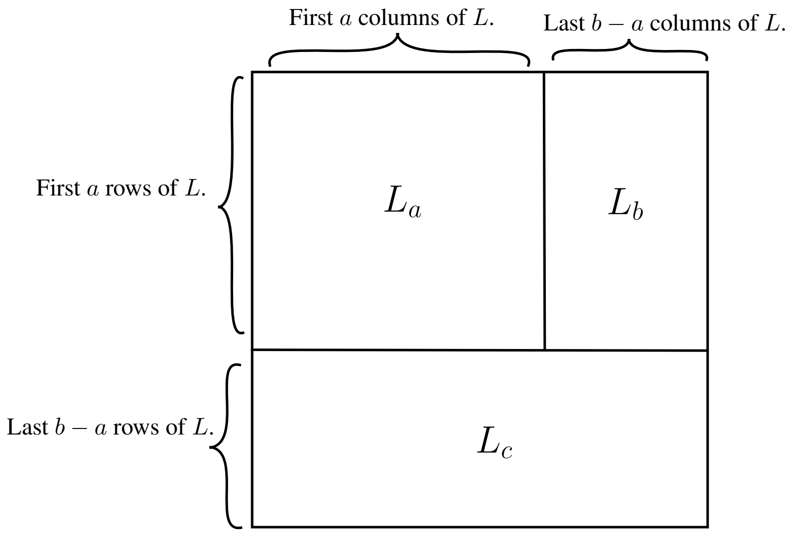

Figure 3: Three partial latin squares of L.

where C' and C'' are as in (3) above.

(5) If $r_1 = 0$ and $s_1 = 4$, then a set of cycles is $(x_{2i-1}, z_{2i-1}, y_{2i-1}, x_{2i}, y_{2i}, z_0)$, $(x_{2i}, z_{2i}, y_{2i}, x_{2i-1}, y_{2i-1}, z_0)$, C', C'', where C' and C'' are as in (3) above.

The subgraphs of $K_{a,b,b}$ corresponding to these L_{ai}'s contain, besides other edges, all the edges corresponding to the cells of the 0^{th} row and 0^{th} column of L_a except the cell $(0,0)$, for which the triangle $C = (x_0, y_0, z_0)$ has already been fixed. The remaining cells of L_a are the cells of the $a'(a'-1)$ 2×2 subsquares of the form (α) (which are not on the "diagonal"). Each of these 2×2 subsquares of the form (α) can be decomposed into two C_6, or one C_6 and two C_3, or four C_3; see (2) in Case (i) above. Thus the edges of $K_{a,b,b}$ corresponding to the cells of L_a are partitioned into r', $1 \le r' \le a^2$, 3-cycles and s_1 (which may be zero) 6-cycles; the value of $r' = 0$ is excluded here as the 3-cycle $C = (x_0, y_0, z_0)$ is available in the decomposition obtained above.

Next we partition the edges of $K_{a,b,b}$ corresponding to the cells of L_b into r'' 3-cycles and s_2 (possibly zero) 6-cycles.

From the construction of L, L_b (see Figure 3) contains $a'(b'-a')$ 2×2 subsquares of the form (α). We partition L_b into L_b^1 and L_b^2, where L_b^1 contains the first three rows of L_b and L_b^2 contains the rest of the rows of L_b. Here L_b^1 is partitioned into $b'-a'$ 3×2 subsquares of the form shown below:

	$c_{2a'+2j-1}$	$c_{2a'+2j}$
r_0	$2a'+2j$	$2a'+2j-1$
r_1	α	$\alpha+1$
r_2	$\alpha+1$	α

,

where $1 \le j \le b'-a'$.

Each of these 3×2 subsquares of the above form corresponds to 18 edges of $K_{a,b,b}$, and possible partitions of these edges into C_3 and C_6 are listed below:

(1) Three 6-cycles: $(x_0, z_{2a'+2j-1}, y_{2a'+2j}, x_1, y_{2a'+2j-1}, z_{2a'+2j})$, $(x_1, z_\alpha, y_{2a'+2j-1}, x_2, y_{2a'+2j}, z_{\alpha+1})$, $(x_2, z_\alpha, y_{2a'+2j}, x_0, y_{2a'+2j-1}, z_{\alpha+1})$.

(2) Two 6-cycles and two 3-cycles: $C' = (x_0, y_{2a'+2j-1}, z_{2a'+2j})$, $C'' = (x_0, y_{2a'+2j}, z_{2a'+2j-1})$, $(x_1, y_{2a'+2j}, x_2, z_{\alpha+1}, y_{2a'+2j-1}, z_\alpha)$, $(x_1, y_{2a'+2j-1}, x_2, z_\alpha, y_{2a'+2j}, z_{\alpha+1})$.

(3) One 6-cycle and four 3-cycles: $(x_1, y_{2a'+2j}, z_{\alpha+1}, x_2, y_{2a'+2j-1}, z_\alpha)$, C', C'', $(x_1, y_{2a'+2j-1}, z_{\alpha+1})$ and $(x_2, y_{2a'+2j}, z_\alpha)$.

(4) Six 3-cycles: the six 3-cycles correspond to the six entries in the six cells.

This proves that L_b^1 can be decomposed into a suitable number of C_3 and C_6. Next we consider L_b^2.

The cells in L_b^2 can be partitioned into $(a'-1)(b'-a')$ 2×2 subsquares of the form (α) and the 12 edges corresponding to each of these subsquares can be decomposed into four C_3, or two C_3 and one C_6, or two C_6; see (2) in Case (i) above. Corresponding to L_b^2 we have obtained r'', $0 \le r'' \le a(b-a)$, C_3 and s_2 (which may be zero) C_6. So far we have obtained $r = r' + r''$, $1 \le r \le ab$, 3-cycles and $s' = s_1 + s_2$ (possibly zero) 6-cycles of $K_{a,b,b}$ corresponding to the cells of L_a and L_b.

Next we shall partition the edges of $K_{a,b,b}$ corresponding to the cells in L_c.

Recall that each of the cells in L_c represents exactly one edge of $K_{b,b} \subset K_{a,b,b}$, as the above rC_3 and $s'C_6$ obtained through L_a, L_b^1 and L_b^2 contain all the edges incident with the partite set of size a. For example, the entry k of the cell (i,j) in L_c represents the edge $y_j z_k$ in $K_{a,b,b}$. Let H be the bipartite subgraph of $K_{b,b} \subseteq K_{a,b,b}$ corresponding to the cells of L_c. Clearly, L_c contains $b'-a'$ 2×2 subsquares of the form:

	$c_{2a'+2i-1}$	$c_{2a'+2i}$
$r_{2a'+2i-1}$	$2a'+2i-1$	0
$r_{2a'+2i}$	0	$2a'+2i$

, $1 \le i \le b'-a'$.

These $b'-a'$ 2×2 subsquares together with the cells in the 0^{th} column of L_c can be partitioned into 2×3 subsquares of the form L_{c_i}, $1 \le i \le b'-a'$, where

$L_{c_i} =$

	c_0	$c_{2a'+2i-1}$	$c_{2a'+2i}$
$r_{2a'+2i-1}$	$2a'+2i$	$2a'+2i-1$	0
$r_{2a'+2i}$	$2a'+2i-1$	0	$2a'+2i$

;

see the structure in Figure 2. Six edges of H corresponding to the six cells of L_{c_i} induce the 6-cycle $(y_0, z_{2a'+2i-1}, y_{2a'+2i-1}, z_0, y_{2a'+2i}, z_{2a'+2i})$. Let H_0 be the subgraph of H corresponding to the entries of the cells of L_c', where L_c' is obtained from L_c by deleting the cells L_{c_i}, $1 \le i \le b'-a'$; see Figure 4. Now partition the cells of L_c' into $(b-a)/6$ partial latin squares L_{c_i}', $1 \le i \le \frac{b-a}{6}$, where L_{c_i}' consists of six consecutive rows, beginning from the first row, of L_c'. We shall now show that the subgraph of H_0 corresponding to the cells of each L_{c_i}' can be decomposed into cycles of length six.

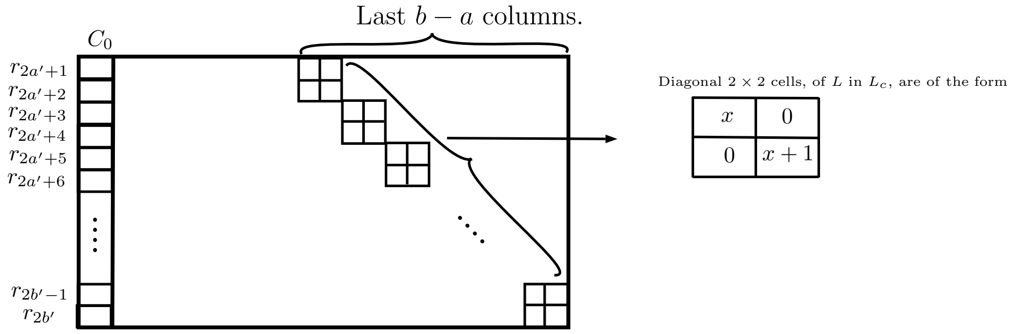

Figure 4: L'_c consists of all the cells of L_c which are not shown explicitly. Part of the 2×2 "diagonal" cells of L and the cells of the 0^{th} column of L_c are shown explicitly.

Subcase 2.1. b' **is odd.**

A 6-cycle decomposition of the subgraph of H_0 corresponding to L'_{c_i}, $1 \le i \le \frac{b-a}{6}$, is determined here. The six rows of L'_{c_i} arise out of three rows of $I_{b'}$, except the three cells of $I_{b'}$; see Figure 5 and Observation 2 of Remark 3.7.

	c_1	c_2	c_3	c_4	$\dots$	c_{t-2}	c_{t-1}	c_t	c_{t+1}	c_{t+2}	c_{t+3}	c_{t+4}	$\dots$	$c_{b'-1}$	$c_{b'}$
r_t	m	$m+n$	$m+2n$	$m+3n$	$\dots$	$m+(t-3)n$	$m+(t-2)n$	$*$	$m+(t)n$	$m+(t+1)n$	$m+(t+2)n$	$m+(t+3)n$	$\dots$	$m-2n$	$m-n$
r_{t+1}	$m+n$	$m+2n$	$m+3n$	$m+4n$	$\dots$	$m+(t-2)n$	$m+(t-1)n$	$m+(t)n$	$*$	$m+(t+2)n$	$m+(t+3)n$	$m+(t+4)n$	$\dots$	$m-n$	m
r_{t+2}	$m+2n$	$m+3n$	$m+4n$	$m+5n$	$\dots$	$m+(t-1)n$	$m+(t)n$	$m+(t+1)n$	$m+(t+2)n$	$*$	$m+(t+4)n$	$m+(t+5)n$	$\dots$	m	$m+n$

Figure 5: The three rows of the partial latin square of $I_{b'}$ corresponding to the six rows of $L'_{c_i}, 1 \le i \le \frac{b-a}{6}$, is given above, wherein the three entries of the cells with $*$ are already used by L_{c_i}. Here t stands for $a' + 3i - 2$, $n = \lceil \frac{b'}{2} \rceil$ and $m = 1 + n(t-1)$.

The cells of $I_{b'}$ in these three rows of it are partitioned into three cells each, according to $t \equiv 1$ or $0 \pmod 2$, where $t = a' + 3i - 2$; see Figure 6(a) or Figure 6(b), respectively. Note that in Figure 6(b) the first two cells in the last column and the first cell of the row t of $I_{b'}$ give rise to twelve entries in L'_{c_i}; similarly, the three cells (r_{t+1}, c_1), (r_{t+2}, c_1) and $(r_{t+2}, c_{b'})$ of $I_{b'}$ yield twelve cells in L'_{c_i}. Each of the three cells of $I_{b'}$ (shown by bold lines in Figure 6) give rise to twelve cells in L'_{c_i}. Each of the subgraphs, having twelve edges, corresponding to these twelve cells, is isomorphic to the graph G (since in the three cells of $I_{b'}$, shown by the bold lines covering three cells, two of the cells have the same symbol); see Figure 7(c), which can be decomposed into two cycles each of length six.

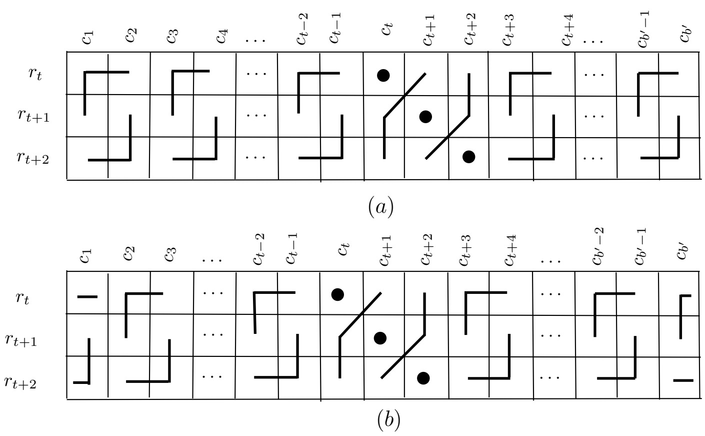

Figure 6: In (a) and (b), the edges of $K_{a,b,b}$ corresponding to the cells with bullets have been used by L_{c_i}.

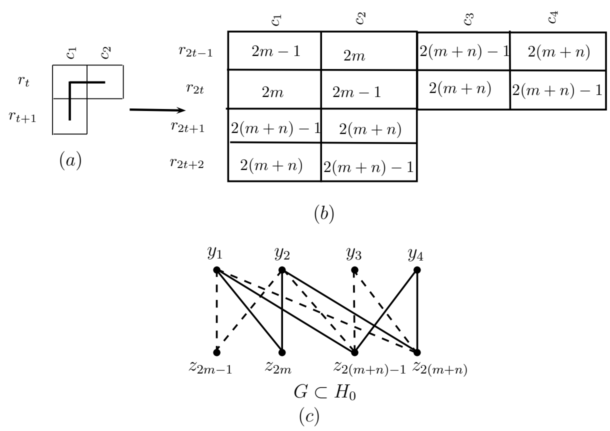

Figure 7: Twelve cells of L'_{c_i} corresponding to the three cells of $I_{b'}$, covered by bold lines of (a), are shown in (b). The subgraph of H_0 corresponding to the twelve cells in (b) is shown in (c) with a C_6-decomposition.

Subcase 2.2. b' **is even.**

First we complete the proof of the case $(a, b) \neq (3, 9)$.

Let $b' = 2b''$ for some $b'' \geq 3$. Here we obtain a C_6-decomposition of the subgraph of H_0 corresponding to the cells of L'_{c_i}, $1 \leq i < (b-a)/6$, and $L'_{c_{(b-a)/6}}$ (note that, by our construction, $L'_{c_{(b-a)/6}}$ is different from L'_{c_i} and so we deal with it separately). The six rows of L'_{c_i} (respectively, $L'_{c_{(b-a)/6}}$) correspond to the three rows t, $t+1$ and $t+2$ (respectively, the last three rows) of $I_{b'}$, except its three cells; see Figure 8 (respectively, Figure 10),

	c_1	c_2	c_3	c_4	$\dots$	c_{t-2}	c_{t-1}	c_t	c_{t+1}	c_{t+2}	c_{t+3}	c_{t+4}	$\dots$	$c_{b'-2}$	$c_{b'-1}$	$c_{b'}$
r_t	m	$m+n$	$m+2n$	$m+3n$	$\dots$	$m+(t-3)n$	$m+(t-2)n$	$*$	b'	$m+(t+1)n$	$m+(t+2)n$	$m+(t+3)n$	$\dots$	$m-2n$	$m-n$	$m+(t)n$
r_{t+1}	$m+n$	$m+2n$	$m+3n$	$m+4n$	$\dots$	$m+(t-2)n$	$m+(t-1)n$	$m+(t)n$	$*$	b'	$m+(t+3)n$	$m+(t+4)n$	$\dots$	$m-n$	m	$m+(t+2)n$
r_{t+2}	$m+2n$	$m+3n$	$m+4n$	$m+5n$	$\dots$	$m+(t-1)n$	$m+(t)n$	$m+(t+1)n$	$m+(t+2)n$	$*$	b'	$m+(t+5)n$	$\dots$	m	$m+n$	$m+(t+4)n$

Figure 8: The entries of the three rows t, $t+1$ and $t+2$ of $I_{b'}$, except the three cells with $*$ symbol, where $n = b''$ and $m = 1 + n(t-1)$ and the entries are taken modulo $b'-1$ except the entries in the cells (r_t, c_{t+1}), (r_{t+1}, c_{t+2}) and (r_{t+2}, c_{t+3}).

see Observation 3 (respectively, Observation 4) of Remark 3.7, where $t = a' + 3i - 2$. Now we partition the cells of Figure 8 (respectively, Figure 10) into three cells each, according to Figure 9 (respectively, Figure 11), where three of the cells with entry α_j, $1 \le j \le 5$, form a member of the partition. Each of these three cells of Figure 8 (respectively, Figure 10) give rise to twelve cells in L'_{c_i} (respectively, $L'_{c_{(b-a)/6}}$) and the subgraph of H_0 corresponding to these twelve cells is isomorphic to the graph G shown in Figure 7(c), which can be decomposed into two cycles of length six.

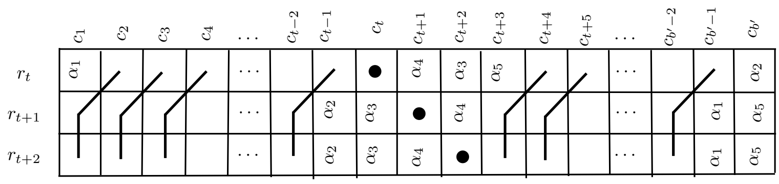

Figure 9

Now we complete the proof for the case when $a = 3$ and $b = 9$.

By the construction of L, the partial latin square L'_c of L_c is given in Figure 12.

A C_6-decomposition of H_0 corresponding to the entries of the cells of L'_c is given below:

$(y_1, z_5, y_3, z_2, y_4, z_6)$, $(y_2, z_5, y_4, z_1, y_3, z_6)$, $(y_1, z_7, y_5, z_2, y_6, z_8)$,
$(y_2, z_7, y_6, z_1, y_5, z_8)$, $(y_1, z_3, y_7, z_1, y_8, z_4)$ and $(y_2, z_3, y_8, z_2, y_7, z_4)$.

This completes the proof. □

	c_1	c_2	c_3	c_4	…	$c_{b''-1}$	$c_{b''}$	$c_{b''+1}$	$c_{b''+2}$	…	$c_{b'-3}$	$c_{b'-2}$	$c_{b'-1}$	$c_{b'}$
$r_{b'-2}$	$b'-1$	b''	$2b''$	$3b''$	…	$(b''-2)b''$	$(b''-1)b''$	$(b'')^2$	$(b''+1)b''$	…	$(b'-5)b''$	$*$	b'	$b''-1$
$r_{b'-1}$	b'	$2b''$	$3b''$	$4b''$	…	$(b''-1)b''$	$(b'')^2$	$(b''+1)b''$	$(b''+2)b''$	…	$b'-2$	$b''-1$	$*$	b''
$r_{b'}$	b''	$b''+1$	$b''+2$	$b''+3$	…	$b'-2$	$b'-1$	1	2	…	$b''-3$	$b''-2$	$b''-1$	$*$

Figure 10: The entries of the last three rows of $I_{b'}$, except the three cells with $*$ symbol, are given above, where the entries are taken modulo $b'-1$ except the entries in the cells $(r_{b'-2}, c_{b'-1})$ and $(r_{b'-1}, c_1)$.

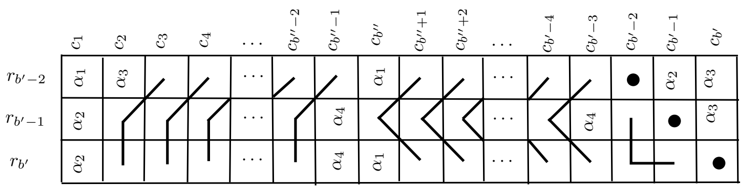

Figure 11

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8
r_3	5	6			7	8	1	2
r_4	6	5			8	7	2	1
r_5	7	8	1	2			3	4
r_6	8	7	2	1			4	3
r_7	3	4	5	6	1	2		
r_8	4	3	6	5	2	1		

Figure 12: The entries of L'_c of L of order 9 are shown above.

Now we are ready to prove our main theorem.

Proof of Theorem 1.2

Clearly, $K_{a,b,c} = K_{a,b,b} \oplus K_{a+b,c-b}$. By hypothesis, $a, b, c \equiv t \pmod 6$, where $t \in \{0, 1, 2, 3, 4, 5\}$; hence $a + b$ is even and $c - b \equiv 0 \pmod 6$. The graph $K_{a+b,c-b}$ admits a C_6-decomposition, by Theorem 3.3. Since the maximum number of triangles in $K_{a,b,c}$ and $K_{a,b,b}$ are the same and $K_{a+b,c-b}$ has a C_6-decomposition, it is enough to consider a $\{C_3^{r_1}, C_6^{s_1}\}$-decomposition of $K_{a,b,b}$. By Theorem 3.8 such a decomposition exists. □

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