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On the graceful polynomials of a graph

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Abstract

Every graph can be associated with a family of homogeneous polynomials, one for every degree, having as many variables as the number of vertices. These polynomials are related to graceful labellings: a graceful polynomial with all even coefficients is a basic tool, in some cases, for proving that a graph is non-graceful, and for generating a possibly infinite class of non-graceful graphs. Graceful polynomials also seem interesting in their own right. In this paper we classify graphs whose graceful polynomial has all even coefficients, for small degrees up to 4. We also obtain some new examples of non-graceful graphs.

1 Introduction

In the present paper we define and study a class $\{\mathcal{S}_G^n\}_{n\in\mathbf{N}\setminus\{0\}}$ of polynomials which can be associated with any given graph G using elementary symmetric functions. The polynomial $\mathcal{S}_G^n$ has as many variables as G has vertices, and it is homogeneous of degree n. We focus on graphs for which the above polynomials vanish (mod 2), for every fixed degree from 1 to 4. On one hand, the vanishing is related to graceful labellings, and it traces back to Rosa's technique yielding non-graceful graphs; on the other hand, the family of forbidden subgraphs arising from this condition, for any fixed degree, and the classification of graphs that satisfy the requirement are expected to raise interesting questions in their own right.

Throughout the paper, congruences are understood (mod 2) unless otherwise specified. Let G be a graph with vertex set $V=\{v_1,\ldots,v_{|V|}\}$ and edge set $E=\{e_1,\ldots,e_{|E|}\}$. The degree of a given vertex v_p will be denoted by δ_p, and any edge $\{v_p,v_q\}$ will be shortened to v_pv_q. We define graceful polynomials as follows.

Definition 1.1. Let us introduce a variable x_i for each vertex v_i and associate any given edge $e_j=v_pv_q$ to the polynomial $\mathcal{P}_j=x_p+x_q$. For any fixed integer

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$n \in \{1, \dots, |E|\}$ we define the *n-graceful polynomial of* G as

$$\mathcal{S}_G^n(x_1, x_2, \dots, x_{|V|}) \equiv \sum_{1 \le j_1 < j_2 < \dots < j_n \le |E|} \mathcal{P}_{j_1} \mathcal{P}_{j_2} \cdots \mathcal{P}_{j_n} .$$

The simplest polynomial, $\mathcal{S}_G^1$, is easily seen to be congruent to $\sum_{v_p \in V} \delta_p x_p$. In general, $\mathcal{S}_G^n$ is a homogeneous polynomial of degree n whose summands can be collected in as many groups as the partitions of n. For example, using suitable symbols $\mathcal{H}_{\cdots}^{\cdots}$ for coefficients, we can write $\mathcal{S}_G^4$ as

$$\sum_p \mathcal{H}_p^4 x_p^4 + \sum_{p,q} \mathcal{H}_{p,q}^{31} x_p^3 x_q + \sum_{pq} \mathcal{H}_{pq}^{22} x_p^2 x_q^2 + \sum_{p,qr} \mathcal{H}_{p,qr}^{211} x_p^2 x_q x_r + \sum_{pqrs} \mathcal{H}_{pqrs}^{1111} x_p x_q x_r x_s ,$$

where commas separate *sets* of indices; therefore, the pairs of indices a, b and b, a are both present in the second summation, whereas only one between ab and ba occurs in the third summation.

Every coefficient $\mathcal{H}_{\cdots}^{\pi}$ is related to a partition π in, say, t parts, and to the specific choice of t vertices. In particular, it is clear that the coefficient varies as the subgraph on the t selected vertices varies. We will write $\mathcal{H}_{\cdots}^{\pi}(S)$ if the subgraph (with distinguished vertices) under examination is S. For example, writing $\mathcal{H}_{p,qr}^{211}(S)$ makes sense if S is any subgraph with 3 vertices; thus, one of the possible choices for S is the graph consisting of an edge $v_p v_q$ together with an isolated vertex v_r. Isomorphic subgraphs may well give rise to different coefficients, due to the permutation of indices (in the present example, which will be part of the proof of Proposition 3.2, $\mathcal{H}_{p,qr}^{211} \not\equiv \mathcal{H}_{q,pr}^{211}$).

As mentioned above, we are interested in the following problem.

Main question: *For any fixed* n*, which conditions on* G *can guarantee that* $\mathcal{S}_G^n$ *vanishes* $(mod\ 2)$?

Obviously, G must have at least n edges. The above question arises from a generalisation of Rosa's counting technique for constructing *non-graceful* graphs (see [4] and [6]). Let us then provide some basic notions on graceful graphs.

Definition 1.2. A graph $G = (V, E)$ is *graceful* if there exists an injective map $f \colon V \to \{0, 1, 2, \dots, |E|\}$ such that the set $\{|f(u) - f(v)| \colon uv \in E\}$ is equal to $\{1, 2, \dots, |E|\}$. The labelling f is termed *graceful* as well. If a graph admits no such labelling, it is *non-graceful.*

Graceful labellings have been intensively studied in the last decades. Currently, there are many questions which attract the interest of researchers. Several classes of graphs have been shown to be graceful, and the efforts of combinatorialists often give rise to nice constructions for specific classes—see the thorough survey [2]. On the other hand, it is also challenging to find necessary conditions for a graph to be graceful, and to discover new classes of non-graceful graphs. One of the main conjectures in these areas is Ringel's (see [3]), asserting that every tree is graceful (see also the author's point of view in [5]). Necessary conditions are definitely rare in the literature; the most important class of graphs which was ruled out by an algebraic argument leading to a necessary condition (Rosa's theorem, see [4]) is that of Eulerian graphs whose number of edges is congruent to 1 or 2 (mod 4). We will show that our definition of graceful polynomial is strongly related to Rosa's approach, as it is a generalisation from the degree 1 to any degree. We remark that the concept of graceful polynomial was already present—without an explicit definition—in [6], where some graceful polynomials were examined for a particular class of trees in order to find necessary conditions on the labels of any graceful labelling. However, the systematic study of these polynomials can be assumed to start with the present paper.

For our purposes we will avail of the following result whose elementary proof is omitted.

Lemma 1.3. *The binomial coefficient* $\binom{k}{2}$ *(resp.* $\binom{k}{3}$, $\binom{k}{4}$ *) is even if and only if* $k \equiv 0, 1 \pmod 4$ *(resp.* $k \not\equiv 3 \pmod 4$, $k \equiv 0, 1, 2, 3 \pmod 8$ *).*

In the present work, many basic definitions on graphs are in accordance with the textbook by Bryant [1].

2 Graceful polynomials of small degree

The first graceful polynomial is the basic tool Rosa employed to establish his theorem on the non-gracefulness of a class of Eulerian graphs. Let us examine it in the present context.

Case $n = 1$. Using the above notation, we consider the polynomial $\sum \mathcal{H}_p^1 x_p$. As already remarked, every coefficient $\mathcal{H}_p^1$ is congruent to δ_p. Notice that these coefficients involve subgraphs having only one vertex (we could write $\mathcal{H}_p^1(\hat{G}) \equiv \delta_p$, where $\hat{G}$ consists of a unique vertex v_p).

In order to guarantee that $\mathcal{S}_G^1 \equiv 0$, the degree of every vertex must be even; that is,

$$\delta_p \equiv 0 \; for\ all\ vertices\ v_p\,.$$

Equivalently, G must be Eulerian. This condition was exploited by Rosa to show that Eulerian graphs with $4c + 1$ or $4c + 2$ edges, for any positive integer c, are non-graceful. For, assuming the contrary, we have a contradiction by evaluating the (odd) parity of $\sum_{1 \le j \le |E|} \mathcal{P}_j \equiv 1 + 2 + \cdots + (4c + h)$ with $h \in \{1, 2\}$ (by virtue of the graceful labelling, every $\mathcal{P}_j$ is congruent to a distinct integer between 1 and $|E|$).

Let us begin the study of graceful polynomials of larger degree, with the above main question in mind.

Case $n = 2$. We have the following result.

Theorem 2.1. *The polynomial* $\mathcal{S}_G^2$ *vanishes* $\pmod 2$ *if and only if* G *is a complete graph on* $4d + 2$ *vertices, for any positive integer* d.

Proof. Reasoning (mod 2), we can write $\mathcal{S}_G^2$ as

$$\sum \mathcal{H}_p^2 x_p^2 + \sum \mathcal{H}_{pq}^{11} x_p x_q \ .$$

The first coefficient depends on single vertices, as in the previous case. However, there are now $\binom{\delta_p}{2}$ ways of selecting two edges containing a vertex v_p (each of these choices contributes to a monomial x_p^2). Therefore, $\mathcal{H}_p^2 \equiv \binom{\delta_p}{2}$.

The second coefficient can be associated with two subgraphs, namely, either the disconnected graph on two vertices v_p, v_q, or the single edge connecting v_p and v_q. Let us denote these graphs as G_1^2 and G_2^2 respectively. We have that $\mathcal{H}_{pq}^{11}(G_1^2) \equiv \delta_p\delta_q$ and $\mathcal{H}_{pq}^{11}(G_2^2) \equiv \delta_p\delta_q - 1$, where the subtraction takes into account the multiple choice of the edge v_pv_q (repetitions of edges are not allowed).

Using Lemma 1.3 we obtain the following conditions (mod 4) on the structure of G:

$$\delta_p \equiv 0, 1\, \forall p \ , \ (\delta_p, \delta_q) \not\equiv (1, 1)\, \forall G_1^2 \ , \ \delta_p \equiv \delta_q \equiv 1\, \forall G_2^2 \, .$$

It is easy to deduce that these conditions are satisfied only by complete graphs as specified in the claim. □

The above theorem can be employed together with Rosa's counting technique, so as to obtain the non-gracefulness of an infinite family of complete graphs. To this end we interpret every polynomial $\mathcal{S}_G^n$ as a function of the $|V|$ variables $\{x_i\}$, as already done in the basic case, with $n = 1$. After counting the edges one can reach a contradiction through the following lemma (see [6]).

Lemma 2.2. *Let G be a graceful graph and let f_i be the label of vertex v_i. Then, for every positive integer $n \le |E(G)|$,*

$$\mathcal{S}_G^n(f_1, \ldots, f_{|V|}) \equiv \binom{\left[\frac{|E(G)|+1}{2}\right]}{n} \ .$$

(In the proof we count all possible products of odd, distinct integers in $[1, |E(G)|\,]$; the symbol $\binom{a}{b}$ is, by definition, 0 if $a < b$.)

If $n = 1$ the above lemma becomes the basic tool for reaching the contradiction in Rosa's theorem. If $n = 2$ we obtain the mentioned non-gracefulness of a class of complete graphs:

Theorem 2.3. *All complete graphs on either $16u + 10$ or $16u + 14$ vertices, for any positive integer u, are non-graceful.*

Proof. The number of edges of any graph in the claim of Theorem 2.1 is $14d + 1$. For any possible graceful labelling $f_1, \ldots, f_{|V|}$ of a given graph G of this family we have that $\mathcal{S}_G^2(f_1, \ldots, f_{|V|}) \equiv 1 \pmod 2$ for $d \equiv 2, 3 \pmod 4$, by virtue of Lemma 2.2 and Lemma 1.3. However, Theorem 2.1 guarantees that the parity of $\mathcal{S}_G^2$ is even, whatever the labelling, thus yielding a contradiction for all the above values of d. □

Although complete graphs with more than 4 vertices are well known to be non-graceful, we have just obtained a different proof of non-gracefulness for an infinite family of such graphs.

Case $n = 3$. We are going to prove the following result.

Theorem 2.4. *Let G be a graph having some vertices of odd degree. The polynomial $\mathcal{S}_G^3$ vanishes (mod 2) on G precisely in one of the following two cases.*

(A) G is a complete graph K_{4t+2}, for some positive integer t, possibly having $4u$ additional vertices and $4u(4t+2)$ additional edges that connect these vertices to the above complete graph.

(B) G is obtained by taking two complete graphs K_a, K_b and n additional vertices satisfying one of the following conditions:

(B_1) $a \equiv b \equiv 2 \pmod 4$ *and* $n = 0$;

(B_2) $a \equiv b \equiv n \equiv 1 \pmod 4$ *with the exclusion of* $a = b = n = 1$;

(B_3) $a \equiv b \equiv n \equiv 3 \pmod 4$; *every additional vertex must then be connected to all the vertices of K_a and K_b.*

Finally, $\mathcal{S}_G^3$ vanishes (mod 2) whenever G has all vertices of even degree.

The following observation will be useful in the proof of the above theorem.

Remark 2.5. If a term of the summation in Definition 1.1 contains some factors $(x_{p_1} - x_{p_2}), (x_{p_2} - x_{p_3}), \ldots, (x_{p_t} - x_{p_1})$ whose corresponding edges form, therefore, a t-cycle, then there is a double contribution of this summand to every monomial of the resulting polynomial $\mathcal{S}_G^n$ which is divisible by $x_{p_1}x_{p_2}\cdots x_{p_t}$ and is generated by that summand.

Let us now proceed with the proof of the theorem.

Proof of Theorem 2.4. The polynomial $\mathcal{S}_G^3$ has the following form:

$$\sum \mathcal{H}_p^3 x_p^3 + \sum \mathcal{H}_{p,q}^{21} x_p^2 x_q + \sum \mathcal{H}_{pqr}^{111} x_p x_q x_r \,.$$

There are four subgraphs on 3 vertices v_p, v_q, v_r. We denote by G_1^3, G_2^3, G_3^3, G_4^3 respectively the graph whose edge set is $\emptyset$, $\{v_q v_r\}$, $\{v_p v_q, v_p v_r\}$, and $\{v_p v_q, v_p v_r, v_q v_r\}$. Now we have:

$\mathcal{H}_p^3 \equiv \binom{\delta_p}{3}$.

$\mathcal{H}_{p,q}^{21}(G_1^2) \equiv \binom{\delta_p}{2}\delta_q$.

$\mathcal{H}_{p,q}^{21}(G_2^2) \equiv \binom{\delta_p}{2}\delta_q - (\delta_p - 1)$ because, as for $\mathcal{S}_G^2$, the edge $v_p v_q$ cannot be repeated, but in the present case there would be $\delta_p - 1$ ways of choosing the third edge.

$\mathcal{H}_{pqr}^{111}(G_1^3) \equiv \delta_p \delta_q \delta_r$.

$\mathcal{H}_{pqr}^{111}(G_2^3) \equiv \delta_p(\delta_q \delta_r - 1)$.

$\mathcal{H}^{111}_{pqr}(G^3_3) \equiv \delta_p\delta_q\delta_r - \delta_q - \delta_r$, where the two subtractions correspond to the forbidden repetitions of either v_pv_r or v_pv_q; we will often write the number of such repetitions as $[pr]$, $[pq]$, or in some cases $[pq, rs, \ldots]$, if we consider multiple occurrences of more than one edge; therefore, in the present case we could have written $\mathcal{H}^{111}_{pqr}(G^3_3) \equiv \delta_p\delta_q\delta_r - [pr] - [pq]$ etc.

$\mathcal{H}^{111}_{pqr}(G^3_4) \equiv \delta_p\delta_q\delta_r - [pq] - [pr] - [qr] \equiv \delta_p\delta_q\delta_r - \delta_r - \delta_q - \delta_p$.

It is important to note that the simultaneous choice of all the three edges of G^3_4 is obtainable in two distinct ways—according to the orientation of the 3-cycle—but also the contribution in $\mathcal{S}^3_G$ is double, according to Remark 2.5.

For every vertex v_p, the condition $\binom{\delta_p}{3} \equiv 0$ is equivalent to $\delta_p \not\equiv 3 \pmod 4$, by Lemma 1.3. As to G^2_1, we have the equation $\binom{\delta_p}{2}\delta_q + \binom{\delta_q}{2}\delta_p \equiv 0$, because we consider $x_p^2x_q$ together with $x_px_q^2$. Notice that these monomials are both equivalent to x_px_q; more generally, it is clear that in every computation (mod 2) all positive exponents reduce to the exponent 1. Also notice that for the first time there are two distinct coefficients for the same subgraph, depending on the permutation of indices.

Reasoning in a similar way, on G^2_2 we have $\binom{\delta_p}{2}\delta_q - (\delta_p - 1) + \binom{\delta_q}{2}\delta_p - (\delta_q - 1) \equiv 0$. In the remaining four cases the corresponding quantities must be congruent to 0, and there is only one monomial to consider in each case, namely, $x_px_qx_r$. We can summarise the equations as follows.

$$\begin{cases} \forall p & \delta_p \not\equiv 3 \,(\mathrm{mod}\, 4) & (I)_3 \\ \forall G^2_1 & \binom{\delta_p}{2}\delta_q + \binom{\delta_q}{2}\delta_p \equiv 0 & (II)_3 \\ \forall G^2_2 & \binom{\delta_p}{2}\delta_q + \binom{\delta_q}{2}\delta_p + \delta_p + \delta_q \equiv 0 & (III)_3 \\ \forall G^3_1 & \delta_p\delta_q\delta_r \equiv 0 & (IV)_3 \\ \forall G^3_2 & \delta_p(\delta_q\delta_r + 1) \equiv 0 & (V)_3 \\ \forall G^3_3 & \delta_p\delta_q\delta_r + \delta_q + \delta_r \equiv 0 & (VI)_3 \\ \forall G^3_4 & \delta_p\delta_q\delta_r + \delta_r + \delta_q + \delta_p \equiv 0 & (VII)_3 \end{cases}$$

One may easily solve the above system by assuming that every degree is even. Let us now assume that some vertices of G have odd degree, that is, $\delta_{p_i} \equiv 1$ (mod 4) with $i = 1, 2, \ldots, s$ (s is necessarily even). We will call them *odd* vertices or, in this particular case, 1-vertices (mod 4). By $(II)_3$, an odd vertex must be adjacent to every vertex of degree $4u + 2$ (shortly, a $(4u + 2)$-vertex). On the other hand, $(III)_3$ implies that an odd vertex cannot be adjacent to any $(4u)$-vertex. By $(IV)_3$, any three odd vertices must have at least one edge between them. Now Equation $(V)_3$ implies that any two $(4u)$-vertices cannot be adjacent (consider these vertices together with an odd vertex v_p in G^3_2).

Equation $(VI)_3$ has two consequences. First, no edge connects a $(4u)$-vertex v_q to a $(4u + 2)$-vertex v_p, for any choice (add an odd vertex v_r as the third vertex of G^3_3). Therefore, we can assume that no $(4u)$-vertex exists, because no edge would contain such a vertex. Second, three odd vertices cannot give rise to a subgraph G^3_3. Finally, Equation $(VII)_3$ rules out all 3-cycles having exactly one odd vertex; it follows that also the subset of all $(4u + 2)$-vertices has no edge.

Let us draw our attention to 1-vertices (mod 4). These vertices necessarily make up either one or two (disjoint) complete graphs. If there are no "even" vertices, each complete graph has 2 vertices (mod 4), so we have either type (A) with no additional vertices, or type (B_1). Otherwise, assuming that there are some 2-vertices (mod 4), we have that 1-vertices (mod 4) are as many as 2 (mod 4). If these vertices make up a unique, complete graph, then we have case (A) with the additional even vertices; if 1-vertices make up two complete graphs, the values of a and b satisfy either (B_2) or (B_3), and the values of n are subsequently determined. The exclusion in the claim prevents the number of edges from being too small. □

We can exploit Theorem 2.4 for generating some classes of non-graceful graphs—see [2] for the known examples.

Theorem 2.6. *The following graphs are non-graceful.*

(A) *Graphs of type* (A) *in the claim of Theorem* 2.4*, with the additional condition that* $t \equiv 2 \pmod 4$.

(B) *Graphs of type* (B_1)*, with the additional condition that* $a + b \equiv 12 \pmod{16}$; *graphs of type* (B_2) *such that* $a + b \equiv 10 \pmod{16}$; *graphs of type* (B_3) *such that* $a + b \equiv 14 \pmod{16}$.

Proof. Graphs of type (A) have $(2t + 1)(4t + 1) + 4u(4t + 2)$ edges, that is, $6t + 1$ edges (mod 8). Lemma 2.2 gives a contradiction if this number is congruent to 5 (mod 8). Therefore, we can easily obtain the first assertion. For graphs of type (B_1), let us write a and b as $4\alpha + 2$ and $4\beta + 2$ respectively. The number of edges is $(2\alpha + 1)(4\alpha + 1) + (2\beta + 1)(4\beta + 1) \equiv 6\alpha + 6\beta + 2 \pmod 8$. By Lemma 2.2 we have a contradiction if $6\alpha + 6\beta + 2 \equiv 6 \pmod 8$, that is, if $\alpha + \beta \equiv 2 \pmod 4$. We can now obtain the condition in the claim. For graphs of type (B_2), we write a, b and n as $4\alpha + 1$, $4\beta + 1$ and $4\nu + 1$ respectively. In this case—considering also the additional $n(a + b)$ edges—it can be shown that the number of edges is again congruent to $6\alpha + 6\beta + 2 \pmod 8$, and the conclusion is similar. Type (B_3) gives rise to similar calculations. □

An original and economic example of non-graceful graph of type (B) is given by two copies of K_7 with 3 additional vertices that are fully connected to the complete graphs ($a = b = 7$, $n = 3$). More generally, one can take $4\nu + 3$ additional vertices, thus obtaining an infinite class. Another infinite class is obtained by taking $a = b = 5$ and any $n \equiv 1 \pmod 4$; for $n = 1$ the example is already known. Notice that the non-graceful graphs arising in this way contain K_6 as the largest complete subgraph, and this is clearly the smallest size we can obtain, over all the possible cases to consider.

If G only has even vertices, it is not possible to obtain original examples from the above theorem, because Lemma 2.2 only yields "half" Rosa's set of non-graceful graphs, namely, Eulerian graphs with $|E|$ congruent to 5 or 6 (mod 8).

3 The 4-graceful polynomial

When dealing with graceful polynomials of degree 4 many distinct cases arise and the techniques become more general and—we believe—suggestive. Although the relevant theorem provides only two very small graphs for which the polynomial vanishes, we think that the structure of the proof is what counts most in the present section.

Theorem 3.1. *Let G be a graph having at least four edges. The polynomial $\mathcal{S}_G^4$ vanishes $(mod\ 2)$ on G if and only if G is a 3-cycle together with a further edge, which can be either pendant or disjoint from the cycle.*

This theorem will follow from a preparatory result which, for the sake of clarity, we have decided to separate from the rest of the proof.

Proposition 3.2. *Let G be a graph having at least four edges. A necessary condition for $\mathcal{S}_G^4$ to vanish $(mod\,2)$ on G is that G be one of the following graphs, for any $h \geq 1$:*

i. *A disjoint union of edges.*

ii. *A $(4h+3)$-star.*

iii. *A complete graph K_{4h+2} having $4h+2$ additional, pendant edges.*

iv. *A complete graph K_{4h-1}, a vertex of which is adjacent to $4i+1$ additional vertices, with $i \geq 0$.*

v. *A complete graph K_{4h-1} having an additional edge, isolated.*

vi. *A complete graph K_{4h+3}.*

Proof. The polynomial $\mathcal{S}_G^4$ has already been developed—see the Introduction. Looking at the first three summations of $\mathcal{S}_G^4$, we have:

$$\begin{aligned}
\mathcal{H}_p^4 &\equiv \binom{\delta_p}{4};\\
\mathcal{H}_{p,q}^{31}(G_1^2) &\equiv \binom{\delta_p}{3}\delta_q;\\
\mathcal{H}_{p,q}^{31}(G_2^2) &\equiv \binom{\delta_p}{3}\delta_q - [pq] \equiv \binom{\delta_p}{3}\delta_q - \binom{\delta_p - 1}{2};\\
\mathcal{H}_{pq}^{22}(G_1^2) &\equiv \binom{\delta_p}{2}\binom{\delta_q}{2};\\
\mathcal{H}_{pq}^{22}(G_2^2) &\equiv \binom{\delta_p}{2}\binom{\delta_q}{2} - [pq] = \binom{\delta_p}{2}\binom{\delta_q}{2} - (\delta_p - 1)(\delta_q - 1).
\end{aligned}$$

Let us stop here and draw the first conclusions before analysing the other coefficients. We have the following system.

$$\begin{cases} \forall p & \binom{\delta_p}{4} \equiv 0 & (I)_4 \\ \forall G_1^2 & \binom{\delta_p}{3}\delta_q + \binom{\delta_q}{3}\delta_p + \binom{\delta_p}{2}\binom{\delta_q}{2} \equiv 0 & (II)_4 \\ \forall G_2^2 & \binom{\delta_p}{3}\delta_q - \binom{\delta_p - 1}{2} + \binom{\delta_q}{3}\delta_p - \binom{\delta_q - 1}{2} & \\ & \quad + \binom{\delta_p}{2}\binom{\delta_q}{2} - (\delta_p - 1)(\delta_q - 1) \equiv 0 & (III)_4 \end{cases}$$

Equation $(I)_4$ is equivalent to $0 \le \delta_p \le 3 \pmod 8$—see Lemma 1.3. As to $(II)_4$, a direct calculation could show that the odd values are attained by pairs of non-adjacent vertices whose degrees are $(1,3)$, $(2,2)$, $(2,3)$, $(3,3) \pmod 4$. Therefore, these configurations cannot be present in the graph. In particular, notice the following:

⋄ *In every admissible graph, all* 2*-vertices are pairwise adjacent.*

Next, let us denote by $\varphi(\delta_p, \delta_q)$ the polynomial on the left side of $(II)_4$. The polynomial in $(III)_4$ can then be written as $\varphi + \binom{\delta_p - 1}{2} + \binom{\delta_q - 1}{2} + (\delta_p - 1)(\delta_q - 1)$. After a lengthy examination it is easy to deduce that the forbidden pairs of degrees for adjacent vertices are $(0,0)$, $(0,1)$, $(3,3) \pmod 4$. In particular, we emphasise the following:

⋄ *Every admissible graph has at most one vertex of degree* 3 $(\mathit{mod}\ 4)$.

Now we examine the subgraphs with three vertices. Due to some symmetries in these subgraphs, it is not necessary to analyse all the coefficients:

$$\begin{aligned}
\mathcal{H}^{211}_{p,qr}(G_1^3) &\equiv \binom{\delta_p}{2}\delta_q\delta_r. \\
\mathcal{H}^{211}_{p,qr}(G_2^3) &\equiv \binom{\delta_p}{2}(\delta_q\delta_r - 1). \\
\mathcal{H}^{211}_{q,pr}(G_2^3) &\equiv \delta_p\left(\binom{\delta_q}{2}\delta_r - (\delta_q - 1)\right). \\
\mathcal{H}^{211}_{p,qr}(G_3^3) &\equiv \binom{\delta_p}{2}\delta_q\delta_r - [pq] - [pr] - [pq,pr] \\
&\equiv \binom{\delta_p}{2}\delta_q\delta_r - ((\delta_p - 1)\delta_r - 1) - ((\delta_p - 1)\delta_q - 1) - 1. \\
\mathcal{H}^{211}_{q,pr}(G_3^3) &\equiv \binom{\delta_q}{2}\delta_p\delta_r - [pq] - [pr] \equiv \binom{\delta_p}{2}\delta_q\delta_r - (\delta_q - 1)\delta_r - \binom{\delta_q}{2}. \\
\mathcal{H}^{210}_{p,qr}(G_4^3) &\equiv \binom{\delta_p}{2}\delta_q\delta_r - [pq] - [pr] - [qr] - [pq,pr] \\
&\equiv \binom{\delta_p}{2}\delta_q\delta_r - ((\delta_p - 1)\delta_r - 1) - ((\delta_p - 1)\delta_q - 1) - \binom{\delta_p}{2} - 1.
\end{aligned}$$

We have the following system (for every subgraph we are summing up the three contributions, one for each vertex, using the three appropriate coefficients).

$$\begin{cases} \forall G_1^3 \ \binom{\delta_p}{2}\delta_q\delta_r + \binom{\delta_q}{2}\delta_p\delta_r + \binom{\delta_r}{2}\delta_p\delta_q \equiv 0 & (IV)_4 \\ \forall G_2^3 \ \binom{\delta_p}{2}\delta_q\delta_r + \binom{\delta_q}{2}\delta_p\delta_r + \binom{\delta_r}{2}\delta_p\delta_q + \delta_p\delta_q + \delta_p\delta_r + \binom{\delta_p}{2} \equiv 0 & (V)_4 \\ \forall G_3^3 \ \binom{\delta_p}{2}\delta_q\delta_r + \binom{\delta_q}{2}\delta_p\delta_r + \binom{\delta_r}{2}\delta_p\delta_q + \delta_p\delta_q + \delta_p\delta_r + \binom{\delta_q}{2} + \binom{\delta_r}{2} + 1 \equiv 0 & (VI)_4 \\ \forall G_4^3 \ \binom{\delta_p}{2}\delta_q\delta_r + \binom{\delta_q}{2}\delta_p\delta_r + \binom{\delta_r}{2}\delta_p\delta_q + \binom{\delta_p}{2} + \binom{\delta_q}{2} + \binom{\delta_r}{2} + 1 \equiv 0 & (VII)_4 \end{cases}$$

Equation $(IV)_4$ fails to be satisfied if and only if either one or three monomials are odd. The latter case can be excluded because it would yield $\delta_p \equiv \delta_q \equiv \delta_r \equiv 3 \pmod 4$. For the former case to hold, one vertex must have degree 2 and the others 1 (mod 4)—consider also the effect of $(II)_4$. In conclusion we have another remarkable constraint:

$\diamond$ *Three vertices with degrees* 1, 1, 2 (*mod* 4) *must have at least one edge between them.*

Let us denote by ψ the polynomial in $(IV)_4$. When analysing $(V)_4$ we notice that this equation can be written as $\psi + \delta_p\delta_q + \delta_p\delta_r + \binom{\delta_p}{2} \equiv 0$. We can evaluate separately $\delta_p\delta_q + \delta_p\delta_r$ as a function, say χ, of the three degrees—this will also be useful for the next case. A routine calculation shows that all the configurations we should rule out using $(V)_4$ have already been discarded, because each of them contains a forbidden configuration of two vertices (for example, if $\delta_p \equiv \delta_q \equiv 1$ and $\delta_r \equiv 0$ (mod 4), then in particular v_q and v_r cannot be adjacent, by $(III)_4$).

The polynomial in $(VI)_4$ can be written as $\psi + \chi + \binom{\delta_q}{2} + \binom{\delta_r}{2} + 1$. After studying the behaviour of $\binom{\delta_q}{2} + \binom{\delta_r}{2}$, one can easily obtain the forbidden values of $(\delta_p, \delta_q, \delta_r)$ (mod 4) for the subgraph G_3^3; they are $(1,1,1)$, $(2,0,0)$, $(2,1,0)$—together with $(2,0,1)$—and $(3,0,0)$. For the same reason as above, some other cases have been discarded.

For the last subgraph with three vertices—the 3-cycle—we have the equation $\psi + \binom{\delta_p}{2} + \binom{\delta_q}{2} + \binom{\delta_r}{2} + 1 \equiv 0$. In this case the routine calculations rule out six configurations up to symmetries; the forbidden triples are $(0,2,2)$, $(0,2,3)$, $(1,1,1)$, $(1,1,2)$, $(1,1,3)$, and $(1,2,2)$.

Let us now determine all graphs that satisfy the constraints so far obtained. First, by virtue of $(II)_4$, we can record the following:

$\diamond$ *If the* 3*-vertex is present, then it is adjacent to all* 1*-vertices and* 2*-vertices.*

Next, consider the relationship between 1-vertices and 2-vertices, assuming that both types of vertex are present. By $(VI)_4$ and $(VII)_4$, the only way to have some adjacent 1-vertices is that $2s$ 1-vertices give rise to s disjoint edges; furthermore, there are possibly t additional, mutually non-adjacent 1-vertices. We know that the triple $(1,1,2)$ is forbidden both for non-adjacent vertices and for a 3-cycle. The former constraint forces any two non-adjacent 1-vertices to be adjacent (one or both) to any given 2-vertex, say v. However, if $s \geq 2$, attempting to satisfy the latter constraint for any two fixed edges yields a contradiction (we leave the details to the reader). Therefore, 1-vertices make up at most one edge between them. If such an edge exists, then let us assume that there is at least an isolated vertex. It follows that precisely one 1-vertex, w, of the edge must be adjacent to the 2-vertex v. No other vertex of the whole graph G is adjacent to w, save possibly the unique 3-vertex, because 0-vertices are not available, by $(III)_4$. This is a contradiction, because the degree of w would not be congruent to 1 (mod 4). We have thus reached the following conclusion:

$\diamond$ *If there is an edge consisting of* 1*-vertices, then no further* 1*-vertex is present, nor the* 3*-vertex, and this edge is isolated from* 2*-vertices; if there are all mutually non-adjacent* 1*-vertices and no edge, then either all of them are adjacent to the* 3*-vertex, or there is no* 3*-vertex and they are adjacent to t* 2*-vertices, one for each* 1*-vertex, thus giving rise to t mutually disconnected edges.*

Now we analyse 0-vertices. By $(III)_4$, these vertices are mutually non-adjacent. Some of them can be adjacent to some 2-vertices and also to the 3-vertex, if it exists. None is adjacent to any 1-vertex, as already noted. Now we recall that 2-vertices make up a complete graph. By $(VII)_4$ we deduce that each 0-vertex can be adjacent to a unique 2-vertex, and this is not enough if we ask for a degree congruent to 0 (mod 4). Therefore, we can emphasise the following property:

$\diamond$ *No* 0*-vertex exists—we do not consider totally isolated vertices.*

We can summarise the above discussion and examine all the possibilities for the number, t, of non-adjacent 1-vertices and the number of 2-vertices, which we denote by u, thus taking t and u greater than zero. As a first sub-case, we assume that the 3-vertex is not present. Then, every 2-vertex must be adjacent to some 1-vertex, for otherwise its degree would not be congruent to 2 (mod 4)—we recall that 2-vertices form a complete graph and that t of them are known to be adjacent to 1-vertices. Therefore, the only possibility is given by a complete graph K_{4h+2} having $4h + 2$ additional, pendant edges, with $h \geq 1$. In the second sub-case, according to which the 3-vertex exists and 1-vertices are disjoint from 2-vertices, we have that $t + u \equiv 3$ and $u \equiv 2 \pmod 4$, whence $t \equiv 1 \pmod 4$. We are, therefore, in the presence of a complete graph K_{4h-1} with $h \geq 1$, a vertex of which is adjacent to $4i + 1$ additional vertices, with $i \geq 0$.

We deal with the remaining cases, leaving the elementary proofs to the reader. If G has no 2-vertices, then either it is a disjoint union of s edges, or it is a $(4h+3)$-star with $h \geq 1$. If, instead, G has no mutually non-adjacent 1-vertices, then it is a complete graph K_{4h+3} with $h \geq 1$, and there is possibly an additional edge, isolated. If this is the case, h can also be equal to 0. Finally, it is not possible that both 1-vertices and 2-vertices are missing in G. $\square$

By virtue of Proposition 3.2 we have discarded several graphs for which the 4-graceful polynomial does not vanish. Now the remaining candidates have to undergo the last test, by examining the coefficient $\mathcal{H}^{1111}_{pqrs}$. This will be the main task in the proof of Theorem 3.1.

Proof of Theorem 3.1. The complete graph K_{4h+2} endowed with $4h+2$ pendant edges contains a subgraph S having 4 mutually adjacent vertices, all of even degree. The coefficient $\mathcal{H}^{1111}_{pqrs}(S)$ is congruent to $\delta_p\delta_q\delta_r\delta_s - [pq] - [pr] - [ps] - [qr] - [qs] - [rs] - [pq, rs] - [pr, qs] - [ps, qr] \equiv \delta_p\delta_q\delta_r\delta_s - (\delta_r\delta_s - 1) - (\delta_q\delta_s - 1) - \ldots - 1 - 1 - 1 \equiv 1$. Therefore, we have to discard this class of graphs. For the same reason we can eliminate every K_{4h-1} with or without the $4i + 1$ pendant edges, except K_3 with pendant edges. For all $i \geq 1$ this last type of graph contains 4 isolated vertices of degree 1; the evaluation of $\mathcal{H}^{1111}_{pqrs}$ on the corresponding subgraph simply gives the product of degrees, which is odd. Remarkably, if $i = 0$ the graph is admissible. Let v_p, v_q, v_r, v_s be the vertices of degree 2, 2, 3, 1 respectively; the coefficient $\mathcal{H}^{1111}_{pqrs}$ depends on the entire graph, and it is congruent to $\delta_p\delta_q\delta_r\delta_s - [pq] - [pr] - [qr] - [rs] - [pq, rs] \equiv 0 - (3 \cdot 1 - 1) - 2 \cdot 1 - 2 \cdot 1 - (2^2 - 1) - 1 \equiv 0$.

Note also that the 3-cycle having a further, separate edge v_sv_t is an admissible graph. To see this, we have two subgraphs to test. The 3-cycle on v_p, v_q, v_r together with an isolated vertex v_s gives $\delta_s(\delta_p\delta_q\delta_r - \delta_p - \delta_q - \delta_r) \equiv 1(8 - 6) \equiv 0$; the disjoint edges v_pv_q and v_sv_t whose vertices have degree 2, 2, 1, 1 give $(2^2 - 1)(1^1 - 1) \equiv 0$.

If G is the disjoint union of edges, then no more than 3 edges can exist, because otherwise we could find 4 isolated 1-vertices. However, this number of edges is not admissible, as it is smaller than 4. Finally, $(4h + 3)$-stars are not admissible because each of them contains at least 4 isolated 1-vertices.

We have come to the end of the analysis, thus obtaining the claimed result. □

Unlike in the previous cases, Lemma 2.2 cannot be joined to the above theorem to prove that one or both the graphs in the claim are non-graceful, because the products of 4 distinct integers in $[1, 4]$ (that is, only one product) or in $[1, 5]$ (5 products) are both even, so we have no contradiction by assuming gracefulness. However, it is easy to provide a graceful labelling of the former graph, and to show that the latter cannot admit any such labelling.

We conclude this section with the following note. When analysing forbidden subgraphs, vertices have been labelled by integers (mod 4) corresponding to the vertex degree. In the case of graceful polynomials of larger degree, a generalisation of Lemma 1.3 would lead to congruences (mod a) with a larger than 4 for the vertex degrees; consequently, the subgraphs would be labelled by a richer set of integers. In such a context, regarding these as *colours* might simplify and streamline some proofs.

4 Conclusion

The vanishing problem (mod 2) is only one of the many questions that arise from considering graceful polynomials. Far from making a comparison with the well-known chromatic polynomial—which would sound peculiar at this stage—we nevertheless believe that an algebraic object is always welcome if associated with a graph. In this spirit, some other properties of graceful polynomials might turn out to be connected to the combinatorial structure and to classical invariants of the graph. As a first task in the near future, however, we plan to investigate the vanishing problem for graceful polynomials of larger degree, but calculations would become more complex and perhaps some new analyses would need to be developed.

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