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Coverings, matchings and the number of maximal independent sets of graphs

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Abstract

We determine the maximum number of maximal independent sets of an arbitrary graph in terms of its covering number, and we completely characterize the extremal graphs. As an application, we give a similar result for König–Egerváry graphs in terms of their matching numbers.

1 Introduction

Throughout this paper let G be a simple (i.e. finite, undirected, loopless and without multiple edges) graph. An independent set in G is a set of vertices no two of which are adjacent to each other. An independent set in G is maximal (with respect to set inclusion) if the set cannot be extended to a larger independent set. Let $\mathrm{m}(G)$ be the number of maximal independent sets of a simple graph G. Around 1960, Erdős and Moser raised the problem of determining the largest value of $\mathrm{m}(G)$ in terms of the order of G, which we shall denote by n in this paper, and determining the extremal graphs. In 1965, Moon and Moser [14] solved this problem.

Since then, research has been focused on investigating $\mathrm{m}(G)$ for various classes of graphs such as: connected graphs by Füredi [5] and independently Griggs et al. [8]; triangle-free graphs by Hujter and Tuza [10] and connected triangle-free graphs by Chang and Jou [3]; graphs with at most r cycles by Sagan and Vatter [16] and Goh et al. [6]; connected unicyclic graphs by Koh et al. [11]; trees independently by Cohen [4], Griggs and Grinstead [7], Sagan [15], Wilf [17]; bipartite graphs by Liu [13] and bipartite graphs with at least one cycle by Li et al. [12].

A subset of the vertices of a graph G is called a vertex cover if every edge in G is incident to at least one vertex of the set. The *covering number* of G, denoted

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by $\tau(G)$, is the minimum size of a vertex cover of G. The goal of this paper is to determine the maximum value of $\mathrm{m}(G)$ for an arbitrary simple graph G in terms of its covering number, and to characterize the extremal graphs. Our results improve certain results among those mentioned above. Before stating our results, recall that a matching in G is a set of edges, no two of which meet a common vertex. The *matching number* $\nu(G)$ of G is the maximum size of a matching of G. An induced matching M in a graph G is a matching where no two edges of M are joined by an edge of G. The *induced matching number* $\nu_0(G)$ of G is the maximum size of an induced matching of G. We always have $\nu_0(G) \le \nu(G)$; and if $\nu_0(G) = \nu(G)$ then we call G a *Cameron–Walker* graph. This definition is similar to the one in Hibi et al. [9] including both disconnected graphs and star graphs and star triangle graphs. The main result of the paper is as follows:

Theorem (Theorem 2.7 and Theorem 3.3) *Let G be a graph. Then* $\mathrm{m}(G) \le 2^{\tau(G)}$, *and the equality holds if and only if G is a Cameron–Walker bipartite graph.*

A graph G is called a *König–Egerváry graph* if the matching number is equal to the covering number, that is, $\tau(G) = \nu(G)$. As an application, we determine the maximum value of $\mathrm{m}(G)$ for König–Egerváry graphs G, and characterize the extremal graphs, in the following corollary.

Corollary (Corollary 3.4) *Let G be a König–Egerváry graph. Then*

$$\mathrm{m}(G) \le 2^{\nu(G)},$$

and the equality holds if and only if G is a Cameron–Walker bipartite graph.

It is well-known that all bipartite graphs are König–Egerváry (see [1, Theorem 8.32]). In general, $\nu(G) \leqslant \lfloor \frac{n}{2} \rfloor$, where n is the order of G. Thus Corollary 3.4 improves the main result of Liu (see [13, Theorem 2.1]) for bipartite graphs.

2 Bounds for $m(G)$

We now recall some basic concepts and terminology from graph theory (see [1]). Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. An edge $e \in E(G)$ connecting two vertices x and y will be also written as xy (or yx). For a subset S of $V(G)$, we denote by $G[S]$ the induced subgraph of G on the vertex set S, and use $G \setminus S$ to denote $G[V(G) \setminus S]$. The neighborhood of S in G is the set

$$N_G(S) := \{y \in V(G) \setminus S \mid xy \in E(G) \text{ for some } x \in S\},$$

and the closed neighborhood of S is $N_G[S] := S \cup N_G(S)$. Let $G_S := G \setminus N_G[S]$. If $S = \{x\}$, we write $N_G(x)$ (respectively, $N_G[x]$, G_x, $G \setminus x$) instead of $N_G(\{x\})$ (respectively, $N_G[\{x\}]$, $G_{\{x\}}$, $G \setminus \{x\}$). The number $\deg_G(x) := |N_G(x)|$ is called the *degree* of x in G. A vertex in G of degree zero is called an *isolated vertex* of G. A vertex x of G is called a *leaf adjacent to y* if $\deg_G(x) = 1$ and xy is an edge of G. A complete graph with n vertices is denoted by K_n. A graph K_3 is called a *triangle*. The union of two disjoint graphs G and H is the graph $G \cup H$ with vertex set $V(G \cup H) = V(G) \cup V(H)$ and edge set $E(G \cup H) = E(G) \cup E(H)$. The union of t copies of disjoint graphs isomorphic to G is denoted by tG, where t is a positive integer.

A graph is called *totally disconnected* if it is either a null graph or contains no edge. Thus, $\mathrm{m}(G) = 1$ whenever G is totally disconnected. The following basic lemmas on determining $\mathrm{m}(G)$ for an arbitrary graph G will be used frequently later.

Lemma 2.1. [10, Lemma 1] *Let G be a graph. Then*

1. $\mathrm{m}(G) \le \mathrm{m}(G_x) + \mathrm{m}(G \setminus x)$, *for any vertex x of G.*
2. *If x is a leaf adjacent to y of G, then* $\mathrm{m}(G) = \mathrm{m}(G_x) + \mathrm{m}(G_y)$.
3. *If $G_1, \dots, G_s$ are connected components of G, then*

$$\mathrm{m}(G) = \prod_{i=1}^{s} \mathrm{m}(G_i).$$

Lemma 2.2. *If H is an induced subgraph of G, then* $\mathrm{m}(H) \le \mathrm{m}(G)$.

We first give an upper bound for $\mathrm{m}(G)$ in terms of $\nu(G)$, and the extremal graphs.

Proposition 2.3. *Let G be a graph. Then,* $\mathrm{m}(G) \le 3^{\nu(G)}$ *and the equality holds if and only if $G \cong sK_3 \cup tK_1$, where $s = \nu(G)$ and $t = |V(G)| - 3s$.*

Proof. We prove the proposition by induction on $\nu(G)$. If $\nu(G) = 0$, then G is totally disconnected, and then the assertion is trivial.

If $\nu(G) = 1$, let xy be an edge of G and let $S := V(G) \setminus \{x, y\}$. Then $G[S]$ is totally disconnected and if we have two vertices in S, say u and v, such that xu and yv are edges of G, then $\{xu, yv\}$ is a matching in G, a contradiction. Thus, there is at most one vertex in S that is adjacent to both x and y. We now consider two cases:

Case 1: There is no vertex in S which is adjacent to both x and y. In this case, G is a star union of some number of isolated vertices. Thus we have $\mathrm{m}(G) = 2$, and the proposition holds.

Case 2: There is a vertex in S, say z, that is adjacent to both x and y. In this case, no other vertex of S is adjacent to either x or y. Thus $G = K_3 \cup tK_1$, where $t = |V(G)| - 3$ and $\mathrm{m}(G) = 3 = 3^{\nu(G)}$. Therefore the proposition is proved in this case.

Assume that $\nu(G) \geqslant 2$. Let xy be an edge of G. Since neither x nor y are vertices of the following graphs: G_x, G_y and $G \setminus \{x, y\}$, we deduce that

$$\nu(G_x) \leqslant \nu(G) - 1,\ \nu(G_y) \leqslant \nu(G) - 1 \ \text{ and } \ \nu(G \setminus \{x, y\}) \le \nu(G) - 1.$$

Thus, by the induction hypothesis, we obtain

$$\mathrm{m}(G_x) \leqslant 3^{\nu(G)-1},\ \mathrm{m}(G_y) \leqslant 3^{\nu(G)-1} \text{ and } \ \mathrm{m}(G \setminus \{x,y\}) \leqslant 3^{\nu(G)-1}.$$

Note that $(G \setminus x)_y = G_y$. Combining with Lemma 2.1, we obtain

$$\begin{aligned} \mathrm{m}(G) &\leq \mathrm{m}(G_x) + \mathrm{m}(G \setminus x) \\ &\leq \mathrm{m}(G_x) + \mathrm{m}(G_y) + \mathrm{m}(G \setminus \{x,y\}) \\ &\leq 3^{\nu(G)-1} + 3^{\nu(G)-1} + 3^{\nu(G)-1} = 3^{\nu(G)}. \end{aligned}$$

This proves the first conclusion of the proposition. The equality $\mathrm{m}(G) = 3^{\nu(G)}$ occurs if and only if

$$\begin{aligned} \mathrm{m}(G) &= \mathrm{m}(G_x) + \mathrm{m}(G \setminus x),\ \mathrm{m}(G \setminus x) = \mathrm{m}(G_y) + \mathrm{m}(G \setminus \{x,y\}), \\ \mathrm{m}(G_x) &= \mathrm{m}(G_y) = \mathrm{m}(G \setminus \{x,y\}) = 3^{\nu(G)-1}, \end{aligned}$$

and

$$\nu(G_x) = \nu(G_y) = \nu(G \setminus \{x,y\}) = \nu(G) - 1.$$

If $G = sK_3 \cup tK_1$, then $s = \nu(G)$ and $\mathrm{m}(G) = 3^{\nu(G)}$. This establishes the necessary condition of the second conclusion of the proposition. Now, it remains to prove that if $\mathrm{m}(G) = 3^{\nu(G)}$ then $G \cong sK_3 \cup tK_1$.

Indeed, by the induction hypothesis, it follows that when the isolated vertices of G_x, G_y and $G \setminus \{x,y\}$ are removed, the remaining graphs are isomorphic, namely $(s-1)K_3$, where $s = \nu(G)$. In particular, x and y are not adjacent to any vertex of $(s-1)K_3$. Let H be an induced subgraph of G on the vertex set $V(G) \setminus V((s-1)K_3)$. Then, H and $(s-1)K_3$ are disjoint subgraphs of G. By Lemma 2.1, we infer $\mathrm{m}(G) = \mathrm{m}(H)\,\mathrm{m}((s-1)K_3) = \mathrm{m}(H)3^{s-1}$. Since $\mathrm{m}(G) = 3^s$, $\mathrm{m}(H) = 3$. Note that $\nu(H) = 1$, so the induction hypothesis again yields $H = K_3 \cup tK_1$. Thus, $G = sK_3 \cup tK_1$. The proof is complete. □

The following lemma gives a lower bound for $\mathrm{m}(G)$ in terms of the induced matching number $\nu_0(G)$.

Lemma 2.4. *Let G be a graph. Then,* $\mathrm{m}(G) \geq 2^{\nu_0(G)}$.

Proof. Let $\{x_1y_1, \ldots, x_ry_r\}$ be an induced matching of G, where $r = \nu_0(G)$. Set $H := G[\{x_1, \ldots, x_r, y_1, \ldots, y_r\}]$. By Lemma 2.2, $\mathrm{m}(G) \geq \mathrm{m}(H) = 2^{\nu_0(G)}$. □

Recall that a vertex cover of G is a subset S of $V(G)$ such that for each $xy \in E(G)$, either $x \in S$ or $y \in S$. The following two lemmas are obvious.

Lemma 2.5. *Let H be an induced subgraph of G. Then,*

1. *If S is a vertex cover of G, then $S \cap V(H)$ is a vertex cover of H; and*
2. $\tau(H) \leq \tau(G)$.

Lemma 2.6. *Assume S is a vertex cover of G. If $U \subseteq S$, then*

1. *$S \setminus U$ is a vertex cover of $G \setminus U$; and*
2. *$\tau(G \setminus U) \leq \tau(G) - |U|$.*

We conclude this section by giving an upper bound for $\mathrm{m}(G)$ in terms of $\tau(G)$.

Theorem 2.7. *Let G be a graph. Then,* $\mathrm{m}(G) \leq 2^{\tau(G)}$.

Proof. We prove the theorem by induction on $\tau(G)$. If $\tau(G) = 0$, then G is totally disconnected, and so the assertion is trivial.

Assume that $\tau(G) \geqslant 1$. Let S be a vertex cover of G such that $|S| = \tau(G)$. Let $x \in S$. By Lemma 2.6, we have $\tau(G \setminus x) \leq \tau(G) - 1$. Hence, $\mathrm{m}(G \setminus x) \leq 2^{\tau(G \setminus x)}$ by the induction hypothesis.

Since G_x is an induced subgraph of $G \setminus x$, $\mathrm{m}(G_x) \leq \mathrm{m}(G \setminus x)$ by Lemma 2.2. Together with Lemma 2.1, we obtain

$$\begin{aligned} \mathrm{m}(G) &\leq \mathrm{m}(G \setminus x) + \mathrm{m}(G_x) \\ &\leq 2\,\mathrm{m}(G \setminus x) \leq 2^{\tau(G \setminus x)+1} \leq 2^{\tau(G)}, \end{aligned}$$

as required. □

3 Extremal graphs

A graph G is called *bipartite* if its vertex set can be partitioned into two subsets A and B so that every edge has one end in A and one end in B; such a partition is called a *bipartition* of the graph, and denoted by (A, B). If every vertex in A is joined to every vertex in B then G is called a complete bipartite graph, which is denoted by $K_{|A|,|B|}$. A *star* is the complete bipartite graph $K_{1,m}$ $(m \geq 0)$ consisting of $m+1$ vertices. A *star triangle* is a graph consisting of some triangles joined at one common vertex.

Cameron and Walker [2] gave firstly a classification of the connected graphs G with $\nu(G) = \nu_0(G)$. Hibi et al. [9] modified their result slightly and gave a full generalization with some corrections.

Lemma 3.1. ([2, Theorem 1] or [9, p.258]) *A connected graph G is Cameron–Walker if and only if it is one of the following graphs:*

1. *a star;*
2. *a star triangle;*
3. *a finite graph consisting of a connected bipartite graph with bipartition (A, B) such that there is at least one leaf edge attached to each vertex $i \in A$ and that there may be possibly some pendant triangles attached to each vertex $j \in B$.*

Example 3.2. Let G be a Cameron–Walker graph with 8 vertices, as x in i Figure 1. Then $\nu(G) = 2$ and the maximal independent sets of G are

$$\{1, 2, 5, 6, 7, 8\}; \{3, 4\}; \{3, 5, 6\}; \{4, 7, 8\}.$$

Hence $\mathrm{m}(G) = 4$.

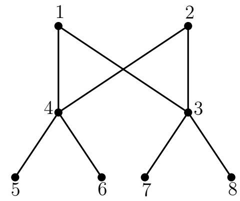

Figure 1.

Theorem 3.3. *Let G be a graph. Then $\mathrm{m}(G) = 2^{\tau(G)}$ if and only if G is a Cameron–Walker bipartite graph.*

Proof. If G is a Cameron–Walker bipartite graph, then $\nu_0(G) = \nu(G) = \tau(G)$. Together with Lemma 2.4 and Theorem 2.7, this fact yields $\mathrm{m}(G) = 2^{\tau(G)}$.

Conversely, assume that $\mathrm{m}(G) = 2^{\tau(G)}$. We will prove that G is Cameron–Walker bipartite by induction on $\tau(G)$.

If $\tau(G) = 0$, then G is totally disconnected and so the assertion is trivial. If $\tau(G) = 1$, then G is a union of a star and isolated vertices. In this case, G is a Cameron–Walker bipartite graph by Lemma 3.1.

Assume that $\tau(G) \geq 2$. Let S be a minimal vertex cover of G such that $|S| = \tau(G)$. We first prove two following claims.

Claim 1: S is an independent set of G.

Assume to the contrary that there is an edge, say xy, with $x, y \in S$. By Lemma 2.5, $S \cap V(G_x)$ is a vertex cover of G_x. Since $S \cap V(G_x) \subseteq S \setminus \{x, y\}$, we deduce that

$$\tau(G_x) \leqslant |S| - 2 = \tau(G) - 2.$$

Similarly, $S \setminus \{x\}$ is a vertex cover of $G \setminus x$. Thus $\tau(G \setminus x) \leqslant \tau(G) - 1$.

Together those inequalities with Lemma 2.1 and Theorem 2.7, we have

$$\mathrm{m}(G) \leqslant \mathrm{m}(G_x) + \mathrm{m}(G \setminus x) \leqslant 2^{\tau(G)-2} + 2^{\tau(G)-1} < 2^{\tau(G)}.$$

This inequality contradicts our assumption. Therefore, S is an independent set of G.

Claim 2: $\mathrm{m}(G_U) = 2^{\tau(G_U)}$ and $\tau(G_U) = \tau(G) - |U|$ for any $U \subseteq S$.

We prove the claim by induction on $|U|$. If $|U| = 0$, i.e. U is empty, then there is nothing to prove.

If $|U| = 1$, then $U = \{x\}$ for some vertex x. Since $x \in S$, by Lemmas 2.5 and 2.6, we have $\tau(G_x) \leq \tau(G \setminus x) \leq \tau(G) - 1$. By Theorem 2.7, $\mathrm{m}(G \setminus x) \leq 2^{\tau(G \setminus x)}$ and

$\mathrm{m}(G_x) \le 2^{\tau(G_x)}$. Together these inequalities with equality $\mathrm{m}(G) = 2^{\tau(G)}$, Lemma 2.1 gives

$$\begin{aligned} 2^{\tau(G)} = \mathrm{m}(G) &\le \mathrm{m}(G \setminus x) + \mathrm{m}(G_x) \le 2^{\tau(G\setminus x)} + 2^{\tau(G_x)} \\ &\le 2^{\tau(G)-1} + 2^{\tau(G)-1} = 2^{\tau(G)}. \end{aligned}$$

Hence, $\mathrm{m}(G_x) = 2^{\tau(G_x)}$ and $\tau(G_x) = \tau(G) - 1$, and the claim holds in this case.

We now assume $|U| \ge 2$. Let $x \in U$ and let $T := U \setminus \{x\}$. Note that T is a nonempty independent set of S and $|T| = |U| - 1$. By the induction hypothesis of our claim, $\mathrm{m}(G_T) = 2^{\tau(G_T)}$ and $\tau(G_T) = \tau(G) - |T|$.

Note that, by Claim 1, S is an independent set of G. Thus $S \setminus T = S \setminus N_G[T]$. By Lemma 2.5, $S \setminus T$ is a vertex cover of G_T. Since $x \in S \setminus T$, by the same argument in the inductive step of our claim with G_T replacing by G, we have $\mathrm{m}((G_T)_x) = 2^{\tau((G_T)_x)}$ and $\tau((G_T)_x) = \tau(G_T) - 1$.

Since $G_U = (G_T)_x$, we obtain $\mathrm{m}(G_U) = 2^{\tau(G_U)}$ and

$$\tau(G_U) = \tau(G_T) - 1 = \tau(G) - (|T| + 1) = \tau(G) - |U|,$$

as claimed.

We turn back to the proof of the theorem. By Claim 1, S is both a vertex cover and an independent set of G. Therefore G is a bipartite graph with bipartition $(S, V(G) \setminus S)$. It remains to prove that G is a Cameron–Walker graph.

For each $x \in S$, let $U := S \setminus \{x\}$. By Claim 2, $\tau(G_U) = \tau(G) - |U| = 1$. Hence, G_U is a union of a star with bipartition $(\{x\}, Y)$, where $\emptyset \ne Y \subseteq V(G) \setminus S$ and isolated vertices. Thus, there is a vertex $y \in Y$ such that $\deg_{G_U}(y) = 1$ and $xy \in E(G)$. Since $V(G) \setminus S$ is an independent set, the equality $\deg_{G_U}(y) = 1$ forces $\deg_G(y) = 1$. By using Lemma 3.1, we conclude that G is a Cameron–Walker graph, and the proof is complete. ☐

If G is a König–Egerváry graph, then $\tau(G) = \nu(G)$. Together with Theorems 2.7 and 3.3, this fact yields:

Corollary 3.4. *Let G be a König–Egerváry graph. Then*

$$\mathrm{m}(G) \le 2^{\nu(G)},$$

and the equality holds if and only if G is a Cameron–Walker bipartite graph.

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