AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **74(1)** (2019), Pages 17–32

Some properties of the Knödel graph $W(k, 2^k)$, $k \geq 4$

R. Balakrishnan

Department of Mathematics, Bharathidasan University
Tiruchirappalli 620024, Tamil Nadu
India
mathrb13@gmail.com

P. Paulraja

Department of Mathematics, Kalasalingam Academy of Research and Education
Krishnankoil 626126, Tamil Nadu
India
ppraja56@gmail.com

Wasin So

Department of Mathematics, San Jose State University
San Jose, CA 95192-0103
U.S.A.
wasin.so@sjsu.edu

M. Vinay

Department of Mathematics, Manipal Institute of Technology
Manipal 576104, Karnataka
India
vinay.m2000@gmail.com

Abstract

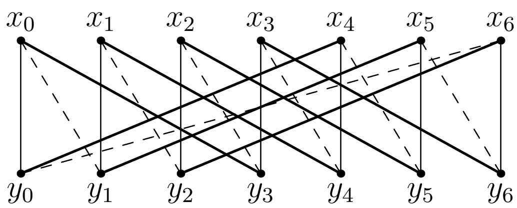
Knödel graphs have, of late, come to be used as strong competitors for hypercubes in the realms of broadcasting and gossiping in interconnection networks. For an even positive integer n and $1 \leq \Delta \leq \lfloor \log_2 n \rfloor$, the general Knödel graph $W_{\Delta,n}$ is the Δ-regular bipartite graph with bipartition sets $X = \{x_0, x_1, \ldots, x_{\frac{n}{2}-1}\}$ and $Y = \{y_0, y_1, \ldots, y_{\frac{n}{2}-1}\}$ such that x_j is adjacent to y_j, y_{j+2^1-1}, y_{j+2^2-1}, $\ldots$, $y_{j+2^{\Delta-1}-1}$, with suffixes being taken modulo $\frac{n}{2}$. The edge $x_j y_{j+2^i-1}$ at x_j and the edge $y_j x_{j-(2^i-1)}$ at y_j are called edges of dimension i at the stars centered at x_j and y_j respectively. In this paper, we concentrate on the Knödel graph $W_k = W_{k,2^k}$

ISSN: 2202-3518

with $k \geq 4$. We show that for $k \geq 4$, any automorphism of W_k fixes the set of 0-dimensional edges of W_k. We determine the automorphism group $\mathrm{Aut}(W_k)$ of W_k and show that it is isomorphic to the dihedral group $D_{2^{k-1}}$. In addition, we determine the spectrum of W_k and prove that it is never integral. As a by-product of our results, we obtain three new proofs showing that, for $k \geq 4$, W_k is not isomorphic to the hypercube H_k of dimension k, and a new proof for the result that W_k is not edge-transitive.

1 Introduction

For an even positive integer n, and $1 \leq \Delta \leq \lfloor \log_2 n \rfloor$, the Knödel graph $W_{\Delta,n}$ is defined to be the bipartite graph with the bipartition (X, Y), where $X = \{x_0, x_1, \ldots, x_{\frac{n}{2}-1}\}$ and $Y = \{y_0, y_1, \ldots, y_{\frac{n}{2}-1}\}$, and x_j is adjacent to y_j, y_{j+2^1-1}, y_{j+2^2-1}, $\ldots$, $y_{j+2^{\Delta-1}-1}$, the suffixes being taken modulo $\frac{n}{2}$. Thus, $W_{\Delta,n}$ is a Δ-regular bipartite graph on n vertices with $|X| = |Y| = \frac{n}{2}$. Figure 1.1 displays the Knödel graph $W_{3,14}$.



Normal lines denote 0-dimensional edges, broken lines represent 1-dimensional edges and bold lines represent 2-dimensional edges.

Figure 1.1: Knödel graph $W_{3,14}$

The edge $x_j y_{j+2^i-1}$ at x_j and the edge $y_j x_{j-(2^i-1)}$ at y_j (suffixes being taken modulo $\frac{n}{2}$) are called the edges of dimension i at the stars at x_j and y_j, respectively; see Figure 1.1.

The Knödel graphs have many interesting properties. The Knödel graphs have, of late, come to be used as competitors for hypercubes in the domains of broadcasting and gossiping. This is explained below in detail. The diameter of W_k is known but the diameter of the general Knödel graph $W_{\Delta,n}$ is not yet known; a tight lower and upper bounds on the diameter of Knödel graph $W_{\Delta,n}$ is obtained by Grigoryan and Harutyuanyan, see [8, 9]. However, the exact diameter of the graph W_k is $\lceil \frac{k+2}{2} \rceil$ (see [4]). The spectrum of W_k is studied by Harutyuanyan and Morosan, see [16]; using the spectrum of W_k, they have also obtained an upper bound on the number of spanning trees of W_k. The graph W_k is vertex transitive but not edge transitive, see [3]. In [27], Paulraja and Sampath Kumar have shown that W_k is almost Hamilton cycle decomposable, that is, $W_k \setminus E(H)$ is Hamilton cycle decomposable, where H is a 2-factor or a 3-factor according to whether k is even or odd.

The graphs W_k are the most popular in the family of interconnection networks along with the hypercubes H_k (see [22]) and the recursive bicirculant graphs $G(2^k, 4)$ introduced by Park and Chwa (see [26]), all of order 2^k. Both H_k and W_k, besides having the same order, are regular of degree k, and consequently have the same number of edges. However, they are not isomorphic since H_k has diameter k and W_k has diameter $\lceil \frac{k+2}{2} \rceil$ (see [4]).

The gossiping problem, as described by Knödel in [20] is as follows: "Given n persons, each with a bit of information, wishing to distribute their information to one another in binary calls, each call taking a fixed time, how long must it take before each knows everything?". Broadcasting is a similar problem where only one person (the originator) has all the information that needs to be distributed to all the others in binary calls. In essence, they deal with problems in dissemination of information in interconnection networks.

Every interconnection network can be represented by means of a graph. If this graph has n vertices, the minimum time required for broadcasting is $\lceil \log_2 n \rceil$. Such graphs are known as minimal broadcasting graphs. Both H_k and W_k are minimal broadcast networks of time k. A broadcast graph with minimum number of edges is called minimum broadcast graph. The number of edges in a minimum broadcast graph on n vertices is denoted by $B(n)$.

Broadcasting and gossiping have been extensively studied in literature. There are several papers dealing with Knödel graphs as some subfamilies of Knödel graphs have good properties in terms of broadcasting, gossiping and fault-tolerance, see [11, 13, 14, 17, 21]. In particular, W_k has been proved to be a minimum broadcast graph. For more details on minimum broadcast and gossip graphs, see [6, 17, 19]. In [12], new dimensional broadcast schemes for Knödel graphs are given. In the same paper, a general upper bound for $B(n)$ for almost all odd n is obtained. In [25], exact lower and upper bounds for the number of broadcast schemes in arbitrary networks is dealt with.

Fraigniaud and Lazard, see [5], deal with various methods and problems in communication networks such as the complete network, the torus, the grid, the ring, the underlying de Bruijn graph. In [14], two broad category of problems, namely, finding the best network for a given high level goal and finding the best protocol for a given goal are discussed. Also construction of sparse broadcast graphs is explained. For an elaborate discussions on Knödel graphs, see [3, 5, 14, 17]. Some more properties of Knödel graphs and modified Knödel graphs are given in [2, 10].

In this paper, we deal with the subfamily $W_k = W(k, 2^k)$, $k \geq 4$, of Knödel graphs. In our main result (Theorem 2.13), we determine the automorphism group of W_k and show that it is isomorphic to the dihedral group $D_{2^{k-1}}$. We prove this by showing that any automorphism of W_k takes a star (that is, a $K_{1,k}$ subgraph) to a star, preserving the dimensions of the corresponding edges.

We follow [1] for standard graph-theoretic notation and terminology.

This paper is organized as follows: In Section 2, we establish the fact that the only perfect matching of W_k, $k \geq 4$, whose removal disconnects W_k is the one obtained by joining the corresponding vertices of X and Y. This provides a new proof of the fact that W_k and H_k are not isomorphic. Next, we use this fact to determine the automorphism group of W_k and this yields a second new proof for the non-isomorphic nature of W_k and H_k. Our results show that the only automorphism that fixes a vertex of W_k, $k \geq 4$, is the identity automorphism. Finally, in Section 3, we determine the spectrum of W_k and show that it is never integral. This incidentally yields a third new proof of the fact that W_k and H_k, $k \geq 4$, are not isomorphic. We also determine lower and upper bounds for the number of spanning trees of W_k. These bounds are better than what are known as on date.

Before we close this section, we mention that the Knödel graphs W_k are vertex-transitive, but for $k \geq 4$, they are not edge-transitive [4]. Indeed, our results provide a new proof of the fact that the Knödel graphs W_k, $k \geq 4$, are not edge-transitive but they are vertex transitive.

In the rest of the paper, we denote 2^{k-1} by n, p pairwise disjoint copies of a graph G by pG, and the set of edges of dimension i in W_k by E_i.

Observation 1.1 ([3]). Yet another important structural property of W_k is that the set of edges of dimension zero in W_k, namely, $E_0 = \{x_0y_0, x_1y_1, \ldots, x_{n-1}y_{n-1}\}$, is a perfect matching of W_k with the property that $W_k \setminus E_0$ consists of two disjoint copies of W_{k-1}, which we denote by $W_{k-1}^{(1)}$ and $W_{k-1}^{(2)}$. See Figures 1.2 and 1.3. Denote the bipartitions of $W_{k-1}^{(1)}$ and $W_{k-1}^{(2)}$ by $\left(X_{k-1}^{(1)}, Y_{k-1}^{(1)}\right)$ and $\left(X_{k-1}^{(2)}, Y_{k-1}^{(2)}\right)$ respectively. Then

$$\begin{aligned}
X_{k-1}^{(1)} &= \{x_0, x_2, \ldots, x_{2^{k-1}-2}\},\\
Y_{k-1}^{(1)} &= \{y_1, y_3, \ldots, y_{2^{k-1}-1}\},\\
X_{k-1}^{(2)} &= \{x_1, x_3, \ldots, x_{2^{k-1}-1}\}\\
Y_{k-1}^{(2)} &= \{y_2, y_4, y_6, \ldots, y_{2^{k-1}-2}, y_0\}.
\end{aligned}$$

(Observe that y_0 is given at the end in $Y_{k-1}^{(2)}$).
If we relabel the vertex subsets $X_{k-1}^{(1)}$ and $Y_{k-1}^{(1)}$ of W_k by $\{u_0, u_1, \ldots, u_{2^{k-2}-1}\}$ and $\{v_0, v_1, \ldots, v_{2^{k-2}-1}\}$ respectively, preserving the orders of the vertices, and join the edges $u_jv_{j+2^i-1}$, $0 \leq j \leq 2^{k-2}-1$, $0 \leq i \leq k-2$, the resulting graph is isomorphic to $W_{k-1}^{(1)}$. A similar statement applies for the next two vertex subsets $X_{k-1}^{(2)}$ and $Y_{k-1}^{(2)}$, and in this case the resulting graph is isomorphic to $W_{k-1}^{(2)}$. (See Figure 1.2). Notice further that x_0y_1 and y_7x_6 are 0-dimensional edges at x_0 and y_7 of $W_3^{(1)}$ respectively. A redrawing of W_4 is given in Figure 1.3, from which it is clear that $W_4 \setminus E_0$ consists of two "identical" copies of W_3. A similar statement applies to W_k ($k \geq 4$) as well.

2 The automorphism group of the Knödel graph W_k, $k \geq 4$

We begin by establishing a structure theorem on Knödel graphs.

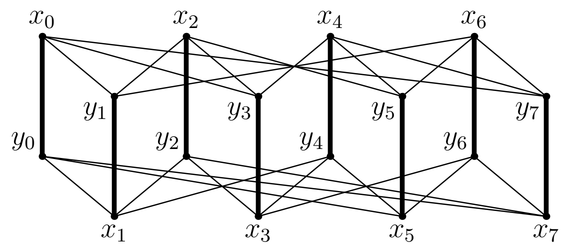


Figure 1.2: W_4 and the perfect matching E_0 (in bold lines)

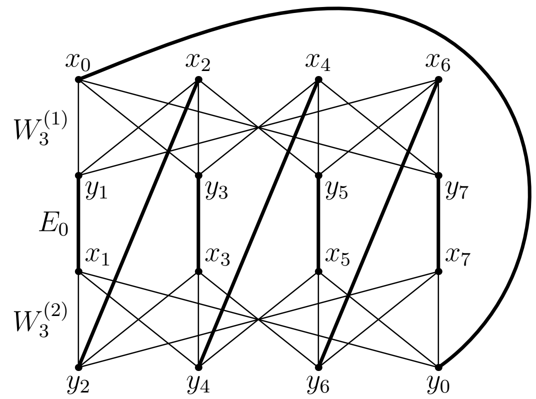


Figure 1.3: Another drawing of W_4 (bold lines represent the perfect matching E_0 whose removal results in two "identical" copies of W_3).

Theorem 2.1. *Let E_0 be the perfect matching of the Knödel graph W_k, $k \geq 4$, consisting of the 0-dimensional edges of W_k. Then E_0 is the only perfect matching such that $W_k \setminus E_0$ consists of two isomorphic copies of W_{k-1}.*

Proof. Let $X = \{x_0, x_1, \dots, x_{n-1}\}$ and $Y = \{y_0, y_1, \dots, y_{n-1}\}$ (recall: $n-1 = 2^{k-1} - 1$) be the bipartition of W_k. For each i, $0 \leq i \leq n-1$, x_i and y_i are the corresponding vertices of W_k. By choice, E_0 is the set of edges $\{x_0y_0, x_1y_1, \dots, x_{n-1}y_{n-1}\}$. By Observation 1.1, E_0 is a perfect matching with the property that $W_k \setminus E_0$ is a disjoint union of two copies of W_{k-1}. We claim that E_0 is the only perfect matching of W_k with this property.

In our proof, the suffix i in x_i and y_i is always taken modulo $2^{k-1} = n$. We note that x_i (respectively y_i) is adjacent to y_{i-1}, y_i, y_{i+1} (respectively to x_{i-1}, x_i, x_{i+1}).

If possible, assume that there is a perfect matching $E_0' \neq E_0$ of W_k such that $W_k \setminus E_0'$ has two components, each isomorphic to W_{k-1}. We call these two components as W_{k-1}' and W_{k-1}'' with (X_1', Y_1') and (X_2'', Y_2'') as their respective bipartitions, where $X_1' \subset X$, $X_2'' \subset X$ and $Y_1' \subset Y$, $Y_2'' \subset Y$. The vertices x_0 and y_0 may be in the same or different components of $W_k \setminus E_0'$.

Case 1. x_0 and y_0 are in different components of $W_k \setminus E_0'$ (so that $x_0y_0 \in E_0'$).

Assume that $x_0 \in X_1' \subset W_{k-1}'$, so that $y_0 \in Y_2'' \subset W_{k-1}''$. As $x_0 \in X_1'$, the other neighbors of x_0, namely, $y_1, y_3, y_7, \dots, y_{n-1}$ must all belong to Y_1' (See Figure 2.1). As the edges between W_{k-1}' and W_{k-1}'' are the edges of E_0', and since $x_0y_0 \in E_0'$

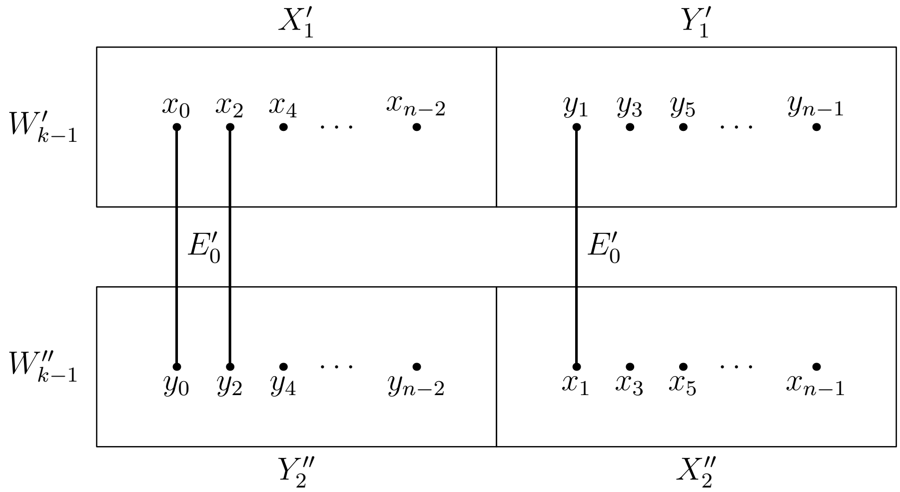


Figure 2.1: Case when $x_0y_0 \in E'_0$

and $x_1y_0 \in \mathrm{E}(W_k)$, x_1 must belong to X''_2. Again, as $x_1 \in W''_{k-1}$, $y_1 \in W'_{k-1}$ and $x_1y_1 \in \mathrm{E}(W_k)$, x_1y_1 must belong to E'_0 (see Figure 2.1). As x_1y_1 and x_2y_1 are edges of W_k, and as $x_1y_1 \in E'_0$, $x_2y_1 \notin E'_0$, and so x_2 must be in X'_1. Again, $x_1y_1 \in E'_0$ and $x_1y_2 \in \mathrm{E}(W_k)$ imply that $y_2 \in Y''_2$, and therefore $x_2y_2 \in E'_0$. By induction, it is clear that $x_i \in X'_1$ or X''_2 according to whether i is even or odd, and $y_j \in Y'_1$ or Y''_2 according to whether j is odd or even. Thus $x_iy_i \in E'_0$ for each i, $0 \leq i \leq n-1$. In other words, $E'_0 = E_0$.

Case 2. x_0 and y_0 are in the same component of $W_k \setminus E'_0$, say, W'_{k-1}.

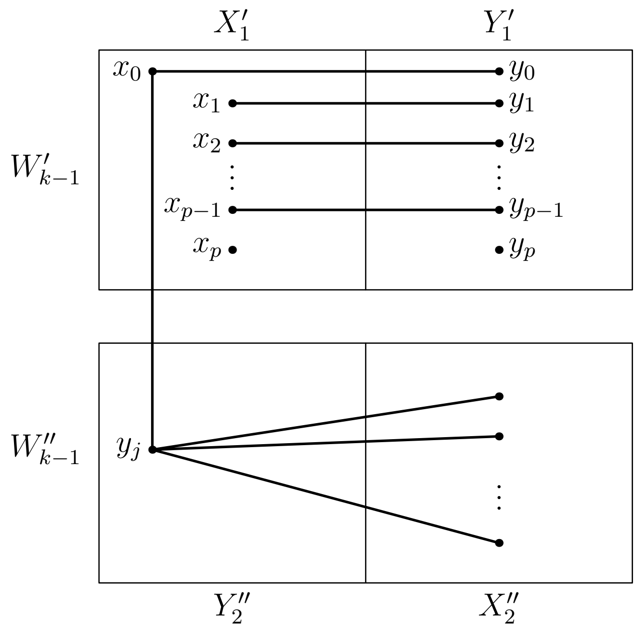


Figure 2.2: Case when $x_0y_0 \notin E'_0$

As E'_0 is a perfect matching, only one edge of E'_0, say, x_0y_j, $j \neq 0$, is incident to x_0. Hence $y_j \in Y_2''$ (see Figure 2.2). Assume, for the moment, that $j \neq 1, n-1$. The neighbors of x_0 in W_k, other than y_j, must all be in Y_1', and the neighbors of y_j, other than x_0, must all be in X_2''. In particular, y_0, y_1 and y_{n-1} are all in Y_1'. *This situation is possible as* $k \geq 4$. Consequently, $x_1 \in X_1'$ (else, $x_1 \in X_2''$ and x_1y_0 and x_1y_1 would be in E_0', which is impossible) and $x_1y_1 \in W_{k-1}'$. Thus both x_0y_0 and x_1y_1 are in W_{k-1}'. Recall that $x_{n-1}y_{n-1}$, $x_{n-1}y_0$ are both edges of W_k, and so $x_{n-1} \in X_1'$. As y_2 is adjacent to both x_1 and x_{n-1} (note that $x_{n-1}y_2$ is an edge of dimension 3 of W_k), $y_2 \notin Y_2''$ (otherwise, there will be two edges of E_0' incident at y_2), and so $y_2 \in Y_1'$. This forces that $x_2 \in X_1'$ (otherwise, x_2y_2 and x_2y_1 must be in E_0'), and hence $x_2y_2 \in W_{k-1}'$.

We claim that $x_0y_0, x_1y_1, \ldots x_{n/2-1}y_{n/2-1}$ are in W_{k-1}'. Since x_0y_0, x_1y_1 and x_2y_2 are in W_{k-1}', assume, by induction, that $x_0y_0, x_1y_1, x_2y_2, \ldots, x_{p-1}y_{p-1}$ where $p \leq n/2-1$, are all edges of W_{k-1}'. (See Figure 2.2). Now, as y_px_{p-1} and y_px_{p-3} are edges of W_k, and as x_{p-1} and x_{p-3} are in X_1', $y_p \in Y_1'$. Again, as x_py_{p-1} and x_py_{p-3} are edges of W_k, and since y_{p-1} and y_{p-3} are in Y_1', $x_p \in X_1'$. As $x_py_p \in \mathrm{E}(W_k)$, $x_py_p \in W_{k-1}'$. This completes the proof of the induction step. By induction $x_{n/2-1}y_{n/2-1}$ is an edge of W_{k-1}'. Consequently, W_{k-1}' contains the vertices $\{x_0, x_1, \ldots x_{n/2-1}\} \cup \{y_0, y_1, \ldots y_{n/2-1}\}$. Since W_{k-1}' has only n vertices, the veritces $x_j, y_j, n/2 \leq j \leq n-1$, must be in W_{k-1}''. Since the vertex $x_{n/2-1}$ of W_{k-1}' is adjacent to the vertices $y_{n/2}$ and $y_{n/2+2}$ of W_{k-1}'', E_0' cannot be an edge cut of W_k, a contradiction to the choice of E_0'.

Finally, we consider the cases when $j = 1$ and $j = n-1$. If $j = 1$, as $x_0y_1 \in E_0'$, $y_1 \in Y_2''$, and since both x_1y_1 and x_2y_1 are edges of W_k, both $x_1, x_2 \in X_2''$. The neighbors y_0 and y_3 of x_0 must belong to Y_1'. Hence, $x_1y_0, x_2y_3 \in E_0'$, and therefore, y_4, y_5 (which are neighbors of x_1 and x_2 respectively) $\in Y_2''$. Now, $x_3 \notin X_2''$, since otherwise, there will be two edges of E_0', namely y_3x_2 and y_3x_3 at y_3. Hence, $x_3 \in X_1'$. For a similar reason, $y_2 \in Y_2''$. But then, x_3y_2 and x_3y_4 become matching edges of E_0', a contradiction.

A similar argument holds when $j = n-1$. This proves that $E_0' = E_0$ and hence E_0 is the only perfect matching of W_k having the stated property. □

Note 2.2. *Notice that we have used the fact that* $k \geq 4$ *crucially in the proof of Theorem 2.1.*

Corollary 2.3. *If* $k \geq 4$*, the Knödel graph* W_k *is not isomorphic to the hypercube* H_k*.*

Proof. If $k \geq 4$, H_k has more than one edge disjoint perfect matchings, the removal of any one of which results in a disjoint union of two H_{k-1}'s. However, this is not the case with W_k, by virtue of Theorem 2.1. □

We observe that for $k = 3$, $W_3 \cong H_3$, and H_3 has three edge disjoint perfect matchings, E_i', $1 \leq i \leq 3$, such that $H_3 \setminus E_i'$ is isomorphic to $2C_4 \cong 2H_2$.

Let α be any automorphism of a graph G with vertex set $\mathrm{V}(G)$. For $M \subset \mathrm{E}(G)$, the edge set of G, let $\alpha(M)$ denote the set of edges $\{\, \alpha(u)\alpha(v) : u, v \in \mathrm{V}(G), uv \in M \,\}$.

Corollary 2.4. *Every automorphism α of the Knödel graph W_k, $k \geq 4$, maps an edge of dimension zero to an edge of dimension zero. Equivalently, $\alpha(E_0) = E_0$, where E_0 is the set of edges of W_k of dimension zero.*

Proof. We know that $W_k \setminus E_0 = 2W_{k-1}$, a disjoint union of two copies of W_{k-1}. Let β be any automorphism of W_k. Then $\beta(E_0) =$ (say) F is also a perfect matching of W_k, and $W_k \setminus F$ is a disjoint union of two copies of W_{k-1}. By Theorem 2.1, this means that $F = E_0$. Thus $\beta(E_0) = E_0$ and hence every automorphism of W_k fixes E_0; equivalently, every automorphism of W_k maps an edge of dimension zero of W_k to an edge of dimension zero. □

An immediate consequence of Corollary 2.4 is the following result of Fertin and Raspaud [3] which had been proved by considering sums of powers of 2.

Corollary 2.5 ([3])**.** *The Knödel graphs W_k, $k \geq 4$, are not edge-transitive.*

Proof. By Corollary 2.4, no automorphism can take an edge of dimension zero to an edge of dimension not equal to zero. □

Theorem 2.6. *Let W'_k and W''_k be two disjoint copies of W_k, $k \geq 4$. Let ϕ be any isomorphism of W'_k onto W''_k. Then ϕ maps an edge of dimension zero of W'_k to an edge of dimension zero of W''_k.*

Proof. Let E'_0 and E''_0 be the sets of 0-dimensional edges of W'_k and W''_k respectively. As E'_0 is a perfect matching of W'_k, and since ϕ is an isomorphism, $\phi(E'_0)$ is a perfect matching of W''_k. As $W'_k \setminus E'_0$ is a disjoint union of two copies of W_{k-1}, the same must be true of $W''_k \setminus \phi(E'_0)$. By Theorem 2.1, this implies that $\phi(E'_0) = E''_0$. □

Proposition 2.7. *Let α be an automorphism of W_k, $k \geq 4$. Then either α fixes $W^{(1)}_{k-1}$ and $W^{(2)}_{k-1}$ or else interchanges them.*

Proof. Let v be any vertex of $W^{(1)}_{k-1}$. Suppose $\alpha(v) \in W^{(1)}_{k-1}$. We claim that α fixes $W^{(1)}_{k-1}$. Let $w \neq v$ be any vertex of $W^{(1)}_{k-1}$. As $W^{(1)}_{k-1}$ is connected, there is a $w-v$ path P in W_k which is completely contained in $W^{(1)}_{k-1}$. As $\alpha(v)$ is in $W^{(1)}_{k-1}$, $\alpha(P)$ should be completely contained in $W^{(1)}_{k-1}$; otherwise, it should contain an edge of E_0 but $\alpha(E_0) = E_0$ by Corollary 2.4 and P does not contain an edge of E_0. □

Proposition 2.8. *If α is an automorphism of W_k, $k \geq 4$, that induces the identity automorphism on $W^{(1)}_{k-1}$, then α is the identity automorphism of W_k.*

Proof. Let v be any vertex of $W^{(1)}_{k-1}$. Since α fixes all the vertices of $W^{(1)}_{k-1}$, α fixes all the neighbors of v in $W^{(1)}_{k-1}$. α has exactly one neighbor v' in $W^{(2)}_{k-1}$ which also belongs to E_0. As $k \geq 4$, α fixes E_0 by Theorem 2.1, and hence α fixes v'. This means that α is the identity automorphism of W_k. □

Corollary 2.9. *If the automorphisms α_1 and α_2 of W_k, $k \geq 4$, induce the same automorphism on $W^{(1)}_{k-1}$, then $\alpha_1 = \alpha_2$.*

Proof. This is because $\alpha_1\alpha_2^{-1}$ induces the identity automorphism on $W_{k-1}^{(1)}$. Now apply Proposition 2.8. □

Theorem 2.10. *The maps $\phi = (x_0x_1 \dots x_{n-1})(y_0y_1 \dots y_{n-1})$ and $\psi = (x_0y_{n-1})(x_1y_{n-2}) \dots (x_{n-1}y_0)$ on $V(W_k)$, $k \geq 4$, define two automorphisms of W_k which generate a group $\mathcal{A}$ of order $2n$.*

Proof. Obvious. □

Note that $\mathcal{A}$ acts transitively on W_k and, consequently, W_k is a vertex-transitive graph, a result originally proved by Heydemann et al. [18].

We now proceed to show that $\mathcal{A}$ is indeed the automorphism group of W_k, $k \geq 4$.

Lemma 2.11. *Let α be any automorphism of W_4. Suppose that α fixes some vertex of W_4. Then α is the identity automorphism of W_4.*

Proof. Without loss of generality (as W_4 is vertex-transitive), assume that α fixes x_0. By Theorem 2.1, $\alpha(E_0) = E_0$. Hence $\alpha(y_0) = y_0$, where $x_0y_0 \in E_0$. By Proposition 2.7, $\alpha(W_3^{(i)}) = W_3^{(i)}$, $i = 1, 2$. Now $\alpha(N(x_0)) = N(x_0)$, where N stands for the neighbor set in W_k. Hence $\alpha(N(x_0)) = \alpha(\{y_1, y_3, y_7\}) = \{y_1, y_3, y_7\}$, which implies that y_5 and hence x_5 are both fixed by α. Now $N(x_5) = \{y_0, y_4, y_6\}$ is fixed by α. Therefore, y_2 and hence x_2 are both fixed by α. Again, $N(x_2) = \{y_1, y_3, y_5\}$ is fixed by α and hence $\{y_1, y_3\}$ is fixed by α (as y_5 is fixed by α). This means that $\alpha(y_7) = y_7$ and hence $\alpha(x_7) = x_7$. Again, $N(y_0) = \{x_1, x_3, x_7\}$, and as x_5 and x_7 are fixed by α, α fixes x_1 and therefore y_1. Further, $\alpha(\{x_4, x_6\}) = \{x_4, x_6\}$, $d(x_6, y_1) = 1$ and $d(x_4, y_1) \neq 1$. Hence α must fix x_4, x_6 and hence y_4, y_6. Finally, α must fix the left out pair of vertices of W_4, namely, x_3 and y_3. Hence α is the identity map of W_4. □

Theorem 2.12. *Let α be an automorphism of W_k, $k \geq 4$. If α fixes some vertex of W_k, then α is the identity automorphism of W_k.*

Proof. As W_k is vertex-transitive, we can assume that $\alpha(x_0) = x_0$. Now α induces an automorphism on $W_{k-1}^{(1)}$ which again fixes x_0. This induced automorphism of $W_{k-1}^{(1)}$ induces an automorphism of $W_{k-2}^{(1)}$ which again fixes x_0. By repeating the argument, we reach the stage when the restriction α' of α is an automorphism of W_4 which fixes x_0. As α' is an automorphism of W_4 which fixes x_0, apply Proposition 2.8 repeatedly to conclude that α is the identity automorphism of W_k. □

Theorem 2.13. *For $k \geq 4$, the automorphism group $\mathcal{A}$ of W_k is isomorphic to the dihedral group $D_{2^{k-1}}$ of order 2^k.*

Proof. Let $\alpha \in \mathcal{A}$, and let $\alpha(a) = b$ for some vertices a and b of W_k. Now there exists an automorphism $\beta \in \langle \phi, \psi \rangle$, where ϕ and ψ are as in Theorem 2.10, with $\beta(a) = b$ so that $a = \beta^{-1}(b) \Rightarrow \alpha\beta^{-1}(b) = b$. Hence by Theorem 2.12, $\alpha = \beta$. Thus $\mathcal{A} \subseteq \langle \phi, \psi \rangle$, and therefore $\mathcal{A} = \langle \phi, \psi \rangle$. But then ϕ and ψ generate the dihedral group $D_{2^{k-1}}$ of order 2^k. (We observe that we have essentially used Burnside's Orbit-Stabilizer Lemma [7].) □

We observe that Theorem 2.10 provides a second new proof of the fact that when $k \geq 4$, $W_k \ncong H_k$, as $|\operatorname{Aut}(W_k)| = 2^k$ while $|\operatorname{Aut}(H_k)| = 2^k k!$.

We conclude this section with a result on the dimensions of edges of W_k.

Theorem 2.14. *Let α be any automorphism of the Knödel graph W_k, $k \geq 4$. Then for $i = 0, 1, \ldots, k-4$, α preserves the edges of dimension i in W_k.*

Proof. We have seen that $\alpha(E_0) = E_0$, so that α takes a 0-dimensional edge to a 0-dimensional edge. Now, $W_k \backslash E_0 \cong 2W_{k-1} =$ (say) $W_{k-1}^{(1)} \cup W_{k-1}^{(2)}$. Therefore, $\alpha \restriction_{(W_k \backslash E_0)}$ maps a 0-dimensional edge of $W_{k-1}^{(1)}$ to a 0-dimensional edge of $W_{k-1}^{(1)}$ or $W_{k-1}^{(2)}$. Now, the 0-dimensional edges of $W_k \setminus E_0$ are the 1-dimensional edges of W_k, see [3]. This proves that α fixes the 1-dimensional edges of W_k. As $W_k \setminus (E_0 \cup E_1) \cong 4W_{k-2}$, and the 0-dimensional edges of W_{k-2} are the 2-dimensional edges of W_k and vice versa, α fixes all the 2-dimensional edges of W_k. We now repeat this procedure until we reach copies of W_4 for which the result has already been established in Theorem 2.6. Consequently, we conclude that α preserves the dimensions of edges. □

3 Spectrum of Knödel graphs

In this section, we determine the spectrum of the general Knödel graph $W_{\Delta,n}$ using a method different from the one of Harutyunyan and Morosan [15]. Then we deduce the spectrum of the special Knödel graph W_k and use it to obtain (i) yet another proof of the fact that W_k is not isomorphic to H_k for $k \geq 4$, and (ii) a better upper bound and a new lower bound for the number of spanning trees of W_k for $k \geq 2$. We begin by reviewing some basic properties of circulant matrices over complex numbers.

Definition 3.1. A matrix is a circulant if each successive row is obtained by shifting the current row to the right with wrap around.

Hence a circulant matrix is determined by its first row. Denote by Z the special $n \times n$ circulant matrix with first row $[0,\ 1,\ 0, \ldots,\ 0]$.

Lemma 3.2. *Let C be an $n \times n$ circulant matrix with first row $[c_0,\ c_1,\ \ldots,\ c_{n-1}]$. Then $C = c_0 I + c_1 Z + \cdots + c_{n-1} Z^{n-1}$.*

Proof. Straight-forward verification. □

Let $\mathcal{C}_n$ be the collection of all $n \times n$ circulant matrices. Then, by Lemma 3.2,

$$\mathcal{C}_n = \{\, p(Z) : p \text{ is a polynomial of degree at most } n-1 \text{ in the matrix } Z \,\}.$$

Corollary 3.3. *$\mathcal{C}_n$ is closed under matrix multiplication, transpose, and conjugate transpose.*

Proof. Note that $Z^n = I$ and $Z^T = Z^{n-1}$. Hence, if $C = c_0 I + c_1 Z + \cdots + c_{n-1} Z^{n-1}$, then $C^T = c_0 I + c_1 Z^T + \cdots + c_{n-1} (Z^T)^{n-1} = c_0 I + c_1 Z^{n-1} + \cdots + c_{n-1} Z^{(n-1)(n-1)}$, which can be simplified to a polynomial in Z of degree at most $n-1$. □

Corollary 3.4. *Any two circulant matrices commute.*

Lemma 3.5. *If C is circulant and invertible, then C^{-1} is also circulant.*

Proof. By the Cayley-Hamilton Theorem, C^{-1} is a polynomial in C, and so it is also circulant. □

Lemma 3.6. *Let $C \in \mathcal{C}_n$ with first row $[c_0, c_1, \ldots, c_{n-1}]$. Then the spectrum of C is*

$$\operatorname{Sp}(C) = \left\{ c_0(\omega^t)^0 + c_1(\omega^t)^1 + \cdots + c_{n-1}(\omega^t)^{n-1} : 0 \le t \le n-1 \right\}$$

where $\omega = e^{2\pi \mathbf{i}/n}$.

Proof. Note that Z has eigenvalues $\{\omega^t : 0 \le t \le n-1\}$ where $\omega = e^{2\pi \mathbf{i}/n}$. By Lemma 3.2,

$$C = c_0 I + c_1 Z + \cdots + c_{n-1} Z^{n-1},$$

and so

$$\operatorname{Sp}(C) = \left\{ c_0(\omega^t)^0 + c_1(\omega^t)^1 + \cdots + c_{n-1}(\omega^t)^{n-1} : 0 \le t \le n-1 \right\}. \qquad \square$$

Spectrum of the Knödel graph $W_{\Delta,n}$

Let $W_{\Delta,n}$ be the general Knödel graph of order n (even) and regularity Δ with $\Delta \le \lfloor \log_2 n \rfloor$, i.e., $2^\Delta \le n$. Then its adjacency matrix can be taken in the form

$$A = \begin{bmatrix} 0 & C \\ C^T & 0 \end{bmatrix}$$

where C is an $\frac{n}{2} \times \frac{n}{2}$ circulant matrix with the first row $[1,\ 1,\ 0,\ 1,\ \ldots,\ 1,\ 0,\ \ldots,\ 0]$, where the 1's in the first row of C appear at the columns: $1,\ 2,\ 2^2,\ \ldots,\ 2^{\Delta-1}$ of C.

Lemma 3.7. *Let $A = \begin{bmatrix} 0 & C \\ C^T & 0 \end{bmatrix}$. Then $\operatorname{Sp}(A) = \pm \operatorname{Sv}(C)$, where $\operatorname{Sv}(C)$ is the collection of singular values of C.*

Proof. By Singular Value Decomposition [23] $C = UDV^T$ where U and V are orthogonal matrices, and D is a diagonal matrix with singular values of C on its diagonal. Hence

$$A = \begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix} \begin{bmatrix} 0 & D \\ D & 0 \end{bmatrix} \begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix}^T.$$

Consequently, $\operatorname{Sp}(A) = \operatorname{Sp}\left(\begin{bmatrix} 0 & D \\ D & 0 \end{bmatrix} \right) = \pm \operatorname{Sp}(D) = \pm \operatorname{Sv}(C)$. □

Theorem 3.8. *The spectrum of the Knödel graph $W_{\Delta,n}$ is*

$$\operatorname{Sp}(W_{\Delta,n}) = \pm \left\{ \left| (\omega^t)^{2^0} + (\omega^t)^{2^1} + \cdots + (\omega^t)^{2^{\Delta-1}} \right| : 0 \le t \le \frac{n}{2} - 1 \right\},$$

where $\omega = e^{4\pi \mathbf{i}/n}$.

Proof. From the structure of the adjacency matrix of $W_{\Delta,n}$ and Lemma 3.7, we have

$$\mathrm{Sp}(W_{\Delta,n}) = \pm\,\mathrm{Sv}(C)$$

where $\mathrm{Sv}(C)$ denotes the set of singular values of C. By Corollaries 3.3 and 3.4, C is a normal matrix, and so its singular values are the absolute values of its eigenvalues, that is,

$$\begin{aligned}\mathrm{Sp}(W_{\Delta,n}) &= \pm\left\{\left|(\omega^t)^{2^0-1} + (\omega^t)^{2^1-1} + \cdots + (\omega^t)^{2^{\Delta-1}-1}\right| : 0 \le t \le \frac{n}{2}-1\right\}\\ &= \pm\left\{\left|(\omega^t)^{2^0} + (\omega^t)^{2^1} + \cdots + (\omega^t)^{2^{\Delta-1}}\right| : 0 \le t \le \frac{n}{2}-1\right\}.\end{aligned}$$

The last equality is due to the fact that $|\omega^t| = 1$. □

Example 3.9. (i) $\mathrm{Sp}(W_{1,2}) = \pm\{1\}$, $\mathrm{Sp}(W_{1,4}) = \pm\{1,1\}$.

(ii) $\mathrm{Sp}(W_{2,4}) = \pm\{2,0\}$, $\mathrm{Sp}(W_{2,6}) = \pm\{2,1,1\}$.

(iii) $\mathrm{Sp}(W_{3,8}) = \pm\{3,1,1,1\}$, $\mathrm{Sp}(W_{3,10}) = \pm\left\{3, \frac{\sqrt5+1}{2}, \frac{\sqrt5+1}{2}, \frac{\sqrt5-1}{2}, \frac{\sqrt5-1}{2}\right\}$.

(iv) $\mathrm{Sp}(W_{4,16}) = \pm\left\{4, 2, \sqrt{2+\sqrt2}, \sqrt{2+\sqrt2}, \sqrt2, \sqrt2, \sqrt{2-\sqrt2}, \sqrt{2-\sqrt2}\right\}$.

Corollary 3.10. *For $k \ge 2$,*

$$\begin{aligned}\mathrm{Sp}(W_k) &= \mathrm{Sp}(W_{k,2^k})\\ &= \pm\{k, (k-2)\}\cup\\ &\pm\left\{\left|(\omega^t)^{2^0} + (\omega^t)^{2^1} + \cdots + (\omega^t)^{2^{k-1}}\right| : 1 \le t \le 2^{k-2}-1\right\}^{(2)}\end{aligned}$$

where $\omega = e^{2\pi\mathbf{i}/2^{k-1}}$, and the superscript $^{(2)}$ means multiplicity 2.

Proof. By Theorem 3.2, with $\Delta = k$ and $n = 2^k$, we have

$$\begin{aligned}\mathrm{Sp}(W_{k,2^k}) &= \pm\left\{\left|(\omega^t)^{2^0} + (\omega^t)^{2^1} + \cdots + (\omega^t)^{2^{k-1}}\right| : 0 \le t \le 2^{k-1}-1\right\}\\ &= \pm k \cup \pm\left\{\left|(\omega^t)^{2^0} + (\omega^t)^{2^1} + \cdots + (\omega^t)^{2^{k-1}}\right| : 1 \le t \le 2^{k-2}-1\right\}\\ &\cup \pm(k-2)\cup\\ &\left\{\left|(\omega^t)^{2^0} + (\omega^t)^{2^1} + \cdots + (\omega^t)^{2^{k-1}}\right| : 2^{k-2}+1 \le t \le 2^{k-1}-1\right\}\\ &= \pm k \cup \pm(k-2) \cup \pm\\ &\left\{\left|(\omega^t)^{2^0} + (\omega^t)^{2^1} + \cdots + (\omega^t)^{2^{k-1}}\right| : 1 \le t \le 2^{k-2}-1\right\}\\ &\cup \pm\left\{\left|(\omega^t)^{2^0} + (\omega^t)^{2^1} + \cdots + (\omega^t)^{2^{k-1}}\right| : 1 \le t \le 2^{k-2}-1\right\}.\end{aligned}$$

The last equality is due to the fact that

$$\left|(\omega^t)^{2^0} + (\omega^t)^{2^1} + \cdots + (\omega^t)^{2^{k-1}}\right| = \left|(\omega^{2^{k-1}-t})^{2^0} + (\omega^{2^{k-1}-t})^{2^1} + \cdots + (\omega^{2^{k-1}-t})^{2^{k-1}}\right|.$$

□

Lemma 3.11. *For $k \geq 2$, $k-2 = \max\{\,|\lambda| : \lambda \in Sp(W_k) \setminus \{\pm k\}\,\}$. In other words, the second largest eigenvalue of W_k is $k-2$, for $k \geq 2$.*

Proof. Let $\lambda \in Sp(W_k) \setminus \{\pm k\}$. By Corollary 3.10, there exists a t with $1 \leq t \leq 2^{k-1}-1$ such that

$$\lambda = \pm \left|(\omega^t)^{2^0} + (\omega^t)^{2^1} + \cdots + (\omega^t)^{2^{k-1}}\right|$$

where $\omega = e^{2\pi\mathbf{i}/2^{k-1}}$. Write $t = 2^r q$ for some r with $0 \leq r \leq k-2$ and odd integer q. Hence $(\omega^t)^{2^{k-2-r}} = (e^{\pi\mathbf{i}})^q = (-1)^q = -1$. Consequently,

$$|\lambda| = \left|(\omega^t)^{2^0} + (\omega^t)^{2^1} + \cdots (-1) + \cdots + (\omega^t)^{2^{k-2}} + 1\right| \leq k-2.$$

On the other hand, take $t = 2^{k-2}$, we have $\omega^t = -1$, and so

$$\begin{aligned}\pm \left|(\omega^t)^{2^0} + (\omega^t)^{2^1} + \cdots + (\omega^t)^{2^{k-1}}\right| &= \pm \left|(-1) + (-1)^2 + \cdots + (-1)^{2^{k-1}}\right| \\ &= \pm(k-2)\end{aligned}$$

are eigenvalues of W_k. □

Theorem 3.12. *For $k \geq 4$, $\sqrt{k^2-6k+10} \in \mathrm{Sp}(W_k)$.*

Proof. Take $t = 2^{k-3}$ (here it requires $k \geq 4$), so that $\omega^t = e^{2\pi\mathbf{i}2^{k-3}/2^{k-1}} = e^{\pi\mathbf{i}/2} = \mathbf{i}$. Now, by Theorem 3.8, W_k has an eigenvalue

$$\begin{aligned}\left|(\omega^t)^{2^0} + (\omega^t)^{2^1} + \cdots + (\omega^t)^{2^{k-1}}\right| &= \left|\mathbf{i} + \mathbf{i}^2 + \mathbf{i}^4 + \cdots + \mathbf{i}^{2^{k-1}}\right| \\ &= |\mathbf{i} + (-1) + 1 + \cdots + 1| \\ &= |\mathbf{i} + (k-3)| \\ &= \sqrt{1^2 + (k-3)^2} \\ &= \sqrt{k^2 - 6k + 10}.\end{aligned}$$

□

Corollary 3.13. *For $k \geq 4$, W_k is not an integral graph. That is, not all its eigenvalues are integers.*

Proof. For $k \geq 4$, $\sqrt{k^2-6k+10}$ is never an integer. □

Corollary 3.14. *If $k \geq 4$, the Knödel graph W_k and the hypercube H_k are not isomorphic.*

Proof. For simple graphs G and H with $\mathrm{Sp}(G) = \{a_1, a_2, \ldots, a_p\}$ and $\mathrm{Sp}(H) = \{b_1, b_2, \ldots, b_q\}$, $\mathrm{Sp}(G \,\square\, H) = \{\, a_i + b_j : 1 \leq i \leq p,\ 1 \leq j \leq q \,\}$, where $\square$ stands for the Cartesian product [1]. As $\mathrm{Sp}(K_2) = \{-1, 1\}$, $\mathrm{Sp}(K_2 \,\square\, K_2) = \{2, 0^{(2)}, -2\}$, and by induction and the fact that $\square$ is associative, we find that $\mathrm{Sp}(H_k) = \{-k, -(k-2), -(k-4), \ldots, (k-2), k\}$ with respective multiplicities ${}^kC_0$, ${}^kC_1, \ldots,$ ${}^kC_k$, and so $\mathrm{Sp}(H_k)$ is integral for all k. However, by Corollary 3.13, $\mathrm{Sp}(W_k)$ is never integral. □

Note that, for $k = 1, 2, 3$, $W_k \cong H_k$, since $W_1 \cong H_1 = K_2$, $W_2 \cong H_2 = K_2 \,\Box\, K_2 = C_4$, $W_3 \cong H_3 = K_2 \,\Box\, K_2 \,\Box\, K_2$. Finally, recall that the number of spanning trees $\tau(G)$ of a k-regular connected graph G of order n can be computed by the formula using the spectrum [24].

$$\tau(G) = \frac{1}{n} \prod_{\lambda \in \mathrm{Sp}(G)\setminus\{k\}} (k - \lambda)$$

Using this formula, Harutyunyan and Morosan [15] gave an upper bound

$$\tau(W_k) \leq \frac{1}{2^{k-1}} k^{2^k - 1}.$$

Using Corollary 3.10, we have

$$\begin{aligned} &\tau(W_k) \\ =&\frac{1}{2^k}[(k-(-k)]\left[(k^2-(k-2)^2\right] \\ &\prod_{1\leq t\leq 2^{k-2}-1}\left[k^2-\left|(\omega^t)^{2^0}+(\omega^t)^{2^1}+\cdots+(\omega^t)^{2^{k-1}}\right|^2\right]^2 \\ =&\frac{1}{2^k}[2k][4(k-1)]\prod_{1\leq t\leq 2^{k-2}-1}\left[k^2-\left|(\omega^t)^{2^0}+(\omega^t)^{2^1}+\cdots+(\omega^t)^{2^{k-1}}\right|^2\right]^2 \end{aligned}$$

Hence we obtain a better upper bound than the one in Harutyunyan and Morosan [15]. Moreover we also give a new lower bound for $\tau(W_k)$ by using Lemma 3.11.

Theorem 3.15. *For $k \geq 2$, $k(k-1)^{2^{k-1}-1} 2^{2^k-k-1} \leq \tau(W_k) \leq \frac{1}{2^{k-3}}(k-1)k^{2^k-3}$.*

Acknowledgments

The authors thank the referees for their careful reading and bringing some more references to our notice. A part of this work was done when the first three authors were visiting Reva University, Bengaluru. The authors thank Reva University for its hospitality.

References

[1] R. Balakrishnan and K. Ranganathan, *A Textbook of Graph Theory*, 2nd ed., Springer, 2012.

[2] J.-C. Bermond, H. A. Harutyunyan, A. L. Liestman and S. Perennes, A note on the dimensionality of modified Knödel graphs, *Internat. J. Found. Comput. Sci.* 8 (1997), 109–116.

[3] G. Fertin and A. Raspaud, A survey on Knödel graphs, *Discrete Appl. Math.* 137 (2004), 173–195

[4] G. Fertin, A. Raspaud, H. Schröder, O. Sýkora and I. Vrťo, Diameter of the Knödel Graph, in: *Proc. 26th Int. Workshop on Graph-Theoretic Concepts in Comp. Sci.* (WG 2000), (Eds.: U. Brandes and D. Wagner), vol. 1928 of Lec. Notes in Comp. Sci., Springer, Berlin, 2000, pp. 149–160.

[5] P. Fraigniaud and E. Lazard, Methods and problems of communication in usual networks, *Discrete Appl. Math.* 53 (1994), 79–133.

[6] G. Gauyacq, *Routages Uniformes dans les Graphes Sommet-transitifs*, Ph.D. Thesis, Université Bordeaux 1 (1995).

[7] C. Godsil and G. Royle, *Algebraic Graph Theory*, Springer-Verlag New York, 2001.

[8] H. Grigoryan and H. A. Harutyunyan, Tight bound on the diameter of the Knödel graph, in:*Combinatorial Algorithms*—24th Int. Workshop, IWOCA 2013, Rouen, France, 2013; Revised Selected Papers, 2013, pp. 206–215.

[9] H. Grigoryan and H. A. Harutyunyan, The shortest path problem in the Knödel graph, *J. Discrete Algorithms* 31 (2015), 40–47.

[10] H. A. Harutyunyan, Multiple message broadcasting in modified Knödel graph, in: SIROCCO 7, *Proc. 7th Int. Coll. Structural Information and Communication Complexity*, Laquila, Italy, 2000, pp. 157–165.

[11] H. A. Harutyunyan, Minimum multiple message Broadcast graphs, *Networks* 47 (2006), 218–224.

[12] H. A. Harutyunyan and Z. Li, Broadcast graphs using new dimensional broadcast schemes for Knödel graphs, in: *Algorithms and Discrete Appl. Math.* Third Int. Conf., CALDAM 2017, Sancoale, Goa, India, 2017, Proceedings, 2017, pp. 193–204.

[13] H. A. Harutyunyan and A. L. Liestman, Upper bounds on the broadcast function using minimum dominating sets, *Discrete Math.* 312 (2012), 2992–2996.

[14] H. A. Harutyunyan, A. L. Liestman, J. G. Peters and D. Richards, Broadcasting and Gossiping, in: *Handbook of Graph Theory*, 2nd ed., (Eds.: J. L. Gross, J. Yellen and P. Zhang), CRC Press, 2013, pp. 1477–1494.

[15] H. A. Harutyunyan and C. D. Morosan, The spectra of Knödel graphs, *Informatica* (Slovenia) 30 (2006), 295–299.

[16] H. A. Harutyunyan and C. D. Morosan, On the minimum path problem in Knödel graphs, *Networks* 50 (2007), 86–91.

[17] S. M. Hedetniemi, S. T. Hedetniemi and A. L. Liestman, A survey of gossiping and broadcasting in communication networks, *Networks* 18 (1988), 319–349.

[18] M.-C. Heydemann, N. Marlins and S. Pérennes, Cayley graphs with complete rotations, *Technical report*, Laboratoire de Recherche en Informatique (Orsay) (1997) TR–1155.

[19] J. Hromkovič, R. Klasing, B. Monien and R. Peine, Dissemination of Information in Interconnection Networks (Broadcasting and Gossiping), in: *Combinatorial Network Theory*, (Eds.: D.-Z. Du and D. Hsu), Kluwer Academic Publishers, 1996, pp.125–212.

[20] W. Knödel, New gossips and telephones, *Discrete Math.* 13 (1975), 95.

[21] R. Labahn, Some minimum gossiping graphs, *Networks* 23 (1993), 333–341.

[22] F. T. Leighton, *Introduction to parallel algorithms and architectures: arrays, trees, hypercubes*, Los Altos, CA, Morgan Kaufmann Publishers, 1992.

[23] D. W. Lewis, *Matrix Theory*, World Scientific, 1991.

[24] B. Mohar, The Laplacian spectrum of graphs, in: *Graph theory, Combinatorics, and Applications*, Vol. 2, (Eds.: Y. Alavi, G. Chartrand, O. Ollerman and A. Schwenk), Wiley, 1991, pp. 871–898.

[25] C. D. Morosan, On the number of broadcast schemes in networks, *Inform. Process. Lett.* 100 (2006), 188–193.

[26] J.-H. Park and K.-Y. Chwa, Recursive circulant: a new topology for multicomputer networks (extended abstract), in: *Proc. Int. Symp. Parallel Architectures, Algorithms and Networks* ISPAN94, Kanazawa, Japan, 1994, pp. 73–80.

[27] P. Paulraja and S. Sampathkumar, Hamilton cycle decompositions of Knödel graphs, *Discrete Math. Theoret. Comput. Sci.* 17 (2016), 263-284.

(Received 10 Aug 2017; revised 11 Dec 2018, 5 Mar 2019)