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# Stirling permutations containing a single pattern of length three

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**Abstract**

We derive explicit formulæ for the number of $k$-Stirling permutations containing a single occurrence of a single pattern of length three as well as expressions for the corresponding generating functions. Furthermore, asymptotic results for these numbers are given.

## 1 Introduction

Given an integer $k \in \mathbb{N}$, a *$k$-Stirling permutation* of order $n$ is a permutation of the multiset[1] $\{1^k, 2^k, \ldots, n^k\}$ such that, for each $i$, $1 \leq i \leq n$, the elements occurring between two occurrences of $i$ are at least $i$, or alternatively, that the elements occurring between two consecutive occurrences of $i$ are larger than $i$. As an example, the permutations 112233, 331221 and 221331 are 2-Stirling permutations of order 3, whereas the permutations 331212 and 321123 of the multiset $\{1^2, 2^2, 3^2\} = \{1, 1, 2, 2, 3, 3\}$ are not. Originally, Stirling permutations were introduced by Gessel and Stanley [8] for the instance $k = 2$ in the context of finding combinatorial interpretations of the coefficients of certain polynomials, where the Stirling numbers appear. Later, Park [19]

[1]Here and throughout this work we use in this context $j^k$ as the shorthand notation for $j$ appearing $k$ times consecutively: $j, \ldots, j$, for $k \geq 1$.

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considered the general case $k \geq 1$. In this work we denote the combinatorial family of $k$-Stirling permutations by $\mathcal{Q}_k$; note that $k = 2$ yields exactly Stirling permutations as defined by Gessel and Stanley [8], whereas $k = 1$ gives just ordinary permutations.

Recently there has been an increasing interest in the study of properties of $k$-Stirling permutations. A reason for that is that $k$-Stirling permutations often yield a natural interpretation of generalizations of quantities for ordinary permutations. See, e.g., the recent work [3, 14] dealing with generalizations of the classical Eulerian numbers and Eulerian polynomials, respectively, via interpretations of ascent statistics in $k$-Stirling permutations. Another example is the generalization of the classical correspondence between permutations and so-called binary increasing trees (i.e., labelled binary trees, where the label of each child node is larger than the label of the parent node) to a correspondence between $k$-Stirling permutations and $(k+1)$-ary increasing trees, which naturally yields combinatorial interpretations of generalizations of a variety of permutation statistics, see [10, 13].

The current work is devoted to pattern occurrence in $k$-Stirling permutations. The interest in such studies also stems from the question of what happens if the fruitful concept for ordinary permutations is lifted to permutations of multisets? Let us first give the well-known notion of a subsequence pattern occurrence for ordinary permutations. Let $\alpha = (\alpha_1, \dots, \alpha_m)$ and $\beta = (\beta_1, \dots, \beta_n)$ denote two sequences of numbers, then the sequence $\alpha$ is said to be contained in $\beta$ as a pattern if there is a subsequence $\beta_{i_1}, \dots, \beta_{i_m}$ of $\beta$, with $1 \leq i_1 < i_2 < \cdots < i_m \leq n$, which is order-isomorphic to $\alpha$, i.e., it holds $\alpha_p \leq \alpha_q$ if and only if $\beta_{i_p} \leq \beta_{i_q}$. If $\beta$ does not contain $\alpha$ one says that $\beta$ *avoids the pattern* $\alpha$, and if there are exactly $r$ subsequences of $\beta$ order-isomorphic to $\alpha$, we say that $\alpha$ is contained in $\beta$ $r$ times. There are quite a few studies devoted to ordinary permutations containing a given pattern a given number of times, see, e.g., [5, 16, 17, 18] and also the review [21]. There are a few works dealing with such questions for permutations of a multiset [1, 9, 12], closely related to $k$-Stirling permutations, and all those, in fact, deal with the avoidance of patterns of length three. In contrast, there is a huge literature for pattern avoidance on words: see for example [6, 22, 24], and the book [11] for many more pointers to the literature.

In this work we treat enumerative questions concerning $k$-Stirling permutations containing a pattern $\alpha \in \{123, 132, 213, 231, 312, 321\}$ once. As a consequence of the 212-avoidance property, using above definition any permutation pattern $\alpha$ in a $k$-Stirling permutation of order $n \geq 1$ occurs at least $k$-times. In what follows we introduce a new notion of pattern containment, which seems to be more appropriate when studying $k$-Stirling permutations and that is compatible with the case of $k = 1$ of ordinary permutations. Let $\mathcal{S}_\alpha = \mathcal{S}_\alpha(\sigma)$ denote the *value-set* of subsequences in $\sigma \in \mathcal{Q}_k$ order-isomorphic to $\alpha$. It consists of all subsequences $s = (s_1, \dots, s_{|\alpha|})$ of $\sigma$ order-isomorphic to $\alpha$; in other words $1 \leq s_i \leq n$, for $1 \leq i \leq n$, such that there exist $1 \leq i_1 < \cdots < i_{|\alpha|} \leq n$ with $\sigma_{i_\ell} = s_\ell$, $1 \leq \ell \leq n$. Then, we say that the pattern $\alpha$ is contained $r$ times in $\sigma$ if the cardinality of the value-set $\mathcal{S}_\alpha$ is equal to $r$, $|\mathcal{S}_\alpha| = r$. Thus, two occurrences of a pattern $\alpha$ will only be considered different if the sets of values of the elements in the corresponding subsequences differ. From here on and throughout this work, pattern containment is always used with respect to this definition and we denote by $\mathcal{Q}_k(\alpha; r)$ the family of $k$-Stirling permutations containing the pattern $\alpha$ exactly $r$times. As an example we consider the 3-Stirling permutation $\sigma = 555444113366631222$ of order 6: the value-set of the pattern 123 contains a single element due to the subsequence 136, $\mathcal{S}_{123} = \{(1, 3, 6)\}$, thus $\sigma \in \mathcal{Q}_3(123; 1)$; furthermore, the value-set of the pattern 312 is given by $\mathcal{S}_{312} = \{(3, 1, 2), (4, 1, 2), (4, 1, 3), (5, 1, 2), (5, 1, 3), (6, 1, 2)\}$ and thus has size 6, i.e., $\sigma \in \mathcal{Q}_3(312; 6)$; moreover, the value-set of the pattern 231 has size 8, thus $\sigma \in \mathcal{Q}_3(231; 8)$ and it is given by $\mathcal{S}_{231} = \{(3, 6, 1), (3, 6, 2), (4, 6, 1), (4, 6, 2), (4, 6, 3), (5, 6, 1), (5, 6, 2), (5, 6, 3)\}$.

Throughout this work we use the abbreviations $\mathcal{G} := \mathcal{G}_k(\alpha) = \mathcal{Q}_k(\alpha; 1)$ and $\mathcal{F} := \mathcal{F}_k(\alpha) = \mathcal{Q}_k(\alpha; 0)$ for the families of $k$-Stirling permutations with a single occurrence and without occurrence of a specified pattern $\alpha$, respectively. As main results we obtain explicit formulæ for the number $G_n = G_n^{[k]}(\alpha) = |\{\sigma \in \mathcal{G}_k(\alpha) \colon |\sigma| = n\}|$ of elements of order $n$ contained in $\mathcal{G}_k(\alpha)$, as well as expressions for the generating function $G(z) = \sum_{n\geq 0} G_n z^n$ in terms of the corresponding quantity $F(z) = \sum_{n\geq 0} F_n z^n$ for the family $\mathcal{F}_k(\alpha)$. Note that we often write $G_n$ or $G_n^{[k]}$ and drop the dependence on $k$ as well as $\alpha$ for the sake of simplicity. Table 1 summarizes our findings on $G_n$ and compares it with the known results for $k = 1$. For the readers convenience we also collect for ordinary permutations, $k = 1$, and Stirling permutations, $k = 2$, the corresponding sequences or their entry in the OEIS (if available).

| *Members* $\alpha$ *in class* | *Enumeration formula* $G_n = \lvert\{\sigma \in \mathcal{G}_k(\alpha) \colon \lvert\sigma\rvert = n\}\rvert$ | $G_n^{[1]}$ |
|---|---|---|
| 312, 213 | $\binom{kn+n-2k-1}{n-1} - \frac{1}{k(n-1)+1}\binom{(k+1)(n-1)}{n-1}$ | $\binom{2n-3}{n-3}$ |
| 231, 132 | $\sum_{j=0}^{n-1}\binom{n-1}{j}\Big[k^2\binom{n+(k-1)j+k-4}{n-j-4} + (k-1)\binom{n+(k-1)j+k-3}{n-j-2} + \binom{n+(k-1)j+k-2}{n-j-1} - (2k-1)\binom{n+(k-1)j-3}{n-j-2} - \binom{n+(k-1)j-2}{n-j-1}\Big]$ | $\binom{2n-3}{n-3}$ |
| 123, 321 | $\sum_{j=0}^{k}\binom{k}{j}(-1)^j\Big[(k-1)\sum_{\ell=0}^{n-j}\binom{n-j-1+(k-1)\ell}{n-j-\ell}\Big(\frac{2\binom{n-j+2}{\ell}}{n-j+2} - \frac{\binom{n-j+1}{\ell}}{n-j+1}\Big) + \sum_{\ell=0}^{n+1-j}\binom{n-j+(k-1)\ell}{n-j+1-\ell}\Big(\frac{2\binom{n+1-j+2}{\ell}}{n+1-j+2} - \frac{\binom{n+1-j+1}{\ell}}{n+1-j+1}\Big)\Big] + \frac{2(2-k)}{n+1}\sum_{\ell=0}^{n-1}\binom{n+1}{\ell}\binom{n-2+(k-1)\ell}{n-1-\ell} - \frac{4}{n+2}\sum_{\ell=0}^{n}\binom{n+2}{\ell}\binom{n-1+(k-1)\ell}{n-\ell} + \sum_{\ell=0}^{n}\frac{\binom{n}{\ell}\binom{n+(k-1)\ell-1}{n-\ell}}{n+1-\ell}$ | $\frac{3}{n}\binom{2n}{n-3}$ |

Table 1: $k$-Stirling permutations containing a single permutation pattern $\alpha$ of length 3.

The derivation of the results in Sections 3-5 somehow reflects the increasing "complexity" of the treatment of these patterns. For the pattern 312, a combinatorial de-

| $\alpha$ | $k=1$ | OEIS | $k=2$ | OEIS |
|---|---|---|---|---|
| 312, 213 | 0,0,1,3,10,35,126,462 | A001700 | 0,0,3,23,155,1014,6580 | not contained |
| 231, 132 | 0,0,1,3,10,35,126,462 | A001700 | 0,0,5,26,135,685,3453,17379 | not contained |
| 123, 321 | 0,0,1,6,27,110,429 | A003517 | 0,0,5,33,180,919,4560,22332 | not contained |

Table 2: $G_n^{[k]}$: OEIS entries and sequences for permutations $k = 1$ and Stirling permutations $k = 2$.

composition of $k$-Stirling permutations according to the $k$ occurrences of the smallest element 1 is successful. The same approach is feasible also for the pattern 231 but requires considerably more care. Finally, for the pattern 123, we must inspect the possible ways to insert the string $(n+1)^k$ into a $k$-Stirling permutation of order $n$; this analysis requires the so-called kernel method, which is detailed in [2, 20]. We are also interested in the asymptotic growth behaviour of these quantities and in describing the influence of the value $k$ on the occurrence of a pattern $\alpha$; thus, in Section 6 we also give asymptotic results. We note that at least in principle the approaches presented also work when studying families $\mathcal{Q}_k(\alpha; r)$, with $r \geq 2$, but even for the case $r = 2$ the computations are rather involved. In particular, so far we are not able to get results on the structure of the generating function for the number of elements of order $n$ in $\mathcal{Q}_k(\alpha; r)$, for general $r$, which have been found for ordinary permutations and the pattern 231 (or 312) in [4, 15], and we have to leave this for future research.

## 2 Preliminaries

We collect a few basic properties of the family $\mathcal{Q}_k$. The number $Q_n := Q_n^{[k]}$ of different $k$-Stirling permutations of order $n$ is given by the following enumeration formula:

$$Q_n = \prod_{j=1}^{n} \bigl(1 + (j-1)k\bigr) = n!\, k^n \binom{n-1+\frac{1}{k}}{n}. \qquad (1)$$

Note that it is sometimes convenient to allow also $n = 0$, i.e., the empty sequence, with $Q_0 = 1$. This result can be shown by induction in a straightforward manner, since, due to the 212-avoidance property, the $k$ copies of $n+1$ have to form a substring and thus each such $k$-Stirling permutation can be obtained uniquely by inserting the string $(n+1)^k$ into a $k$-Stirling permutation of the multiset $\{1^k, 2^k, \ldots, (n-1)^k, n^k\}$ at one of the $1+n\cdot k$ possible positions, i.e., anywhere in the string, including the first or last position. This does not only show the above enumeration formula, but also gives a *simple recursive algorithm* to generate all $k$-Stirling permutations of a given order. This algorithm, in a fittingly adapted form, will be used in Section 5 for the pattern 123 to generate such pattern-restricted $k$-Stirling permutations of arbitrary order, and further, to derive functional equations for suitably defined generating functions.

Another important basic property is the combinatorial decomposition of the family $\mathcal{Q}_k$ with respect to the $k$ occurrences of the smallest label 1: namely, let $\sigma \in \mathcal{Q}_k$ of order $n \geq 1$, then it holds $\sigma = S_1 1 S_2 1 S_3 \dots 1 S_{k+1}$, with (possibly empty) substrings $S_1, \dots, S_{k+1}$. Moreover, it must hold that these substrings do not contain common labels, i.e., $S_i \cap S_j = \emptyset$, for $i \neq j$ (where we assume $S_i$, $S_j$ as multisets), otherwise the property of avoiding the pattern 212 would be violated. Thus, each substring $S_i$, $1 \leq i \leq k+1$, is, after an order-preserving relabelling, itself a member of $\mathcal{Q}_k$. This immediately yields the following formal description of the family $\mathcal{Q} := \mathcal{Q}_k$:

$$\mathcal{Q} = \{\epsilon\} + \mathcal{Z}^{\square} * \mathcal{Q}^{k+1},$$

where $\epsilon$ denotes the empty string, the so-called atomic element $\mathcal{Z}$ corresponds to a multiset $j^k$ of a label, $+$ and $*$ are the disjoint union and the partition product of labelled object families, respectively. Furthermore, $\mathcal{A}^{\square} * \mathcal{B}$ denotes the so-called boxed product of families $\mathcal{A}$ and $\mathcal{B}$ meaning that the smallest label is constrained to lie in $\mathcal{A}$, see [7]. Thus, by a straightforward application of the symbolic method (see also [7]), one immediately gets that the exponential generating function $Q(z) := \sum_{n\geq 0} Q_n \frac{z^n}{n!}$ satisfies the differential equation $Q'(z) = Q(z)^{k+1}$, from which the enumeration result (1) follows.

When considering pattern containment in the family $\mathcal{Q}$, it is a trivial fact that the number of elements in $\mathcal{Q}_k(\alpha; r)$ of order $n$ is equal to the number of elements in $\mathcal{Q}_k(\alpha', r)$ of order $n$, where $\alpha'$ denotes the reversal of the pattern $\alpha$. Thus, we can indeed restrict ourselves to consider the patterns 123, 312 and 231. We conclude this section with two remarks on notation. It is convenient to write $A_1 \prec A_2 \prec \cdots \prec A_q$, for (possibly empty) strings $A_1, \dots, A_q$, if each label contained in $A_i$ is smaller than every label contained in $A_j$, for $i < j$. When decomposing a $k$-Stirling permutation $\sigma$ we use the capital letters $S_1$, $S_2$ etc.; similarly, we use capital letters $T$ when decomposing $\tau$.

## 3 Containing the pattern 312 only once

### 3.1 Avoiding the pattern 312

We consider the combinatorial family $\mathcal{G} := \mathcal{G}_k(312)$ of $k$-Stirling permutations with a single occurrence of the pattern 312 and treat the enumeration problem for it. Let $G_n$ denote the number of $k$-Stirling permutations of order $n$ with a single occurrence of the pattern 312 and $G(z) := \sum_{n\geq 0} G_n z^n$ its generating function. We will compute formulæ for $G_n$ and $G(z)$ by a combinatorial decomposition of these objects with respect to the smallest label and establishing relations to $k$-Stirling permutations avoiding the pattern 312. Therefore, let us introduce the family $\mathcal{F} := \mathcal{F}_k(312)$ of 312-avoiding $k$-Stirling permutations, the number $F_n$ of 312-avoiding $k$-Stirling permutations of order $n$, and its generating function $F(z) := \sum_{n\geq 0} F_n z^n$. It has been proven in [12] via relations to increasing trees that $F_n$ is enumerated by the generalized Catalan numbers. However, in order to get a link between the families

$\mathcal{G}$ and $\mathcal{F}$ we will first give a more direct proof of the results for $\mathcal{F}$, which afterwards will be extended.

Let $\sigma$ be a 312-avoiding $k$-Stirling permutation of order $n \geq 1$ and consider its decomposition with respect to the $k$ occurrences of the smallest label 1: $\sigma = S_1 1 S_2 1 S_3 \dots 1 S_{k+1}$, with (possibly empty) substrings $S_1, \dots, S_{k+1}$, which are, after order-preserving relabellings, themselves 312-avoiding $k$-Stirling permutations. Furthermore, due to the property of avoiding the pattern 312 it must hold $1 \prec S_1 \prec S_2 \prec \cdots \prec S_{k+1}$. This immediately yields the following formal equation for the family $\mathcal{F}$:

$$\mathcal{F} = \{\epsilon\} + \mathcal{Z} \times \mathcal{F}^{k+1},$$

where $+$ and $\times$ denote the disjoint union and the Cartesian product of object families, respectively. Thus, by a straightforward application of the symbolic method, we get the functional equation

$$F(z) = 1 + zF(z)^{k+1} \tag{2}$$

for the generating function $F(z)$. Extracting coefficients, $F_n = [z^n]F(z)$, can be done easily by applying the Lagrange-Bürmann inversion theorem, see [23], and yields $F_n = \frac{1}{kn+1}\binom{(k+1)n}{n}$, i.e., $F_n$ is given by the generalized Catalan numbers.

### 3.2 Single occurrence of the pattern 312

Now let us consider a $k$-Stirling permutation $\tau$ with a single occurrence of the pattern 312. Again we consider the decomposition of $\tau$ with respect to the smallest label 1: $\tau = T_1 1 T_2 1 T_3 \dots 1 T_{k+1}$, with $T_1, \dots, T_{k+1}$ (possibly empty) substrings, which are (after order-preserving relabellings) themselves $k$-Stirling permutations. We distinguish two cases according to the labels $\ell m r$ forming the 312-pattern of $\tau$:

- The 312-pattern $\ell m r$ does not contain label 1, i.e., $m \neq 1$. Then the substrings $T_1, \dots, T_{k+1}$ must satisfy the following properties:
  - The subsequence $\ell m r$ must be contained entirely in a substring $T_j$, $1 \leq j \leq k+1$, otherwise, if $\ell \in T_i$, $r \in T_j$, with $i < j$, then $\ell 1 r$ would be another occurrence of the pattern 312.
  - The substrings must satisfy $1 \prec T_1 \prec T_2 \prec \cdots \prec T_{k+1}$, since otherwise, if there were $\ell' \in T_i$, $r' \in T_j$, with $i < j$ and $\ell' > r'$, then $\ell' 1 r'$ would be another occurrence of the pattern 312.
  - After order-preserving relabellings, $T_j$ is itself a $k$-Stirling permutation with a single occurrence of the pattern 312, whereas $T_1, \dots, T_{j-1}, T_{j+1}, \dots, T_{k+1}$ are 312-avoiding $k$-Stirling permutations.

  Thus, this object family $\mathcal{G}^{[0]}$, where the occurrence of the 312-pattern does not include the letter 1, can be described in the following formal way:

$$\mathcal{G}^{[0]} = (k+1) \cdot \mathcal{G} \times \mathcal{Z} \times \mathcal{F}^k. \tag{3}$$

- The 312-pattern $\ell m r$ contains label 1, i.e., $m = 1$. Let us assume that $\ell \in T_i$ and $r \in T_j$, with $i < j$. Then the following properties must be satisfied:
  - $T_{i+1}, \ldots, T_{j-1} = \emptyset$, otherwise there would exist $\ell' \in T_q$, with $i+1 \leq q \leq j-1$, such that either $\ell' > \ell$, which causes due to $\ell' 1 r$ a further occurrence of the pattern 312, or $\ell' < \ell$, which causes due to $\ell 1 \ell'$ a further occurrence of the pattern 312.
  - $\ell = r + 1$, otherwise if $\ell > r + 1$ then label $r + 1$ would be contained in a substring $T_q$ and then either $q \leq i$, which causes due to $(r+1)1r$ a further occurrence of the pattern 312, or $q \geq j$, which causes due to $\ell 1 (r+1)$ a further occurrence of the pattern 312.
  - Define $T_i' := T_i - \{\ell^k\}$; then it must hold that $T_1 \prec \cdots \prec T_{i-1} \prec T_i' \prec T_{i+1} \prec \cdots \prec T_{k+1}$, since otherwise another 312-pattern would occur.
  - $T_i = T_i' \ell^k$, otherwise there would be an element $m' < \ell$ and consequently $m' < r = \ell - 1$ to the right of $\ell$, which causes due to $\ell m' r$ a further occurrence of the pattern 312.
  - After order-preserving relabellings, $T_1, \ldots, T_{i-1}, T_i', T_{j+1}, \ldots, T_{k+1}$ are 312-avoiding $k$-Stirling permutations and $T_j \neq \emptyset$ is a non-empty 312-avoiding $k$-Stirling permutation.

  Let $\mathcal{G}^{[1]}$ denote the family, where the occurrence of the 312-pattern includes the letter 1. According to our considerations it can be described in the following formal way:

$$\mathcal{G}^{[1]} = \sum_{1 \leq i < j \leq k+1} \mathcal{Z} \times \mathcal{Z} \times \mathcal{F}^{k+1-j+i} \times (\mathcal{F} - \{\epsilon\}). \tag{4}$$

  Here, the atoms $\mathcal{Z}$ correspond to label 1 and label $\ell$ of $T_i = T_i' \ell^k$. Since all parts $T_1, \ldots, T_{i-1}$, $T_i'$, as well as $T_{j+1}, \ldots, T_{k+1}$ are 312-avoiding $k$-Stirling permutations, we get the corresponding term $\mathcal{F}^{i+k+2-(j+1)} = \mathcal{F}^{k+1-j+i}$. Moreover, since $T_j \neq \emptyset$ is a non-empty 312-avoiding $k$-Stirling permutation, we get the additional combinatorial family $\mathcal{F} - \{\epsilon\}$.

Combining the two cases (3) and (4) via $\mathcal{G} = \mathcal{G}^{[0]} + \mathcal{G}^{[1]}$, one obtains after straightforward simplifications the following symbolic equation connecting the families $\mathcal{G}$ and $\mathcal{F}$:

$$\mathcal{G} = (k+1) \cdot \mathcal{G} \times \mathcal{Z} \times \mathcal{F}^k + \mathcal{Z}^2 \times (\mathcal{F} - \{\epsilon\}) \times \sum_{p=1}^{k} p \cdot \mathcal{F}^p. \tag{5}$$

Thus, applying the symbolic method to (5) yields the following functional equation relating the generating functions $G(z)$ and $F(z)$:

$$G(z) = (k+1) z G(z) F(z)^k + z^2 (F(z) - 1) \sum_{p=1}^{k} p F(z)^p,$$

or

$$G(z) = \frac{z^2(F(z)-1)\sum_{p=1}^{k} pF(z)^p}{1-(k+1)zF(z)^k} = \frac{z^2F(z)^2}{(F(z)-1)(1-k(F(z)-1))} - zF(z), \quad (6)$$

where the last expression follows after simple manipulations using the functional equation (2).

Extracting coefficients from (6) yields an explicit formula for $G_n = [z^n]G(z)$. For the first summand we introduce $\hat{F}(z) := F(z) - 1$, which thus satisfies the functional equation $\hat{F} = z(1+\hat{F})^{k+1}$, and apply Cauchy's integration formula [7]. Using $\frac{d\hat{F}}{dz} = \frac{(1+\hat{F})^{k+2}}{1-k\hat{F}}$, this gives

$$\begin{aligned}
[z^n]\frac{z^2F(z)^2}{(F(z)-1)(1-k(F(z)-1))} &= \frac{1}{2\pi i}\oint \frac{(1+\hat{F})^2}{\hat{F}(1-k\hat{F})}\frac{dz}{z^{n-1}} \\
&= \frac{1}{2\pi i}\oint \frac{(1+\hat{F})^2}{\hat{F}(1-k\hat{F})}\frac{(1+\hat{F})^{(k+1)(n-1)}}{\hat{F}^{n-1}}\frac{1-k\hat{F}}{(1+\hat{F})^{k+2}}d\hat{F} = \frac{1}{2\pi i}\oint \frac{(1+\hat{F})^{kn+n-2k-1}}{\hat{F}^n}d\hat{F} \\
&= [\hat{F}^{n-1}](1+\hat{F})^{kn+n-2k-1} = \binom{kn+n-2k-1}{n-1}.
\end{aligned}$$

Furthermore, the second summand of (6) simply yields

$$[z^n]zF(z) = F_{n-1} = \frac{1}{k(n-1)+1}\binom{(k+1)(n-1)}{n-1}.$$

Combining our findings leads to the following theorem.

**Theorem 1** *The generating function $G(z) = \sum_{n\ge 0} G_n z^n$ of the number $G_n := G_n^{[k]}$ of $k$-Stirling permutations of order $n$ with a single occurrence of the pattern 312 is given as follows:*

$$G(z) = \frac{z^2F(z)^2}{(F(z)-1)(1-k(F(z)-1))} - zF(z),$$

*with $F(z) = \sum_{n\ge 0} F_n z^n$ the generating function of the generalized Catalan numbers $F_n := F_n^{[k]} = \frac{1}{kn+1}\binom{(k+1)n}{n}$, i.e., the number of 312-avoiding $k$-Stirling permutations of order $n$, which satisfies the functional equation $F = 1 + zF^{k+1}$.*

# 4 Containing the pattern 231 only once

## 4.1 Avoiding the pattern 231

In this section we consider the combinatorial family $\mathcal{G} := \mathcal{G}_k(231)$ of $k$-Stirling permutations with a single occurrence of the pattern 231. Let $G_n$ denote the number of $k$-Stirling permutations of order $n$ with a single occurrence of the pattern 231 and

$G(z) := \sum_{n\ge0} G_n z^n$ its generating function. Again we will show relations to the family $\mathcal{F} := \mathcal{F}_k(231)$ of 231-avoiding $k$-Stirling permutations and, by using formulæ for the number $F_n$ of 231-avoiding $k$-Stirling permutations and its generating function $F(z) := \sum_{n\ge0} F_n z^n$, we are able to show enumeration results also for the family $\mathcal{G}$. Explicit results for $F_n$ and $F(z)$ (as pointed out before, they are different from the corresponding formulæ for 312-avoiding $k$-Stirling permutations) are obtained in [12] by what is called there component block decomposition of $k$-Stirling permutations. However, in order to show relations between the families $\mathcal{F}$ and $\mathcal{G}$ it seems preferable to use the decomposition of these objects with respect to the smallest label 1. Thus, first we will use this decomposition to reprove the results for $\mathcal{F}$ and afterwards we extend the considerations to cover the family $\mathcal{G}$.

Let $\sigma$ be a 231-avoiding $k$-Stirling permutation of order $n \ge 1$ and consider its decomposition with respect to the $k$ occurrences of the smallest label 1, which we denote by

$$\sigma = S_k 1 S_{k-1} 1 \cdots S_1 1 S,$$

with (possibly empty) substrings $S_1, \dots, S_k$ and $S$, which are themselves (after order-preserving relabellings) $k$-Stirling permutations. Due to the property of avoiding the pattern 231 it must hold that the concatenated string $A := S_k S_{k-1} \dots S_1$ is forming a non-increasing sequence. Thus, there are $q_1, \dots, q_k \ge 0$, denoting the number of different labels contained in $S_1, \dots, S_k$, respectively, and labels $1 < a_1 < a_2 < \cdots < a_{q_1+q_2+\cdots+q_k}$, such that

$$S_j = a^k_{q_1+\cdots+q_j} a^k_{q_1+\cdots+q_j-1} \dots a^k_{q_1+\cdots+q_{j-1}+1}, \quad \text{for } 1 \le j \le k.$$

Furthermore, each element $s' > a_i$ in the substring $S$ must be to the right of each element $s'' < a_i$ in $S$, since otherwise $a_i s' s''$ would cause an occurrence of the pattern 231. Thus, it holds that $S$ can be decomposed itself: $S = C_0 C_1 \dots C_{q_1+\cdots+q_k}$, with substrings $C_0, \dots, C_{q_1+\cdots+q_k}$ satisfying $1 \prec C_0 \prec a_1 \prec C_1 \prec a_2 \prec C_2 \prec \cdots \prec a_{q_1+\cdots+q_k} \prec C_{q_1+\cdots+q_k}$. Moreover, the substrings $C_0, \dots, C_{q_1+\cdots+q_k}$ must be (after order-preserving relabellings) themselves 231-avoiding $k$-Stirling permutations. Thus, apart from 1 and $C_0$, $\sigma$ is decomposed into $k$ sequences of pairs consisting of an element $a_i$ and a 231-avoiding $k$-Stirling permutation $C_i$, which immediately yields the following formal equation for the family $\mathcal{F}$:

$$\mathcal{F} = \{\epsilon\} + \mathcal{Z} \times \mathcal{F} \times \big(\textsc{Seq}(\mathcal{Z} \times \mathcal{F})\big)^k,$$

with $\textsc{Seq}$ denoting the sequence construction for a family of combinatorial objects. An application of the symbolic method to this equation immediately gives that the generating function $F(z)$ satisfies the following functional equation:

$$F(z) = 1 + \frac{zF(z)}{(1 - zF(z))^k}. \tag{7}$$

To extract coefficients one may set $\hat{F}(z) := zF(z)$ and apply the Lagrange-Bürmann inversion theorem, which gives after routine calculations the explicit formula for $F_n$

stated already in [12]:

$$F_n = \frac{1}{n+1}\sum_{j=0}^{n}\binom{n+1}{j}\binom{n+(k-1)j-1}{n-j}. \tag{8}$$

### 4.2 Single occurrence of the pattern 231

Now we treat the family $\mathcal{G}$ and consider a $k$-Stirling permutation $\tau$ with a single occurrence of the pattern 231, where we assume that this pattern is caused by the subsequence $\ell m r$. The decomposition of $\tau$ with respect to the smallest label 1 can be written as

$$\tau = T_k 1 T_{k-1} 1 \cdots T_1 1 T,$$

by $T_1, \dots, T_k$ and $T$ (possibly empty) substrings which are (after order-preserving relabellings) themselves $k$-Stirling permutations. We start our examinations with the following two observations:

- $\ell = r+1$, since otherwise, i.e., if $\ell > r+1$, label $(r+1)$ would occur in a further 231-pattern, namely, either $(r+1)$ is to the left of $m$, then $(r+1)mr$ causes a 231-pattern, or $(r+1)$ is to the right of $m$, then $\ell m(r+1)$ causes a 231-pattern.
- The subsequence $\ell m r = (r+1)mr$ is not contained entirely in the concatenated string $A := T_k T_{k-1} \dots T_1$, since otherwise $(r+1)m1$ would cause a second occurrence of the pattern 231.

Let us assume that the substrings $T_1, \dots, T_k$ contain $q_1, \dots, q_k \geq 0$ different labels, respectively, and thus that the string $A$, when considered as a multiset, is given as $A = \{a_1^k, a_2^k, \dots, a_{q_1+\cdots+q_k}^k\}$, with $1 < a_1 < a_2 \cdots < a_{q_1+\cdots+a_k}$. When we distinguish according to the position of the labels $(r+1)mr$ forming the 231-pattern of $\tau$, there are three possible cases, which will be treated separately.

- The 231-pattern $(r+1)mr$ is contained in $T$, i.e., $(r+1), m, r \in T$. Then the strings $A$ and $T$ must satisfy the following properties:
  - The elements in $A$ form a non-increasing sequence, since otherwise, if there were $a', a'' \in A$, with $a' > a''$ and $a'$ left of $a''$, due to $a'a''1$ a further 231-pattern would occur. Thus the following must hold

    $$T_j = a_{q_1+\cdots+q_j}^k a_{q_1+\cdots+q_j-1}^k \dots a_{q_1+\cdots+q_{j-1}+1}^k, \quad \text{for } 1 \leq j \leq k.$$

  - It holds that each element $t' > a_i$ in the substring $T$ must be to the right of each element $t'' < a_i$ in $T$, since otherwise $a_i t' t''$ would cause a further 231-pattern occurrence. Thus $T$ can be decomposed itself: $T = C_0 C_1 \dots C_{q_1+\cdots+q_k}$, where the substrings $C_j$ are themselves (after order-preserving relabellings) $k$-Stirling permutations satisfying $1 \prec C_0 \prec a_1 \prec C_1 \prec a_2 \prec C_2 \prec \cdots \prec a_{q_1+\cdots+q_k} \prec C_{q_1+\cdots+q_k}$.

- After order-preserving relabellings, exactly one of these substrings, let us assume $C_q$, is a $k$-Stirling permutation with a single occurrence of the pattern 231, whereas all remaining substrings $C_0, \dots, C_{q-1}, C_{q+1}, \dots, C_{q_1+\cdots+q_k}$ are 231-avoiding $k$-Stirling permutations.

From these properties a combinatorial description of the family of $k$-Stirling permutations, where the single occurrence of the 231-pattern is contained in $T$ can be obtained. We denote this family in the following by $\mathcal{G}^{[1]}$. Either $(r+1)mr \in C_0$, which means that, apart from label 1 and $C_0$, the string $\tau$ is decomposed into $k$ sequences of pairs consisting of a label $a_i$ and a 231-avoiding $k$-Stirling permutation $C_i$, which can be described via $\mathcal{Z} \times \mathcal{G} \times \big(\textsc{Seq}(\mathcal{Z} \times \mathcal{F})\big)^k$; or otherwise, if $(r+1)mr \in C_q$, with $q > 0$, then $a_q$ is contained in a substring $T_j$, with $1 \le j \le k$; then we may consider $T_j$ as $T_j = T_j^{[L]} a_q^k T_j^{[R]}$, with $T_j^{[L]}$ and $T_j^{[R]}$ non-increasing sequences of labels. Thus, apart from the elements 1, $C_0$, $a_q$ and $C_q$, the string $\tau$ is decomposed into $k+1$ sequences of pairs consisting of a label $a_i$ and a $k$-Stirling permutation $C_i$. Taking into account that $a_q$ can be contained in one of the $k$ substrings $T_j$, these objects can be described formally via $k \cdot \mathcal{Z} \times \mathcal{Z} \times \mathcal{F} \times \mathcal{G} \times \big(\textsc{Seq}(\mathcal{Z} \times \mathcal{F})\big)^{k+1}$. Thus, for the family $\mathcal{G}^{[1]}$ we get the formal description

$$\mathcal{G}^{[1]} = \mathcal{Z} \times \mathcal{G} \times \big(\textsc{Seq}(\mathcal{Z} \times \mathcal{F})\big)^k + k \cdot \mathcal{Z}^2 \times \mathcal{F} \times \mathcal{G} \times \big(\textsc{Seq}(\mathcal{Z} \times \mathcal{F})\big)^{k+1}. \quad (9)$$

- The 231-pattern $(r+1)mr$ is contained in $AT$, but not in $T$, i.e., $(r+1) \in A$ and $r \in T$. This case implies that also $m \in T$, since otherwise, if $m$ would be contained in $A$, $(r+1)m1$ would cause a further occurrence of the 231-pattern. In the following, let us assume that $(r+1) = a_q \in A$, such that $r = (a_q - 1) \in T$. The structure of the strings $A$ and $T$ will be revealed via the following properties, which have to be satisfied.

  - The elements in $A$ form a non-increasing sequence, since otherwise a further 231-pattern would occur. Thus it must hold that

  $$T_j = a^k_{q_1+\cdots+q_j} a^k_{q_1+\cdots+q_j-1} \dots a^k_{q_1+\cdots+q_{j-1}+1}, \quad \text{for } 1 \le j \le k.$$

  - Let $i \neq q$, then it follows that each element $t' > a_i$ in the substring $T$ must be to the right of each element $t'' < a_i$ in $T$, since otherwise $a_i t' t''$ would cause a further 231-pattern occurrence. Thus $T$ can be decomposed itself: $T = C_0 C_1 \dots C_{q-2} D C_{q+1} C_{q+2} \dots C_{q_1+\cdots+q_k}$, where the substrings $C_j$ and $D$ are themselves (after order-preserving relabellings) $k$-Stirling permutations satisfying $1 \prec C_0 \prec a_1 \prec C_1 \prec a_2 \prec \cdots \prec C_{q-2} \prec a_{q-1} \prec D \prec a_{q+1} \prec C_{q+1} \prec \cdots \prec a_{q_1+\cdots+q_k} \prec C_{q_1+\cdots+q_k}$, where $D$ contains all elements between $a_{q-1}$ and $a_{q+1}$ other than $a_q$. In particular it holds $(a_q - 1) \in D$.
  - $m < a_{q+1}$, since otherwise $a_{q+1} m (a_q - 1)$ would give a further occurrence of the pattern 231. Thus, it holds $m \in D$.

- Each element $d \in D$ with $d < (a_q - 1)$ must be left to $m$, since otherwise $a_q m d$ would give a further 231-pattern occurrence. As a consequence, each such $d < (a_q - 1)$ must be also left to all occurrences of $(a_q - 1)$, since due to the 212-pattern avoidance property of $k$-Stirling permutations, $d$ could be either to the left or to the right of all occurrences of $(a_q - 1)$, but the latter case would cause a further 231-pattern due to $a_q m d$.
- Each element $d \in D$ with $d > (a_q - 1)$ (and thus $d > a_q$) must be to the right of $(a_q - 1)$, since otherwise $a_q d (a_q - 1)$ would give a further 231-pattern occurrence.
- Each element $d' \in D$ with $d' > m$ must be to the right of each element $d'' \in D$ with $d'' < m$, since otherwise $m d' d''$ would give a further 231-pattern occurrence.
- At least one occurrence of $(a_q - 1)$ must be to the right of (necessarily all occurrences of) $m$, since we assumed that these elements cause the single occurrence of the 231-pattern. From the previous properties it follows that $D$ can itself be decomposed into $D = C_{q-1} P C_q^{[1]} C_q^{[2]}$, with $a_{q-1} \prec C_{q-1} \prec (a_q - 1) \prec a_q \prec C_q^{[1]} \prec m \prec C_q^{[2]} \prec a_{q+1}$, where $P$ could be one of the following $k$ strings:

$$P = m^k (a_q - 1)^k \text{ or } P = (a_q - 1)^t m^k (a_q - 1)^{k-t}, \text{ with } 1 \le t \le k - 1.$$

- All the substrings $C_0, \dots, C_{q-1}, C_q^{[1]}, C_q^{[2]}, C_{q+1}, \dots, C_{q_1 + \dots + q_k}$ are (after order-preserving relabellings) 231-avoiding $k$-Stirling permutations.

This decomposition can be used to give a combinatorial description of the family of $k$-Stirling permutations $\mathcal{G}^{[2]}$, where the single occurrence of the 231-pattern is contained in $AT$, but not in $T$. Let us assume that $a_q$ is contained in the substring $T_j$, with $1 \le j \le k$. Then we may consider $T_j$ as $T_j = T_j^{[L]} a_q^k T_j^{[R]}$, with $T_j^{[L]}$ and $T_j^{[R]}$ non-increasing sequences of labels. Thus, the string $\tau$ is decomposed into the special labels 1, $a_q$, $(a_q - 1)$ and $m$ and into the special substrings of 231-avoiding $k$-Stirling permutations $C_0$, $C_q^{[1]}$ and $C_q^{[2]}$, as well as into $k+1$ sequences of pairs consisting of $a_i$ contained in $T_1, \dots, T_{j-1}, T_j^{[L]}, T_j^{[R]}, T_{j+1}, \dots, T_k$, respectively, and the corresponding substring $C_i$. Taking into account the $k$ possibilities for the string $P$ and independent from it the $k$ possibilities for the substring $T_j$ containing $(r+1) = a_q$, we get the following formal description of $\mathcal{G}^{[2]}$:

$$\mathcal{G}^{[2]} = k^2 \cdot \mathcal{Z}^4 \times \mathcal{F}^3 \times \big(\textsc{Seq}(\mathcal{Z} \times \mathcal{F})\big)^{k+1}. \tag{10}$$

- The 231-pattern $(r+1)mr$ is contained in $A1$, i.e., $r = 1$, $(r+1) = 2 = a_1 \in A$ and $m > 2 \in A$. Then the strings $A$ and $T$ must satisfy the following properties:
  - $m = a_2$, since otherwise, if $m = a_i$, with $i > 2$, this would cause a further occurrence of the 231-pattern, namely, either due to $a_2 m 1$ if $a_2$ is left to 2, or due to $2 a_2 1$ if $a_2$ is to the right of 2.

- The elements $a_3, \dots, a_{q_1+\dots+q_k}$ are left to 2, since otherwise a further occurrence of the 231-pattern would appear due to $2a_i1$.
- The elements $a_3, \dots, a_{q_1+\dots+a_q}$ form a non-increasing sequence, since otherwise a further 231-pattern would occur.
- At least one occurrence of 2 is left to (necessarily all occurrences of) $a_2$, since it is assumed that $2a_21$ is forming the 231-pattern. Due to these properties the string $A$ has the following form: $A = a^k_{q_1+\dots+q_k} a^k_{q_1+\dots+a_k-1} \dots a^k_4 a^k_3 P$, with $P$ one of the following $k$ strings:

$$P = 2^k a_2^k \quad \text{or} \quad P = 2^t a_2^k 2^{k-t}, \text{ with } 1 \le t \le k-1.$$

- In $T$ each element $t' > a_i$ must be to the right of each element $t'' < a_i$, since otherwise $a_i t' t''$ would cause a further 231-pattern. Thus $T$ can be decomposed in the following way: $T = C_1 C_2 \dots C_{q_1+\dots+q_k}$, with substrings $C_1, \dots, C_{q_1+\dots+q_k}$ satisfying $1 \prec a_1 = 2 \prec C_1 \prec a_2 \prec C_2 \prec \dots \prec a_{q_1+\dots+q_k} \prec C_{q_1+\dots+q_k}$.
- The substrings $C_1, \dots, C_{q_1+\dots+q_k}$ are (after order-preserving relabellings) themselves 231-avoiding $k$-Stirling permutations, since otherwise a further 231-pattern would occur.

Combining these properties a combinatorial description of the family of $k$-Stirling permutations, where the single occurrence of the 231-pattern is contained in $A1$, let us denote it by $\mathcal{G}^{[3]}$, can be obtained. To do this we distinguish between two cases according to the substring $P$ contained in $A$ given in above description. Namely, if $P = 2^k a_2^k = a_1^k a_2^k$ then we may think of getting the substrings $T_k, \dots, T_1$ in the decomposition of $\tau$ by starting with the string $\tilde{A} = a^k_{q_1+\dots+q_k} a^k_{q_1+\dots+q_k-1} \dots a^k_2 a^k_1$, partition it into $k$ substrings and finally switch $a_1^k$ with $a_2^k$. This means that $\tau$ is decomposed into 1 and $k$ sequences of pairs consisting of $a_i$ and the corresponding $C_i$, but it must be guaranteed that $A$ consists of at least 2 different labels, otherwise we could not switch the two smallest ones. Thus these two cases have to be excluded and we get the formal description $\mathcal{Z} \times \big(\textsc{Seq}(\mathcal{Z} \times \mathcal{F})\big)^k - \mathcal{Z} - k \cdot \mathcal{Z} \times \mathcal{Z} \times \mathcal{F}$. On the other hand, if $P$ has the shape $P = 2^t a_2^k 2^{k-t}$, with $1 \le t \le k-1$, then 2 and $a_2$ must lie in the same substring $T_i$ and we have to modify the description. Namely, we may think of getting the substrings $T_k, \dots, T_1$ in the decomposition of $\tau$ by starting with the string $\tilde{A} = a^k_{q_1+\dots+a_k} \dots a^k_2$, partition it into $k$ substrings and finally wrap $2^k$ around $a_2$ in of the $k-1$ possible ways. Thus, $\tau$ is decomposed into 1, $a_1 = 2$, $C_1$ and into $k$ sequences of pairs consisting of $a_i$ and the corresponding $C_i$, but it must be guaranteed that $\tilde{A} \ne \emptyset$ (otherwise $2^k$ could not be inserted in the way described above). Thus this case has to be excluded and we get the formal description $(k-1) \cdot \mathcal{Z} \times \mathcal{Z} \times \mathcal{F} \times \big(\textsc{Seq}(\mathcal{Z} \times \mathcal{F})\big)^k - (k-1) \cdot \mathcal{Z} \times \mathcal{Z} \times \mathcal{F}$. Therefore, we end up with the following description of the family $\mathcal{G}^{[3]}$:

$$\mathcal{G}^{[3]} = \mathcal{Z} \times \big(\textsc{Seq}(\mathcal{Z} \times \mathcal{F})\big)^k + (k-1) \cdot \mathcal{Z}^2 \times \mathcal{F} \times \big(\textsc{Seq}(\mathcal{Z} \times \mathcal{F})\big)^k - \mathcal{Z} - (2k-1) \cdot \mathcal{Z}^2 \times \mathcal{F}. \tag{11}$$

Combining the three cases (9), (10) and (11) by using $\mathcal{G} = \mathcal{G}^{[1]} + \mathcal{G}^{[2]} + \mathcal{G}^{[3]}$ we obtain a formal description of the family $\mathcal{G}$:

$$\begin{aligned}\mathcal{G} = \mathcal{Z} \times \mathcal{G} \times \big(\textsc{Seq}(\mathcal{Z} \times \mathcal{F})\big)^k + k \cdot \mathcal{Z}^2 \times \mathcal{F} \times \mathcal{G} \times \big(\textsc{Seq}(\mathcal{Z} \times \mathcal{F})\big)^{k+1} \\ + k^2 \cdot \mathcal{Z}^4 \times \mathcal{F}^3 \times \big(\textsc{Seq}(\mathcal{Z} \times \mathcal{F})\big)^{k+1} + \mathcal{Z} \times \big(\textsc{Seq}(\mathcal{Z} \times \mathcal{F})\big)^k \\ + (k-1) \cdot \mathcal{Z}^2 \times \mathcal{F} \times \big(\textsc{Seq}(\mathcal{Z} \times \mathcal{F})\big)^k - \mathcal{Z} - (2k-1) \cdot \mathcal{Z}^2 \times \mathcal{F}.\end{aligned}$$

Applying the symbolic method this formal description immediately yields an equation connecting the generating functions $G(z)$ and $F(z)$:

$$\begin{aligned}G(z) = \frac{zG(z)}{(1-zF(z))^k} + \frac{kz^2F(z)G(z)}{(1-zF(z))^{k+1}} + \frac{k^2z^4F(z)^3}{(1-zF(z))^{k+1}} + \frac{z}{(1-zF(z))^k} \\ + \frac{(k-1)z^2F(z)}{(1-zF(z))^k} - z - (2k-1)z^2F(z).\end{aligned}$$

Taking into account the defining functional equation for $F(z)$, we obtain, after straightforward computations, the following formula for $G(z)$:

$$\begin{aligned}G(z) = \frac{1}{1+(k-1)zF(z) - kzF(z)^2} \cdot \Big[z^3F(z)^3(k^2F(z) - (k-1)^2) \\ - (k-1)z^2F(z)^2(F(z)+1) - zF(z)(k-1-(k-2)F(z)) + F(z) - 1\Big],\end{aligned}$$

with $F(z) = 1 + \frac{zF(z)}{(1-zF(z))^k}$.

To extract coefficients and thus to obtain an explicit formula for $G_n = [z^n]G(z)$ we introduce $\hat{F} := \hat{F}(z) := zF(z)$ and apply Cauchy's integration formula. First, after simple computations one gets that $G(z)$ can be expressed via $\hat{F}$ as follows:

$$G(z) = \frac{\hat{F}\left(\frac{k^2\hat{F}^3}{(1-\hat{F})^{k+1}} + \frac{(k-1)\hat{F}}{(1-\hat{F})^k} + \frac{1}{(1-\hat{F})^k} - (2k-1)\hat{F} - 1\right)}{1 - \frac{k\hat{F}^2}{(1-\hat{F})^{k+1}}}. \tag{12}$$

Moreover, using

$$\hat{F} = z\Big(1 + \frac{\hat{F}}{(1-\hat{F})^k}\Big) \tag{13}$$

and thus $dz = \frac{1 - \frac{k\hat{F}^2}{(1-\hat{F})^{k+1}}}{\left(1+\frac{\hat{F}}{(1-\hat{F})^k}\right)^2} d\hat{F}$, we obtain

$$\begin{aligned}G_n = [z^n]G(z) = \frac{1}{2\pi i}\oint \frac{G}{z^{n+1}}dz = \frac{1}{2\pi i}\oint \frac{G\Big(1+\frac{\hat{F}}{(1-\hat{F})^k}\Big)^{n+1}}{\hat{F}^{n+1}} \cdot \frac{1 - \frac{k\hat{F}^2}{(1-\hat{F})^{k+1}}}{\Big(1+\frac{\hat{F}}{(1-\hat{F})^k}\Big)^2} d\hat{F} \\ = \frac{1}{2\pi i}\oint \frac{G\Big(1+\frac{\hat{F}}{(1-\hat{F})^k}\Big)^{n-1}\Big(1-\frac{k\hat{F}^2}{(1-\hat{F})^{k+1}}\Big)}{\hat{F}^{n+1}} d\hat{F}\end{aligned}$$

$$= [\hat{F}^n]G\Big(1 + \frac{\hat{F}}{(1-\hat{F})^k}\Big)^{n-1}\Big(1 - \frac{k\hat{F}^2}{(1-\hat{F})^{k+1}}\Big).$$

Therefore,

$$G_n = [\hat{F}^{n-1}]\left(\frac{k^2\hat{F}^3}{(1-\hat{F})^{k+1}} + \frac{(k-1)\hat{F}}{(1-\hat{F})^k} + \frac{1}{(1-\hat{F})^k} - (2k-1)\hat{F} - 1\right)\cdot\Big(1 + \frac{\hat{F}}{(1-\hat{F})^k}\Big)^{n-1},$$

and by using the binomial series expansion one easily obtains an explicit result for $G_n$. We collect these findings in the following theorem.

**Theorem 2** *The generating function $G(z) = \sum_{n\ge0} G_n z^n$ of the number $G_n := G_n^{[k]}$ of $k$-Stirling permutations of order $n$ with a single occurrence of the pattern 231 is given as follows:*

$$G(z) = \frac{1}{1+(k-1)zF(z) - kzF(z)^2}\cdot\Big[z^3F(z)^3(k^2F(z)-(k-1)^2) - (k-1)z^2F(z)^2(F(z)+1) - zF(z)(k-1-(k-2)F(z)) + F(z) - 1\Big],$$

*with $F(z) = \sum_{n\ge0} F_n z^n$ the generating function of the number $F_n := F_n^{[k]}$ of 231-avoiding $k$-Stirling permutations of order $n$, which satisfies the functional equation $F = 1 + \frac{zF}{(1-zF)^k}$.*

# 5 Containing the pattern 123 only once

## 5.1 Avoiding the pattern 123

In this section we consider the combinatorial family $\mathcal{G} := \mathcal{G}_k(123)$ of $k$-Stirling permutations with a single occurrence of the pattern 123. Throughout this section we denote by $E_x$ the evaluation operator at $x = 1$, and with $D_x$ the differentiation operator with respect to $x$. The aim is to show results for the number $G_n$ of $k$-Stirling permutations of order $n$ with a single occurrence of the pattern 123 as well as for its generating function $G(z) := \sum_{n\ge0} G_n z^n$. As in the previous sections for this we require corresponding results for the family $\mathcal{F} := \mathcal{F}_k(123)$ of 123-avoiding $k$-Stirling permutations. It has been shown already in [12] by an extension of the correspondence of Simion and Schmidt that, for each order $n$, the number of 231-avoiding $k$-Stirling permutations is equal to the number of 123-avoiding $k$-Stirling permutations; thus the numbers $F_n$ and its generating function $F(z) := \sum_{n\ge0} F_n z^n$ are given by the formulæ (8) and (7), respectively, stated in Section 4.

However, for the pattern 123 the situation is more involved and in order to get results for $\mathcal{G}$ these formulæ are not sufficient, but we require refinements of the enumeration results for $\mathcal{F}$. Namely, first we will derive the generating function of the number of 123-avoiding $k$-Stirling permutations of order $n$, weighted according to three parameters. Assume that $\sigma \in \mathcal{F}$ is of order $n$: the parameter $c(\sigma)$ denotes the number of possible insertion places for $(n+1)^k$ in $\sigma$ before the first occurrence of the patterns 12 and 21; moreover, the parameter $d(\sigma)$ denotes the number of the remaining insertion places for $(n+1)^k$ in $\sigma$ before the first occurrence of the pattern 12. Furthermore, the parameter $e(\sigma)$ counts the number of positions, where an insertion of $(n+1)^k$ in $\sigma$ would create a single occurrence of the pattern 123. Note that by defining these parameters we exclude an insertion at the beginning of the $k$-Stirling permutation $\sigma$, since this case will always be treated separately. These insertion places can be naturally ordered from left to right.

**Example 1** *Given the* 123*-avoiding* 2*-Stirling permutation* $\sigma = 1221$*, with* $c(\sigma) = 1$*,* $d(\sigma) = 0$ *and* $e(\sigma) = 3$*. We can obtain either* $\tau = 331221$ *with parameters* $c(\tau) = 2$*,* $d(\tau) = 1$ *and* $e(\tau) = 3$ *by inserting* $3^2$ *at the beginning, or* $\tau = 133221$ *with parameters* $c(\tau) = 1$*,* $d(\tau) = 0$ *and* $e(\tau) = 2$*.*

Let $F(z;u,v,w)$ denote the generating function of 123-avoiding $k$-Stirling permutations weighted according to the parameters $c(\sigma)$, $d(\sigma)$ and $e(\sigma)$,

$$F(z;u,v,w) := \sum_{\sigma\in\mathcal{F}} z^{|\sigma|} u^{c(\sigma)} v^{d(\sigma)} w^{e(\sigma)} = \sum_{c,d} F_{c,d}(z;w) u^c v^d. \tag{14}$$

Of course, it holds $F(z) = F(z;1,1,1)$ for the counting series of 123-avoiding $k$-Stirling permutations, with $F(z)$ given by (7) as mentioned above. A key part in our further analysis is the fact that $F(z;u,v,w)$ satisfies a certain functional equation, stated in the Proposition below.

**Proposition 1** *The generating function* $F(z;u,v,w)$ *of the number of* 123*-avoiding* $k$*-Stirling permutations of order* $n$*, weighted according to the insertion places before* 12 *and* 21*, remaining places before* 12*, and places where a single occurrence of* 123 *can be created, which is defined in* (14)*, satisfies*

$$\begin{aligned} F(z;u,v,w) = 1 + zu^k F(z;v,v,w) + \frac{zuw^k}{u-w}\Big(F(z;u,w,1) - F(z;w,w,1)\Big) \\ + \frac{zv}{v-1}\Big(F(z;u,v,1) - F(z;u,1,1)\Big). \end{aligned} \tag{15}$$

In order to show this functional equation for $F(z;u,v,w)$ we use the following result.

**Lemma 1** *Let* $\sigma = \sigma_1 \dots \sigma_{kn} \in \mathcal{F}$ *denote a* 123*-avoiding* $k$*-Stirling permutation of order* $n \geq 1$*. The values of the parameters* $c(\sigma)$*,* $d(\sigma)$ *and* $e(\sigma)$ *of the* $k$*-Stirling permutations* $\tau \in \mathcal{F}$ *of order* $n+1$*, obtained by inserting* $(n+1)^k$ *into* $\sigma$ *in a suitable way, are given by*

$$(c(\tau), d(\tau), e(\tau)) = \begin{cases} \big(k, c(\sigma)+d(\sigma), e(\sigma)\big), \\ \qquad \text{by insertion at the beginning of } \sigma, \\ \big(\gamma, 0, c(\sigma)-\gamma+d(\sigma)+k\big), \\ \qquad \text{by insertion at position } \gamma \text{ of } c(\sigma),\ 1 \leq \gamma \leq c(\sigma), \\ \big(c(\sigma), \delta, 0\big), \\ \qquad \text{by insertion at position } \delta \text{ of } d(\sigma),\ 1 \leq \delta \leq d(\sigma). \end{cases}$$

*Proof of Lemma 1:* In order to create new 123-avoiding $k$-Stirling permutations of order $n+1$ we can insert the $k$-tuple $(n+1)^k$ either at the beginning, or at any of the $c(\sigma)+d(\sigma)$ other possible insertion places.

- If we insert the $k$-tuple $(n+1)^k$ at the beginning of $\sigma$, we obtain $\tau = (n+1)^k\sigma \in \mathcal{F}$ of order $n+1$, with $c(\sigma) = k$ insertion places before the first occurrences of 12 and 21. Moreover, then we have $d(\tau) = c(\sigma) + d(\sigma)$, since the insertion of $(n+1)^k$ creates an occurrence of 21.
- If we insert $(n+1)^k$ at position $\gamma$ of $c(\sigma)$, we obtain $\tau = a^\gamma(n+1)^k a^{c(\sigma)-\gamma}\sigma_2$. Hence, $c(\tau) = \gamma$ and there are no more insertion places before the first occurrence of 12, i.e., $d(\tau) = 0$. Note that we can create a single occurrence of 123 by inserting at any of the $c(\sigma)-\gamma$ places, at any of the additional $k$ places in $(n+1)^k$, or any of the former $d(\sigma)$ insertion places in the part $\sigma_2$ in $\tau$.
- If we insert $(n+1)^k$ at position $\delta$ of $d(\sigma)$, then for the resulting string $\tau$ an insertion of $(n+2)^k$ after this position, i.e., at one of the former $d(\sigma)-\delta$ remaining places, would create more than one occurrence of the pattern 123, hence $e(\tau) = 0$ and furthermore $d(\tau) = \delta$; moreover, the $c(\sigma)$ insertion positions are not influenced.

□

*Proof of Proposition 1:* By Lemma 1 we obtain the following functional equation for $F(z;u,v,w)$:

$$F(z;u,v,w) = 1 + zu^kF(z;v,v,w) + z\cdot\sum_{c,d}F_{c,d}(z;1)\Big(\sum_{\gamma=1}^{c}u^\gamma w^{c-\gamma+k+d} + u^c\sum_{\delta=1}^{d}v^\delta\Big).$$

Using

$$\sum_{\gamma=1}^{c}u^\gamma w^{c-\gamma+k+d} = w^{k+c+d-1}\cdot u\cdot\frac{\left(\frac{u}{w}\right)^c-1}{\frac{u}{w}-1} = uw^{k+d}\frac{u^c-w^c}{u-w}$$

and also $u^c\sum_{\delta=1}^{d}v^\delta = u^cv\frac{v^d-1}{v-1}$, it simplifies to the stated result. □

## 5.2 Single occurrence of the pattern 123

Now we turn to our studies of the family $\mathcal{G}$ and compute formulæ for $G(z)$ and $G_n$. To do this we introduce the refined generating function $G(z;u)$ of the number of $k$-Stirling permutations with a single occurrence of the pattern 123, weighted according to $c(\sigma)+d(\sigma)$, via

$$G(z;u) := \sum_{\sigma\in\mathcal{G}} z^{|\sigma|}u^{c(\sigma)+d(\sigma)} = \sum_{\ell} G_\ell(z)u^\ell, \tag{16}$$

where the parameters $c(\sigma)$ and $d(\sigma)$ are defined as in Section 5.1 by the number of insertion places before 12 and 21, and the number of the remaining insertion places before 12, respectively. Of course, $G(z) = G(z;1)$. The interest of $G(z;u)$ lies in the relationship to the generating function $F(z;u,v,w)$ studied in Section 5.1.

**Proposition 2** *The generating function $G(z;u)$ of the number of $k$-Stirling permutations of order $n$ with a single occurrence of the pattern* 123*, weighted according to the insertion places before the first occurrence of the pattern* 12*, which is defined in* (16)*, satisfies the following functional equation:*

$$G(z;u) = zE_wD_wF(z;u,u,w) + zu^kG(z;u) + \frac{zu}{u-1}\Big(G(z;u) - G(z;1)\Big), \qquad (17)$$

*with $F(z;u,v,w)$ defined in* (14)*.*

*Proof:* Any $k$-Stirling permutation $\tau$ of order $n+1$ with a single occurrence of the pattern 123 can be generated by inserting $(n+1)^k$ into a $k$-Stirling permutation $\sigma$ of order $n$ in one of the following three ways:

- $\sigma \in \mathcal{F}$ is a 123-avoiding $k$-Stirling permutation and we insert $(n+1)^k$ at one of the $e(\sigma)$ possible positions that create a single 123-pattern.
- $\sigma \in \mathcal{G}$ and we insert $(n+1)^k$ at the beginning of the string.
- $\sigma \in \mathcal{G}$ and we insert $(n+1)^k$ at one of the $c(\sigma)+d(\sigma)$ positions before the first occurrence of the pattern 12.

Hence, the generating function $G(z;u)$ satisfies

$$G(z;u) = zE_wD_wF(z;u,u,w) + zu^kG(z;u) + z\sum_{\ell} G_\ell(z)\sum_{\lambda=1}^{\ell} u^\lambda,$$

which readily leads to the stated result. □

Next we show how the generating function $G(z) = G(z;1)$ can be expressed via $F(z;u,v,w)$.

**Lemma 2** *The generating function $G(z)$ satisfies:*

$$G(z) = F(z)\cdot zE_wD_wF(z;U(z),U(z),w), \qquad (18)$$

*where $F(z;u,v,w)$ is defined in* (14)*, $F(z)$ is given by* (7)*, and $U(z)$ has the representation*

$$U(z) = \frac{1}{1-zF(z)} = 1 + \sum_{n\ge1}\frac{z^n\sum_{\ell=0}^{n-1}\binom{n}{\ell}\binom{n+\ell(k-1)}{n-1-\ell}}{n}. \qquad (19)$$

*Proof:* We show above representation by using the kernel method. First, we rewrite (17) as

$$\big(1 - zu^k - \frac{zu}{u-1}\big)G(z;u) = zE_wD_wF(z;u,u,w) - \frac{zu}{u-1}G(z;1). \qquad (20)$$

The kernel $K(z;u) = 1 - zu^k - \frac{zu}{u-1}$ has a single power series solution $u = U(z)$, where it vanishes:

$$K(z;U(z)) = 1 - zU^k(z) - \frac{zU(z)}{U(z)-1} = 0. \qquad (21)$$

Using the Lagrange-Bürmann inversion theorem, we readily obtain the expansion of $U(z)$ stated above when considering the shifted series $\hat{U}(z) = U(z) - 1$ satisfying

$$\hat{U}(z) = z(\hat{U}(z) + 1)\big(1 + \hat{U}(z)(\hat{U}(z) + 1)^{k-1}\big).$$

Consequently, we have

$$z = \frac{\hat{U}(z)}{(\hat{U}(z) + 1)\big(1 + \hat{U}(z)(\hat{U}(z) + 1)^{k-1}\big)},$$

and obtain for the $n$-th coefficient

$$[z^n]\hat{U}(z) = \frac{1}{n}[\hat{U}^{n-1}](1 + \hat{U})^n\big(1 + \hat{U}(\hat{U} + 1)^{k-1}\big)^n = \frac{1}{n}\sum_{\ell=0}^{n-1}\binom{n}{\ell}\binom{n + \ell(k-1)}{n - 1 - \ell}.$$

Substituting the series $u = U(z)$ in (20) cancels the left hand side of the equation and we directly get

$$zE_wD_wF(z; U(z), U(z), w) = \frac{zU(z)}{U(z) - 1}G(z; 1) = \frac{zU(z)}{U(z) - 1}G(z).$$

Furthermore, it holds that the function $\frac{U(z)-1}{zU(z)}$ is exactly the generating function $F(z)$ given in (7); namely, by simple manipulations of (7) and (21) it follows

$$zF = z\left(1 + \frac{zF}{(1 - zF)^k}\right) \quad \text{and} \quad 1 - \frac{1}{U} = z\left(1 + \frac{1 - \frac{1}{U}}{\left(\frac{1}{U}\right)^k}\right),$$

and thus that $zF(z) = 1 - \frac{1}{U(z)}$, since they satisfy the same functional equation. This shows the stated relation (18). $\square$

**Remark 1** *Another application of the Lagrange-Bürmann inversion theorem similar to the corresponding computations for $U(z)$ in Lemma 2 yields the expansion*

$$F^2(z) = \sum_{n\geq 0} z^n \frac{2}{n+2}\sum_{\ell=0}^{n}\binom{n+2}{\ell}\binom{n - 1 + (k-1)\ell}{n - \ell}, \tag{22}$$

*which can be used later for deriving $G_n = [z^n]G(z)$.*

Thus, according to Lemma 2 it remains to determine the generating function $zE_wD_wF(z; u, u, w)$. When we evaluate $F(z; u, v, w)$ at $v = u$ in the functional equation (15) we get

$$\begin{aligned} F(z; u, u, w) = \frac{1}{1 - zu^k}\bigg(1 &+ \frac{zuw^k}{u - w}\Big(F(z; u, w, 1) - F(z; w, w, 1)\Big) \\ &+ \frac{zu}{u - 1}\Big(F(z; u, u, 1) - F(z; u, 1, 1)\Big)\bigg). \end{aligned} \tag{23}$$

In the following we will determine step by step all the evaluations of $F$ appearing in (23). First, we plug $w = 1$ into (23) and it follows that the occurrences of $F(z, u, 1, 1)$ cancel out; we get

$$F(z; u, u, 1) = \frac{1}{1 - zu^k} \left(1 + \frac{zu}{u-1} F(z; u, u, 1) - \frac{zu}{u-1} F(z; 1, 1, 1)\right),$$

and by taking into account $F(z; 1, 1, 1) = F(z)$ we obtain after simple manipulations

$$F(z; u, u, 1) = \frac{1 - \frac{zuF(z)}{u-1}}{1 - z(u^k + \frac{u}{u-1})}. \tag{24}$$

Next we turn to the function $F(z; u, v, 1)$, which, by evaluation $w = 1$ in (15), satisfies the functional equation

$$\begin{aligned} \Big(1 - \frac{zv}{v-1}\Big) F(z; u, v, 1) = 1 + zu^k F(z; v, v, 1) - \frac{zu}{u-1} F(z; 1, 1, 1) \\ + z\Big(\frac{u}{u-1} - \frac{v}{v-1}\Big) F(z; u, 1, 1), \end{aligned} \tag{25}$$

with $F(z; 1, 1, 1) = F(z)$ and $F(z; v, v, 1)$ as given in (24). In order to solve (25) we carry out another application of the kernel method. The kernel on the left hand side cancels for $v = V(z) = 1/(1 - z)$. Since $zV(z)/(V(z) - 1) = 1$, we get

$$F(z; V(z), V(z), 1) = \frac{(1-z)^k}{z} \big(F(z) - 1\big),$$

and consequently

$$F(z; u, 1, 1) = \frac{1}{1 - \frac{zu}{u-1}} \left(1 + u^k (1-z)^k \big(F(z) - 1\big) - \frac{zu}{u-1} F(z)\right). \tag{26}$$

Plugging (26) into (25) determines the required function:

$$\begin{aligned} F(z; u, v, 1) = \frac{1}{1 - \frac{zv}{v-1}} \Bigg(1 + zu^k \frac{1 - \frac{zv}{v-1} F(z)}{1 - z(v^k + \frac{v}{v-1})} - \frac{zu}{u-1} F(z) \\ + z\big(\frac{u}{u-1} - \frac{v}{v-1}\big) \frac{1}{1 - \frac{zu}{u-1}} \Big(1 + u^k (1-z)^k \big(F(z) - 1\big) - \frac{zu}{u-1} F(z)\Big)\Bigg). \end{aligned} \tag{27}$$

Now we are ready to derive $zE_w D_w F(z; u, u, w)$. We get from (23)

$$\begin{aligned} zE_w D_w F(z; u, u, w) &= \frac{z^2 u}{1 - zu^k} E_w D_w \frac{w^k}{u - w} \Big(F(z; u, w, 1) - F(z; w, w, 1)\Big) \\ &= \frac{z^2 u}{(1 - zu^k)(u-1)} \Bigg(\frac{k(u-1) + 1}{u-1} \big(F(z; u, 1, 1) - F(z)\big) \\ &\qquad + E_w D_w \Big(F(z; u, w, 1) - F(z; w, w, 1)\Big)\Bigg), \end{aligned}$$

with $F(z;w,w,1)$, $F(z;u,1,1)$ and $F(z;u,w,1)$ as given in (24), (26) and (27). We substitute $u = U(z)$ given by (19) and obtain

$$zE_wD_wF(z;U(z),U(z),w) = \frac{z^2U(z)}{(1-zU^k(z))(U(z)-1)}\times$$
$$\Bigg(\frac{k(U(z)-1)+1}{U(z)-1}\big(F(z;U(z),1,1)-F(z)\big)+E_wD_w\Big(F(z;U(z),w,1)-F(z;w,w,1)\Big)\Bigg).$$

The calculation of $E_wD_w\Big(F(z;u,w,1)-F(z;w,w,1)\Big)$ is preferably carried out using a computer algebra system. Moreover, for the remaining simplifications the identities $U(z) = \frac{1}{1-zF(z)}$ and $U^k(z) = \frac{F(z)-1}{zF(z)}$ are useful, where the latter follows directly from (7). We finally obtain, by using also the series expansions of $F(z)$ and $F^2(z)$ given in (8) and (22), respectively, the following result.

**Theorem 3** *The generating function $G(z) = \sum_{n\geq 0} G_n z^n$ of the number $G_n := G_n^{[k]}$ of $k$-Stirling permutations of order $n$ with a single occurrence of the pattern 123 is given as follows:*

$$G(z) = \frac{F(z)(F(z)-1)(1-z)^k\big(z(k-1)+1\big)+z(z(2-k)-2)F^2(z)+zF(z)}{z},$$

*with $F(z) = \sum_{n\geq 0} F_n z^n$ the generating function of the number $F_n := F_n^{[k]}$ of 123-avoiding $k$-Stirling permutations of order $n$, which satisfies the functional equation $F = 1 + \frac{zF}{(1-zF)^k}$.*

## 6 Asymptotic results

The explicit results for the numbers $G_n$ and the corresponding generating functions $G(z)$, respectively, obtained in Section 3-5 can be treated by standard techniques from analytic combinatorics to describe the asymptotic growth behaviour of $G_n = G_n^{[k]}$, for $n \to \infty$ and $k$ fixed. In particular, we are also interested in the asymptotic behaviour of the ratio $\frac{G_n}{F_n}$ between the number of $k$-Stirling permutations of order $n$ with a single occurrence and the corresponding ones avoiding a given pattern $\alpha$ of length 3. We collect our findings in Table 3.

We give some quick remarks on the asymptotic computations. The asymptotic results for the pattern 312 immediately follow from the explicit formula for $G_n$ given in Section 3 after applying Stirling's formula for the factorials. For the patterns 231 and 123 it is advantageous to apply basic singularity analysis [7] to the corresponding generating functions $G(z)$ given in Section 4-5. To do this one first considers the function $\hat{F}(z) = zF(z)$ satisfying the functional equation (13) and gets the following local expansion in a complex neighbourhood of the unique dominant singularity $z = \rho$:

$$\hat{F}(z) = \tau - \kappa \cdot \sqrt{1-\frac{z}{\rho}} + \mathcal{O}\Big(1-\frac{z}{\rho}\Big),$$

| *Members* $\alpha$ *in class* | $G_n^{[k]} = \lvert\{\sigma \in \mathcal{G}_k(\alpha)\colon \lvert\sigma\rvert = n\}\rvert$ | *ratio* $\frac{G_n^{[k]}}{F_n^{[k]}}$ | *growth constant* $c_k^{[\alpha]}$ *and behaviour for* $k \to \infty$ |
|---|---|---|---|
| 213, 312 | $\sim \frac{k^{2k-\frac{1}{2}}}{\sqrt{2\pi}\,(k+1)^{2k+\frac{1}{2}}} \left(\frac{(k+1)^{k+1}}{k^k}\right)^n \frac{1}{\sqrt{n}}$ | $\sim c_k^{[\alpha]} \cdot n$ | $c_k^{[\alpha]} = \left(\frac{k}{k+1}\right)^{2k+1} = e^{-2} \cdot \left(1 - \frac{1}{6k^2} + \mathcal{O}\left(\frac{1}{k^3}\right)\right)$ |
| 231, 132 | $\sim \sqrt{\frac{(1-\tau)(1+(k-1)\tau)}{2\pi k\tau(2+(k-1)\tau)}} \cdot \frac{1-\tau-k\tau^2}{\sqrt{n}\,\rho^n}$ | $\sim c_k^{[\alpha]} \cdot n$ | $c_k^{[\alpha]} = \frac{k\tau(1-\tau-k\tau^2)}{1+(k-1)\tau} = 1 - \frac{1}{\ln k} + \mathcal{O}\left(\frac{(\ln\ln k)}{(\ln k)^2}\right)$ |
| 123, 321 | $\sim \frac{\kappa}{2\sqrt{\pi}\rho} \cdot c_k^{[\alpha]} \cdot \frac{1}{n^{\frac{3}{2}}\,\rho^n}$ | $\sim c_k^{[\alpha]}$ | $c_k^{[\alpha]} = \frac{(2\tau-\rho)(1+(k-1)\rho)(1-\rho)^k}{\rho^2} + \frac{\rho(1+(4-2k)\tau)-4\tau}{\rho} = e(\ln k)^2 \cdot \left(1 + \mathcal{O}\left(\frac{(\ln\ln k)^2}{\ln k}\right)\right)$ |

Table 3: Asymptotic results of $G_n^{[k]}$, for $n \to \infty$. Here $\tau = \tau(k)$ denotes the smallest positive solution of the equation $k\tau^2 = (1-\tau)^{k+1}$, $\rho = \rho(k) = \frac{k\tau^2}{1+(k-1)\tau}$ and $\kappa = \kappa(k) = \sqrt{\frac{2(1-\tau)\tau(1+(k-1)\tau)}{k(2+(k-1)\tau)}}$.

where $\tau$, $\rho$ and $\kappa$ are given as in Table 3. Plugging this local expansion into the formulæ for $G(z)$ stated in Theorem 2 and Theorem 3, respectively, immediately yields local expansions of $G(z)$ around the dominant singularity $z = \rho$ and consequently the asymptotic behaviour of the coefficients $G_n$. In order to get the asymptotic results for the growth constant $c_k$, for $k \to \infty$, we use the following asymptotic expansion of $\tau = \tau(k)$:

$$\tau = \frac{\ln k}{k} - \frac{2\ln\ln k}{k} + \mathcal{O}\left(\frac{\ln\ln k}{k \cdot \ln k}\right),$$

which follows from the defining equation $k\tau^2 = (1-\tau)^{k+1}$ after applying the so-called bootstrapping technique (see [7] and references therein).

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