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An invariant for minimum triangle-free graphs

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Abstract

We study the number of edges, $e(G)$, in triangle-free graphs with a prescribed number of vertices, $n(G)$, independence number, $\alpha(G)$, and number of cycles of length 4, $\mathrm{N}(C_4; G)$. In particular we show that

$$3e(G) - 17n(G) + 35\alpha(G) + \mathrm{N}(C_4; G) \geq 0$$

for all triangle-free graphs G. We also characterise the graphs that satisfy this inequality with equality.

As a consequence we improve the previously best known lower bounds on the independence ratio $i(G) = \alpha(G)/n(G)$ for graphs of average degree at most 4 and girth at least 5, 6 or 7.

1 Introduction

1.1 Background

The *(minimum) edge numbers*, $e(3, k, n)$, are defined as the minimum number of edges in a triangle-free graph on n vertices without an independent set of size k. These numbers, and constructions of related graphs, have successfully been used to compute, or bound, the classical two-colour Ramsey numbers $R(3, \ell)$. In particular the edge numbers have been used when studying $R(3, \ell)$ for $\ell = 6$ by Kalbfleisch [8], for $\ell = 7$ by Graver and Yackel [4] and for $\ell = 9$ by Grinstead and Roberts [5]. Among the useful upper bounds on the Ramsey numbers $R(3, \ell)$ that have been obtained by these considerations are those of Radziszowski and Kreher (e.g. [11]).

In particular Radziszowski and Kreher proved, in [11], that $e(3, k+1, n) \geq 6n - 13k$ for all non-negative integers n and k. One may differently phrase their result by saying $t(G) := e(G) - 6n(G) + 13\alpha(G) \geq 0$ for all triangle-free simple graphs $G = (V, E)$, where $e(G) = |E|$ denotes the number of edges, $n(G) = |V|$ the number of vertices and $\alpha(G)$ the independence number of G. Moreover the triangle-free graphs G for which $t(G) = 0$ have been classified in part by Radziszowski and

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Kreher in [11] and completely by Backelin in [2]. The invariant t is just one in a series of invariants of a similar kind, all of which give bounds on the edge-numbers and for which there are classifications of the triangle-free graphs that satisfy them with equality.

In this article we consider a related invariant, $\nu(G)$, which we define as

$$\nu(G) = 3e(G) - 17n(G) + 35\alpha(G) + \mathrm{N}(C_4; G),$$

where $\mathrm{N}(C_k; G)$ denotes the number of cycles of length k in G. We will, in particular, show that $\nu(G) \geq 0$ for all triangle-free graphs G (see Theorem 1.1 in Section 1.2).

This affirmatively answers a question first considered in [1]. We also give a classification of the graphs that satisfy this inequality with equality. We will see that this bound is tight since there are (infinitely many) triangle-free graphs G for which $\nu(G) = 0$. These graphs seem to be closely related to those for which $t(G) = 0$. In particular there are even infinitely many connected triangle-free graphs, G, for which $t(G) = \nu(G) = 0$.

We define the *independence ratio* of a graph G as $i(G) = \alpha(G)/n(G)$. The main result of this article implies that $i(G) \geq \frac{11}{35}$ for all graphs of girth at least 5 and average degree 4. This improves the previously known bounds obtained as special cases of more general theorems by Hopkins and Staton in [6] (where we have the bound $i(G) \geq \frac{7}{23}$ for graphs with maximum degree 4 and girth at least 6). The results of Hopkins and Staton have also been improved and generalised upon by Lichiardopol in [10]. The bound $i(G) \geq \frac{11}{35}$ improves the bounds given by Lichiardopol for graphs with maximum degree 4 and girth 5, 6 or 7.

1.2 Graphs with ν-value zero and the main theorem

We will here describe all graphs with ν-value zero. That these are indeed all such graphs will be demonstrated in the conclusion of this article. All graphs article will be assumed to be non-empty.

We need to define the following class of graphs (which appears in [1, 2] as *chains* denoted by Ch_k, in [11] as F_k and in [7] as H_k). These graphs will play an important role in our proofs.

Definition 1.1. Let Ch_2 be a cycle of length 5. We recursively define Ch_{k+1} for $k \geq 2$. Let $x \in V(Ch_k)$ be some vertex of degree 2. Let $V(Ch_{k+1}) = V(Ch_k) \mathbin{\dot\cup} \{v, w_1, w_2\}$ and $E(Ch_{k+1}) = E(Ch_k) \cup \{vw_1, vw_2, w_1x\} \cup \{w_2y; y \in N(x)\}$.

It is easy to verify that Ch_k is then well-defined for $k \geq 2$, i.e. up to isomorphism the result does not depend on the choice of vertex of degree 2 in the recursive construction. It is also easy to check that $n(Ch_k) = 3k-1$, $e(Ch_k) = 5k-5$, $\alpha(Ch_k) = k$ and $\mathrm{N}(C_4; Ch_k) = k - 2$. Hence, $\nu(Ch_k) = 0$ for all $k \geq 2$.

There are also two connected 3-regular graphs with ν-value 0. These have been characterised in [1]. Using the same notation as there we define the graphs $(2C_7)_{2i}$ and W_5 as follows. Let $V((2C_7)_{2i}) = \{a_0, a_1, \dots, a_6\} \cup \{b_0, b_1, \dots, b_6\}$ and the edges of $(2C_7)_{2i}$ be such that both $a_0, a_1, \ldots, a_6$ and $b_0, b_1, \ldots, b_6$ form cycles of length 7 in $(2C_7)_{2i}$. Connect these two cycles by adding an edge $b_i a_{2i}$ for all $i \in \{0, 1, \ldots, 6\}$, taking indices modulo 7.

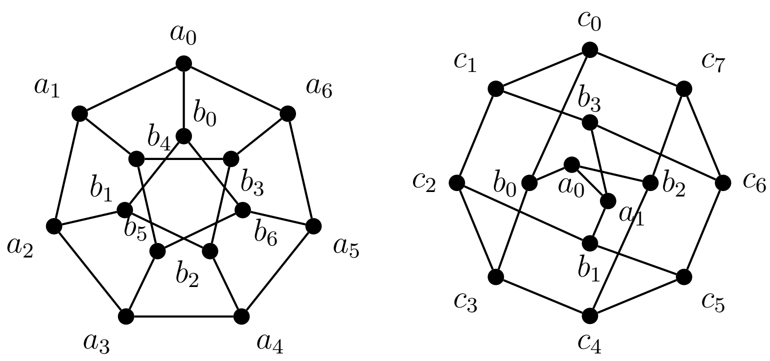

Figure 1: The graphs $(2C_7)_{2i}$ and W_5, respectively..

This graph is also known as a generalised Petersen graph, variously denoted GP(7, 2) or P(7, 2).

Let $V(W_5) = \{a_0, a_1\} \cup \{b_0, \ldots, b_4\} \cup \{c_0, \ldots, c_7\}$ and the edges of W_5 be such that a_0a_1 are adjacent, $b_0, \ldots, b_4$ are independent and $c_0, \ldots, c_7$ form a cycle of length 8. Add edges $b_i a_i$ for $i \in \{1, 2, 3, 4\}$ taking a_i-indices modulo 2. Also add edges $b_i c_{2i}$ and $b_i c_{2i+3}$ for $i \in \{1, 2, 3, 4\}$ taking indices modulo 8.

We will in this article show that these two 3-regular graphs are the only 3-regular connected graphs with ν-value zero. This extends a result in [1] that states that these two graphs are the only two 3-regular graphs with ν-value zero that neither contains cycles of length 3 nor of length 4.

Let BC_k, $k \geq 4$, be a graph consisting of an induced cycle on vertices $c_1, c_2 \ldots, c_{2k}$ and one induced cycle on vertices $d_1, d_2 \ldots, d_k$. Connect the cycles by edges $d_i c_{2i-2}$ and $d_i c_{2i+1}$ for $i \in \{1, \ldots, k\}$, taking indices modulo $2k$ for c_i's and modulo k for d_i's. The graphs BC_k have been called *bicycles* (in [1] and [2]) or *extended k-chains* (in [7], denoted E_k) and G_k (in [11]).

Note that we have $n(BC_k) = 3k$, $e(BC_k) = 5k$. It is not difficult to show that $\alpha(BC_k) = k$. Moreover, for $k \geq 5$ we have $\mathrm{N}(C_4; BC_k) = k$. Hence $\nu(BC_k) = 0$. In the case $k = 4$ we have one "extra" cycle of length 4 formed by the vertices d_1, d_2, d_3 and d_4 and because of this we have $\nu(Ch_4) = 1$.

The main goal of this article is to establish the following theorem. We introduce a notation for the family of graphs with ν-value zero that have been defined in this section.

$$\mathcal{G} := \{W_5, (2C_7)_{2i}\} \cup \{Ch_k; k \geq 2\} \cup \{BC_k; k \geq 5\}.$$

Theorem 1.1. *If G is a triangle-free graph then $\nu(G) \geq 0$, and if $\nu(G) = 0$, with G connected, then $G \in \mathcal{G}$.*

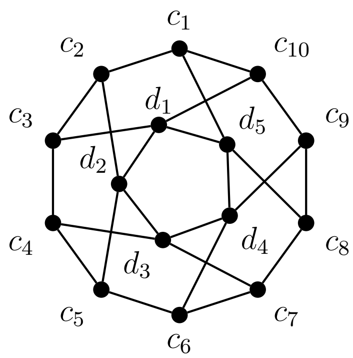

Figure 2: The graph BC_5.

This means that the only triangle-free connected graphs with ν-value zero are those that we have defined in this section. Since ν is linear (in the sense that if $H_1 + H_2$ is the disjoint union of graphs H_1 and H_2 then $\nu(H_1 + H_2) = \nu(H_1) + \nu(H_2)$) it is enough to classify the connected graphs with ν-value zero as in Theorem 1.1.

1.3 Outline of the proof of Theorem 1.1

To establish Theorem 1.1 in Section 1.4 we introduce some preliminary results which we need later while simultaneously exhibiting the bulk of the notation used in the article.

The entirety of Section 2.1 is dedicated to proving Lemma 2.2. Part (xi) of this lemma immediately implies Theorem 1.1. The lemma is proved using simultaneous induction over all of its eleven separate statements. Each statement relates the value of $\nu(G)$ to some structure of the graph G. For example one such statement asserts that the minimum degree of G is at most 4 whenever $\nu(G) \leq 7$. We successively obtain stronger structural properties for G by strengthening the assumed upper bound on $\nu(G)$.

Most of the assertions in the lemma are proved by showing that if the assertion would not hold then G has a proper subgraph which does not satisfy one of the assertions. In proving the crucial part (xi) of the lemma we mimic the work of Radziszowski and Kreher in [11] with the use of a slight modification of their proof mentioned by Backelin in [2]. This part of the proof mostly consists of reformulating their results to make them fit into our particular context.

1.4 Preliminaries and notation

We start by stating some preliminary lemmas which we will use later. These are for the most part easy to prove. For terminology and notation not explained in this article we refer the reader to [3].

For edge sets $T \subseteq E(G)$ we denote by $G - T$ the graph $(V(G), E(G) \setminus T)$ and for vertex sets $S \subseteq V(G)$ we denote by $G \setminus S$ the induced subgraph $G[V(G) \setminus S]$. For this and other notation we may omit the use of set parentheses for singleton sets. Note in particular the distinction between $G - e$ and $G \setminus e$ for $e \in E(G)$.

If $e \in E(G)$ is such that $\alpha(G - e) = \alpha(G)$ we say that e is *redundant*, otherwise it is called *critical*. The graph G is said to be *edge-critical* if all its edges are critical. Its is easily verified that all graphs defined in the previous section are edge-critical. A set of vertices $S \subseteq V(G)$ is said to *destabilise* G if $\alpha(G \setminus S) < \alpha(G)$, and S is then said to be a *destabiliser*. Inclusion-wise minimal destabilising sets are called *minimal destabilisers*. If G has no destabilisers of size r then we say that G is r*-stable*.

For a vertex $v \in V(G)$ the vertices at distance exactly 2 from v in G is denoted by $N_2(v)$. The *second degree of v in G* is defined to be the sum of the degrees of the vertices in $N(v)$ and is denoted by $d^2(G; v)$, whereas the $d(G; v)$ denotes the ordinary degree of v in G.[1]

The *length of a path* will always refer to the number of vertices in the path. We will use $V_k(G)$, for $k \geq 0$, to denote the set of all vertices in G of degree exactly k, i.e. $V_k(G) = \{v \in V(G); d(v) = k\}$, and G_k denotes $G[V_k(G)]$ unless G with subscript k has been separately defined.

Lemma 1.1. *(Lemma 2.2 in [2]) If G is a connected edge-critical triangle-free graph, $v \in V(G)$ and $d(v) \geq 2$ then $N_2(v)$ is a destabiliser of G_v.*

For a set of vertices $W \subseteq V(G)$ we let $N[W]$ denote the *closed neighbourhood* of the vertices in W, which is the set of vertices that are either in W or adjacent to a vertex in W. If S is an independent set of vertices we let G_S denote the graph $G \setminus N[S]$, i.e. the graph obtained by removing all the vertices in S, all their neighbours and edges incident to all such vertices. When it is unambiguous we may drop set parenthesis in these notations, writing e.g. G_v for $G_{\{v\}}$.

We now present a few lemmas, the proofs of which are standard and therefore for the most part omitted.

Lemma 1.2. *(Lemma 2.6 of [2]) Let G be an edge-critical, connected and triangle-free graph. If $v \in V(G)$ is a vertex of degree 2, then G_v is connected.*

Lemma 1.3. *If G is edge-critical, $S \subseteq V(G)$ destabilises G and $v \in V(G) \setminus S$, then $S \cap V(G_v)$ destabilises G_v.*

Proof. Otherwise there would be a maximum independent set T of G_v avoiding $S \cap V(G_v)$. $|T| = \alpha(G_v) = \alpha(G) - 1$ since G is edge-critical, whence $T \cup \{v\}$ would be a maximum independent set of G avoiding S. This contradicts that S destabilises G. □

We now classify the minimal destabilisers of minimum size in Ch_k-graphs. The following two lemmas correspond to parts of a more general lemma in [1] and simple stand-alone proofs may be found in [9].

[1] Here and in other similar notation we may drop the graph G from the notation whenever there is no ambiguity, i.e. $d(v) = d(G; v)$.

Lemma 1.4. *(Lemma 6.2(b) of [1]) If S destabilises $G = Ch_k$, where $k \geq 2$, then $|S| \geq 3$ with equality if and only if $S = N[v]$ for some $v \in V(G)$ of degree 2.*

For minimal destabilisers of size 4 we will not completely classify them, but the following lemma tells us that in all but one case they are connected.

Lemma 1.5. *(Lemma 6.2(e) of [1]) If S is a minimal destabiliser of $G = Ch_k$ such that $|S| = 4$ and S is not connected in G, then $k = 3$ and $S = V_2(Ch_3)$.*

For subsets of vertices $A, B \subseteq V(G)$ we write $E_G(A, B)$ for the set of edges with one endpoint in A and the other endpoint in B, i.e. $E_G(A, B)$ is the set of edges $E(G) \cap \{\{a, b\}; a \in A, b \in B\}$. The cardinality of this set will be denoted by $e_G(A, B)$. We will sometimes abuse the notation by writing $E_G(H_1, H_2)$ for $E_G(V(H_1), V(H_2))$ where H_1 and H_2 are two subgraphs of G.

Lemma 1.6. *Let H be an induced subgraph of G and $M \subseteq V(H)$ be the set of vertices adjacent to $V(G) \setminus V(H)$. If M does not destabilise H then every edge in $E(H, G \setminus H)$ is redundant.*

Proof. Supposing that $e \in E(H, G \setminus H)$ were not redundant, then $\alpha(G-e) = \alpha(G)+1$. Let S be a maximum independent set of $G - e$. $S' = S \cap V(H)$ is independent in H. Since M does not destabilise H there is a maximum independent set S'' of H such that $S'' \cap M = \emptyset$. It follows that $(S \setminus S') \cup S''$ is independent, in $G - e$, of size at least $\alpha(G) + 1$. But since $(S \setminus S') \cup S''$ avoids $e \cap V(H)$ the set $(S \setminus S') \cup S''$ would also be independent in G, a contradiction. □

The *distance* between two vertices $u, v \in V(G)$ is denoted $\text{dist}_G(u, v)$ and is defined to be the least number of edges in a path from u to v, or infinity if there is no such path.

Lemma 1.7. *If G is a triangle-free graph without cycles of length 4 which contains a k-cycle $C = c_1, c_2, \ldots, c_k$, then for all $v \in V(G) \setminus C$ we have $|N(v) \cap C| \leq \lfloor \frac{k}{3} \rfloor$.*

Proof. Let $H := G[C]$ be the induced graph on C. By assumption we have that $\forall u_1, u_2 \in N(v) : \text{dist}_{G \setminus v}(u_1, u_2) \geq 3$. Hence, a fortiori, $\text{dist}_H(u_1, u_2) \geq 3$ for all $u_1, u_2 \in N(v) \cap C$ so indeed at most a third of the vertices of C can be in the neighbourhood of v. □

In particular the previous lemma gives us that if we have a cycle of length 5 in a subgraph of G then any vertex outside the cycle can be adjacent to at most one vertex in the cycle. This fact will be used frequently in what follows.

We now define a collection of graphs with ν-value and minimum degree 2 which will be needed in our proofs.

Definition 1.2. Let a_1, a_2, b_1 and b_2 be the vertices of degree 2 in $G = Ch_3$, where $a_1a_2, b_1b_2 \in E(G)$. We define the *shackled chain* SCh_1 by letting $V(SCh_1) = V(G) \cup \{v, w_1, w_2\}$ and $E(SCh_1) = E(G) \cup \{vw_1, vw_2, w_1a_1, w_1b_1, w_2a_2, w_2b_2\}$. For $k \geq 2$ we define SCh_k recursively as follows. Let a be any vertex of degree 2 in SCh_{k-1} with neighbours b_1 and b_2. We then set $V(SCh_k)$ to be the set $V(SCh_{k-1}) \cupdot \{v, w_1, w_2\}$ and $E(SCh_k) = E(SCh_{k-1}) \cup \{vw_1, vw_2, w_1a, w_2b_1, w_2b_2\}$.

For example in Figure 3 the graphs SCh_1 and SCh_2 are shown.

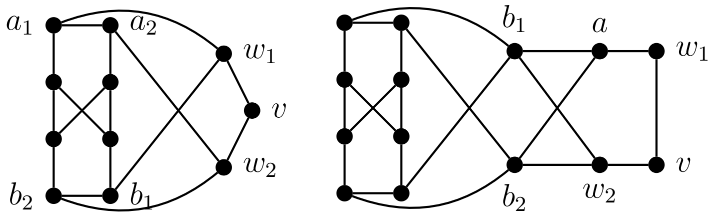

Figure 3: The smallest shackled chains SCh_1 (left) and SCh_2 (right).

It is not difficult to prove that the definition does not depend (up to isomorphism) on the choice of vertex of degree 2 in the recursive construction. It is also not hard to see that $n(SCh_k) = 3k+8$, $e(SCh_k) = 5k+11$, $\alpha(SCh_k) = k+3$ and $\mathrm{N}(C_4; SCh_k) = k$ for all $k \geq 1$. Therefore we get that $\nu(SCh_k) = 2$ for all $k \geq 1$. Furthermore, if S is a destabiliser of size 3 in SCh_k, then $S = N[v]$ for some $v \in V_2(SCh_k)$.

2 Lemmas and the proof

Let $\mathcal{C}(G)$ denote the set of connected components of the graph G. Note in particular that $\nu(G) = \sum_{C \in \mathcal{C}(G)} \nu(C)$. For $S \subseteq V(G)$ we denote by $\mathrm{N}(H; G, S)$ the number of subgraphs of G that are isomorphic to H such that $V(H) \cap S \neq \emptyset$. Recall that $G_v = G \setminus N[v]$.

Lemma 2.1. *If G is a triangle-free graph, then*

$$\forall v \in V(G) : \nu(G_v) \leq \nu(G) - 3d^2(v) + 17d(v) - 18 - \mathrm{N}(C_4; G, N(v)).$$

Proof. Note that $n(G_v) = n(G) - d(v) - 1$ and $e(G_v) = e(G) - d^2(v)$ (since G is triangle-free). Also we have that $\alpha(G_v) \leq \alpha(G) - 1$ since any maximum independent set in G must either contain v or a vertex in the neighbourhood of v. Also, $\mathrm{N}(C_4; G_v) = \mathrm{N}(C_4; G) - \mathrm{N}(C_4; G, N(v))$ since any cycle of length 4 through v also goes through some of the neighbours of v.

Therefore, we get that

$$\begin{aligned}\nu(G_v) &\leq \nu(G) - 3d^2(v) + 17d(v) + 17 - 35 - \mathrm{N}(C_4; G, N(v)) \\ &= \nu(G) - 3d^2(v) + 17d(v) - 18 - \mathrm{N}(C_4; G, N(v)).\end{aligned}$$

□

The rest of this article is dedicated to proving the following lemma which implies Theorem 1.1, since it includes this theorem as statement (xi).

Lemma 2.2. *Let G be a connected triangle-free graph.*

(i) If S destabilises G then

$$e(S) \geq \left\lceil \frac{3\sum_{v\in S} d(v) - 17|S| + 35 - \nu(G)}{3} \right\rceil .$$

(ii) If $\delta(G) = \delta$ and $v \in V(G)$ is a vertex of degree d then either $N_2(v)$ destabilises G_v or $\nu(G) \geq 3\delta d + 18d - 17$.

(iii) If $\nu(G) \leq 17$ then G is 1-stable and $\delta(G) \geq 1$.

(iv) If $\nu(G) \leq 7$ then $\delta(G) \leq 4$.

(v) If $\nu(G) \leq 6$ and G is 2-regular then $G \cong C_5$.

(vi) if $\nu(G) \leq 6$ then either G is 2-stable (and, a fortiori, $\delta(G) \geq 2$) or $G \cong K_2$.

(vii) If $\nu(G) \leq 4$ and $\delta(G) \geq 3$ then G is 3-stable.

(viii) If $\nu(G) \leq 3$, $G \ncong C_5$, $\alpha(G_2) > 1$, and $\mathrm{N}(C_4; N(V_2(G))) = 0$, then there is an edge $e \in E(G)$ such that $G - e \cong C_5 + G'$ where $G' \in \mathcal{G} \setminus \{Ch_k; k \geq 3\}$.

(ix) If $\nu(G) \leq 2$ then G is edge-critical.

(x) If $\nu(G) \leq 2$ and $\delta(G) = 2$, then $G \cong Ch_k$ for some $k \geq 2$ or $G \cong SCh_\ell$ for some $\ell \geq 1$.

(xi) If $\nu(G) \leq 0$ then $G \in \mathcal{G}$ where $\mathcal{G}$ is as defined in Section 1.2.

2.1 Proof of Lemma 2.2

We will prove the above lemma by induction. All the statements hold trivially for $G \cong K_1$. Assume that they hold for all graphs H such that $n(H) < n(G)$ or $n(H) = n(G) \wedge e(H) < e(G)$. In particular all assertions in the lemma hold for $H \subsetneq G$.

We will now establish each of the assertions, in order, also for G. In the process we establish several claims that we may by induction assume holds for all previous graphs (in the inductive order). Note that $\nu(H) \geq 0$ for all $H \subsetneq G$ by (xi). This will be used very frequently, in particular for $H = G_S$ where S is some independent set of vertices.

Proof of (i). Inductively, by (xi), we have $\nu(G \setminus S) \geq 0$. The assertion follows easily by noting that $e(G \setminus S) = e(G) - \sum_{v\in S} d(v) + e(S)$, $n(G \setminus S) = n(G) - |S|$ and $\alpha(G \setminus S) \leq \alpha(G) - 1$. $\square^{(i)}$

Proof of (ii). Supposing that $N_2(v)$ does not destabilise G_v, then there is some independent set, I, of size $\alpha(G_v)$ in G_v such that $I \cap N_2(v) = \emptyset$. Since G is triangle-free we get that $I \cup N(v)$ is an independent set of size $\alpha(G_v) + d$ in G. Therefore $\nu(G_v) \leq \nu(G) + 17 - 18d - 3\delta d$ since $\alpha(G) \geq \alpha(G_v) + d$, $n(G) = n(G_v) - (d+1)$ and $e(G) \geq e(G_v) + \delta d$. The assertion follows inductively from (xi). $\square^{(ii)}$

Proof of (iii). Otherwise $\nu(G \setminus S) \leq 0$ for some singleton set $S = \{v\}$ with strict inequality unless $d(v) = 0$. However, $\nu(K_1) = 18$. $\square^{(iii)}$

Proof of (iv). Follows immediately from Lemma 2.1 since if $v \in V(G)$ has minimal degree then $d^2(v) \geq d(v)^2 = \delta(G)^2$. $\square^{(iv)}$

Proof of (v). Clear, since $\nu(C_k) \geq 7$ for $k = 4$ and $k \geq 6$. $\square^{(v)}$

Proof of (vi) and (vii). These assertions are immediate consequences of (i), for $|S| = 2$ and $|S| = 3$, respectively. $\square^{(vi),(vii)}$

To simplify the proofs of statement (viii) through (xi) we employ some temporary claims. The first two, those that we use to prove (viii), follow.

Claim 1. *If $\nu(G) \leq 3$ and $G \not\cong C_5$ is edge-critical then $C \cong K_1$ or $C \cong K_2$ for all $C \in \mathcal{C}(G_2)$.*

Proof of claim. By (v) and (vi), C is not 2-regular and G is 2-stable. If w is an endpoint of a path component of length at least 3 in G_2 then $\nu(G_w) \leq \nu(G) + 1 \leq 4$ but G_w is connected (by Lemma 1.2) and $\delta(G_w) = 1$, whence $G_w \cong K_2$ by (vi). This is easily seen to be impossible. $\square^{(claim)}$

Claim 2. *If $\nu(G) \leq 3$, $\delta(G) = 2$, $G \not\cong C_5$ is edge-critical, v_1, v_2 are two adjacent vertices of degree 2, each of second degree 5, and $\mathrm{N}(C_4; N(v_i)) = 0$ for both $i \in \{1, 2\}$, then*

(a) if $z \in V(G) \setminus \{v_1, v_2\}, \mathrm{dist}(z, v_1) \leq 3$ then $d(z) \geq 3$, and

(b) if $x \in V_3(G) \cap N_2(v_1)$ then $\delta(G_{v_1,x}) \geq 2$ and $|\mathcal{C}(G_{v_1,x})| = 1$.

Proof of claim. Firstly note that $G_{v_i} \not\cong C_5$ and $\nu(G_{v_i}) \leq \nu(G) + 1 \leq 4$ by Lemma 1.7 and 2.1, respectively. Moreover, G_{v_i} is connected by Lemma 1.2.

Let $z \in V(G) \setminus \{v_1, v_2\}$ be such that $d(z) \leq 2$. If $\mathrm{dist}(z, v_1) = 2$ then $G_{v_1} \cong K_2$ by (vi). This is easily seen to be impossible. On the other hand if $\mathrm{dist}(z, v_1) = 3$ then $\nu(G_{v_2,u_1}) \leq \nu(G_{v_2}) + 1 \leq 5$, where $\{u_1\} = N(v_1) \setminus \{v_2\}$, since $\mathrm{N}(C_4; N(v_2)) = 0$. If G_{v_2,u_1} is connected then, again by (vi), we have $G_{v_2,u_1} \cong K_2$, which is impossible. Thus $G_{v_2,u_1} \cong K_2 + C$ where C is 2-stable, but $e(N(v_2, u_1), C) \leq 2$ contradicting that G is edge-critical. This completes the proof of (a).

Let $x \in V_3(G) \cap N_2(v_1)$. $\delta(G_{v_1,x}) \geq 2$ follows from an argument analogous to (a). Suppose $G_{v_1,x} = G' + G''$ and define $\{x_1, x_2\} := N(x) \setminus N(v_1)$. By (a) we have $d(x_1), d(x_2) \geq 3$ and if $d(G_{v_1}; x_i) \leq 2$ then $e(x_i; N(v_1)) \geq 1$. In particular,

$d(G_{v_1}; x_1) + d(G_{v_1}; x_2) \geq 5$ since otherwise we would have $\mathrm{N}(C_4; N(\{v_1, v_2\})) > 0$. Now, by (vi) and $\delta(G_{v_1,x}) \geq 2$ both G' and G'' are 2-stable.

Now, assume that v_1 and x do not belong to a common cycle of length 5, whence $d(G_{v_1}; x_1) = d(G_{v_1}; x_2) = 3$ and $\nu(G_{v_1,x}) \leq 2$. Define $A_1 := N_2(v_1) \cap V(G_{v_1,x})$, $A_2 := N(x_1) \cap V(G_{v_1,x})$ and $A_3 := N(x_2) \cap V(G_{v_1,x})$. Each of the sets $A_1 \cup A_2$, $A_1 \cup A_3$ and $A_2 \cup A_3$ must destabilise one of G' and G'', and therefore contain at least three vertices from the component that is destabilised. It is then easily seen that this is not possible unless they destabilise the same component, e.g. say $A_1 \cup A_2$ and $A_1 \cup A_3$ destabilises G' then $|V(G'') \cap A_2|, |V(G'') \cap A_3| \leq 1$, and therefore $A_2 \cup A_3$ also must destabilise G'. This either contradicts that G is connected or that G is edge-critical.

Suppose, on the other hand, v_1 and x do belong to a common cycle of length 5. If A_1, A_2, A_3 are defined as before then $e(G_{v_1,x}) \leq e(G) - 6 - (|A_1| + |A_2| + |A_3|)$, $n(G_{v_1,x}) \leq n(G) - 6$ and $\alpha(G_{v_1,x}) \leq \alpha(G) - 2$. Therefore $\nu(G_{v_1,x}) \leq 17 - 3(|A_1| + |A_2| + |A_3|)$, which gives us $|A_1| + |A_2| + |A_3| \leq 5$. Thus we have at most two edges from $N[v_1, x]$ to one of the components G' and G''. Those edges would however be redundant, contradicting that G is edge-critical. $\square^{\text{(claim)}}$

Proof of (viii). If G were not edge-critical then $\nu(G - e) \leq 0$, for some redundant edge e, and $G - e \cong C_5 + G'$ since C_5 is the only graph in $\mathcal{G}$ with minimum degree 2 and $\mathrm{N}(C_4; N(v)) = 0$. Now, $\nu(G') \leq 0$ and, inductively by (xi), we get that $G' \in \mathcal{G} \setminus \{Ch_k; k \geq 3\}$.

Suppose therefore that G is edge-critical. Moreover, suppose $v_1 \in V_2(G)$. If $d^2(v_1) = 6$ then $\nu(G_{v_1}) \leq \nu(G) - 2 \leq 1$ and $\delta(G_{v_1}) \leq 2$ since $\alpha(G_2) > 1$ and therefore $G_{v_1} \cong Ch_k$ for some $k \geq 2$ (by (x) inductively). It is easy to see by Lemmas 1.4 and 1.5 that $\mathrm{N}(C_4; N(v_1)) \neq 0$ unless $k = 3$, in which case we get a contradiction since then $\alpha(G_2) = 1$. Hence, $d^2(v_1) = 5$ by Claim 1. Let v_2 be the neighbour of v_1 of degree 2. The local structure around v_1 and v_2 have been illustrated in Figure 4. Note that the vertices w_{ij} are not, a priori, distinct. If the vertices are not distinct we assume that $w_{11} = w_{21} =: w$, and that y is some neighbour of w, and thus of degree at least 3 by Claim 2.

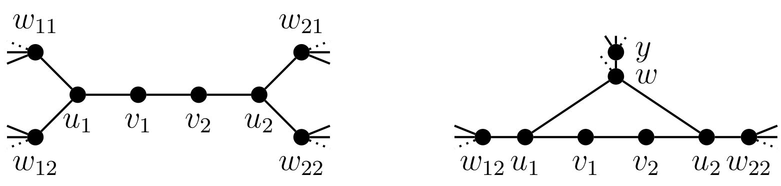

Figure 4: The neighbourhood of v_1 and v_2 in G if $|N(u_1) \cap N(u_2)|$ is 0 or 1, respectively.

Note that either $w_{i2} \in V_3(G)$ and $\nu(G_{v_i,w_{i2}}) \leq \nu(G_{v_i}) - 2 \leq 2$ or $d(w_{i2}) \geq 4$ and $\nu(G_{v_{3-i},u_i}) \leq \nu(G_{v_{3-i}}) - 2 \leq 2$. We can therefore assume that there is some $x \in V_3(G) \cap N_2(v_1)$ such that $\nu(G_{v_1,x}) \leq 2$. By Claim 2 and $\alpha(G_2) > 1$ we have

$\delta(G_{v_1,x}) = 2$ and therefore, by (x), $G_{v_1,x} \cong Ch_k$ for some $k \geq 2$ or $G_{v_1,x} \cong SCh_\ell$ for some $\ell \geq 1$. Some vertex of degree 2 in $G_{v_1,x}$ must also have degree 2 in G, and thus second degree 5 therein. It is easily checked that this is not possible since $\mathrm{N}(C_4; N(V_2(G))) = 0$. $\square^{\text{(viii)}}$

Proof of (ix). Trivial. $\square^{\text{(ix)}}$

Proof of (x). Assume that $G \ncong Ch_k$ for all $k \geq 2$. In particular G is not 2-regular by (v), but G is 2-stable by (vi). By Claim 1 and Lemma 2.1; $d^2(v) \in \{5, 6\}$ for all $v \in V(G_2)$.

Let v be a vertex of degree 2. If $d^2(v) = 6$ then $\nu(G_v) \leq \nu(G) - 2 \leq 0$. Inductively, by (xi), $G_v \in \mathcal{G}$ (since G is edge-critical by (ix) and thus G_v is connected by Lemma 1.2, and $\nu(G_v) = 0$ yields $\mathrm{N}(C_4; N(v)) = 0$). We then have $|N_2(v)| = 4$. $N_2(v)$ is disconnected since if $N_2(v)$ induced $C_4, K_{1,3}$ or P_4 then $\mathrm{N}(C_4; N(v)) \geq 1$. Furthermore, $N_2(v)$ is a minimal destabiliser by Lemma 1.4 and (vii).

It is easily checked that neither W_5 nor $(2C_7)_{2i}$ has disconnected destabilisers of size 4. The same holds for all bicycles BC_k (see e.g. [1, Lemma 6.3(e)]). The only possibility is that $\delta(G_v) = 2$ and $G_v \cong Ch_k$ for some $k \geq 2$. But then, by Lemma 1.5, $k = 3$ and $N_2(v) = V_2(Ch_k)$ which means precisely that $G \cong SCh_1$.

We may therefore assume that $d^2(v) = 5$ for all $v \in V_2(G)$. We will show that there is some $v \in V_2(G)$ such that $\mathrm{N}(C_4; N(v)) \geq 1$. Note that this is sufficient to prove the assertion since this would give $\nu(G_v) \leq 2$ and inductively $G_v \cong Ch_k$ or $G_v \cong SCh_\ell$. It is easy to see by the recursive construction and destabilisers of size 3 that then $G \cong Ch_{k+1}$ or $G \cong SCh_{\ell+1}$.

Suppose that $\mathrm{N}(C_4; N(v)) = 0$ for all $v \in V_2(G)$. Let v_1 and v_2 be two adjacent vertices of degree 2. Then the situation is as illustrated in Figure 4 by Claim 2.

If $d(w_{12}) = 3$ then $\nu(G_{v_1,w_{12}}) \leq \nu(G_{v_1}) - 2 \leq \nu(G) + 1 - 2 \leq 1$. Therefore $d(G_{v_1,w_{12}}; u_2) = 2$ and $H := G_{v_1,w_{12}} \cong Ch_k$, for some $k \geq 2$, by Claim 2 and induction. $S := N_2(w_{12}) \cap V(H) \setminus \{w_{11}\}$ is a destabiliser of H having size 4. If $|N(u_1) \cap N(u_2)| = 0$ then $|S \cap V_2(H)| \leq 2$ and $\alpha(V_2(H) \setminus S) > 1$ (since $\{u_2, w_{11}\} \notin E(H)$). It is easily seen that if S is connected then $\mathrm{N}(C_4; N(w_{12})) \geq 2$ giving $\nu(H) \leq -1$, while for disconnected S we get a contradiction to Lemma 1.4 or 1.5. Hence, $|N(u_1) \cap N(u_2)| = 1$ and in the same way we get $S = N_H[u] \cup \{z\}$ for some $u \in V_2(H)$ and $\mathrm{dist}_H(z, u) \geq 3$. Define x_1, x_2, A_1, A_2, A_3 in the same way as in the proof of Claim 2, where $x = w_{12}$. We may assume that $u \in A_3$ and then $A_1 \cup A_2$ neither contains a destabiliser of size 3, nor all of $V_2(H)$, but $\alpha(V_2(H) \cap (A_1 \cup A_2)) > 1$. Clearly then $A_1 \cup A_2$ does not destabilise H.

Hence, $d(w_{12}) = 4$, analogously $d(w_{22}) = 4$ and, clearly, $d(w) = 3$. Let $H' := G_{v_2,u_1}$. If $\delta(H') = 2$ then $H' \cong Ch_k$ for some $k \geq 2$. Both $S_1 = N(\{w_{12}, w\}) \setminus \{u_1, u_2\}$ and $S_2 = N(\{w_{12}, u_2\}) \setminus \{u_1, u_2, w\}$ destabilises H' and $G[S_i] \subseteq K_{1,3}$. By Lemmas 1.4 and 1.5 we would either have $G[S_i] \cong K_{1,3}$ for some i, or $G[S_i] \cong K_{1,2} + K_1$ for both i. In either case, $\mathrm{N}(C_4; N(w_{12})) \geq 2$ which would imply $\nu(H') \leq -1$. Therefore we may assume that all vertices at distance 3 from $\{v_1, v_2\}$ must have degree at least 4. Hence we get $\nu(G_{v_1,w_{12}}) \leq \nu(G_{v_1}) - 3 \leq \nu(G) + 1 - 3 \leq 0$.

But $d(G_{v_1,w_{12}}; u_2) = 2$ and, by induction, $G_{v_1,w_{12}} \cong Ch_k$ for some $k \geq 2$. It is easily seen that $k \neq 2$, and for $k \geq 3$ we would have $\mathrm{N}(C_4; N(w_{22})) \geq 1$ and therefore $\nu(G_{v_2,w_{22}}) \leq \nu(G_{v_2}) - 3 - \mathrm{N}(C_4; N(w_{22})) \leq -1$. $\square^{(x)}$

To prove the inductive step for the final assertion of Lemma 2.2 and complete the proof we make use of some further claims.

Claim 3. *If* $\nu(G) \leq 0$ *and* $G \notin \mathcal{G}$ *then* $\mathrm{N}(C_4; G) = 0$.

Proof of claim. By (iii), (vi) and (x) we have $\delta(G) \geq 3$. We begin by showing that $\mathrm{N}(C_4; N[V_3(G)] \cup V_4(G) \cup V_5(G)) = 0$ by deriving contradictions in four cases. The claim then follows by induction on d, the minimum degree of a vertex with $\mathrm{N}(C_4; v) > 0$, using $\nu(G_v) \leq \nu(G) - 3d^2(v) + 17d - 19$ by Lemma 2.1.

Case (a): $\mathrm{N}(C_4; V_3(G)) > 0$. Suppose $v_1 \in V_3(G)$ is such that $\mathrm{N}(C_4; v_1) > 0$. Since $\nu(G_{v_1}) \leq 5$ the opposite vertex of v_1 in the 4-cycle must have degree at least 4. It is easy to see that necessarily the local structure in the neighbourhood is as illustrated in Figure 5.

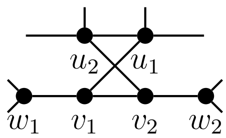

Figure 5: The structure around the vertex v_1 of degree 3 in G.

Note that $\nu(G_{v_1}) \leq 2$ and $N_2(v_1)$ is a destabiliser of size at most 6 in G_{v_1} by e.g. (ii). If G_{v_1} were not connected then $|\mathcal{C}(G_{v_1})| = 2$ and $\delta(C) = 2$ for both $C \in \mathcal{C}(G_{v_1})$ by Lemma 1.6 and (vii). But then $N_2(v_1) = N_{G_{v_1}}[x] \cup N_{G_{v_1}}[x']$, where $x, x' \in V_2(G_{v_1})$, by Lemma 1.4 and (x). It is then easily seen that $\mathrm{N}(C_4; N(v_1)) \geq 3$, which would give $\nu(G_{v_1}) \leq -1$.

Hence, $|\mathcal{C}(G_{v_1})| = 1$. If $d^2(u_1) \leq 13$ then $S := \{w_2\} \cup N(u_1) \setminus \{v_1\}$ would be four degree-2 vertices of $G_{v_1} \cong Ch_k$ for some $k \geq 2$. Either $k = 2$ or $G_{v_1}[S] \cong 2 \cdot K_2$, which are both clearly impossible. Hence, $d^2(u_1) \geq 14$. If $d^2(u_1) \geq 15$ then $|N(u_1) \cup N(w_2)| \geq 2$ and $d^2(w_2) = 11$ by (vi) since v_2 has degree 1 in G_{u_1}, but then $\nu(G_{w_2}) \leq -2$. Thus, $d^2(u_1) = 14$ and $|S \cap V_2(G_{v_1})| \geq 3$. Therefore $G_{v_1} \ncong SCh_\ell$ for any $\ell \geq 1$. Thus, in G_{v_1}, u_2 has one neighbour of degree 2 and one of degree 3 ($G_{v_1} \ncong C_5$ is trivial). It is then readily verified that this is only possible if the situation is as illustrated in Figure 6.

It is then clear that $G_{v_1} \cong Ch_k$ for some $k \geq 4$ by since $\mathrm{N}(C_4; N(v_1)) = 3$. This makes $G \cong BC_{k+1}$ since $(BC_{k+1})_v \cong Ch_k$ where v is any vertex of degree 3. This, however, goes against the assumption $G \notin \mathcal{G}$.

Case (b): $\mathrm{N}(C_4; N(V_3(G))) > 0$. Suppose v is a vertex of degree 3 such that $\mathrm{N}(C_4; N(v)) > 0$. By (a) at least one of the neighbours, w_1, w_2 and w_3, of v must have degree ≥ 4. We may assume $d(w_1) = d(w_2) = 3$, $d(w_3) = 4$, since $d^2(v) < 11$.

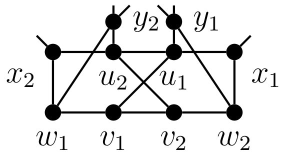

Figure 6: The structure around the vertex v_1 of degree 3 in G.

Not both w_1 and w_2 can have second degree 11, since if they did w_2 would belong to a Ch_k-component for some $k \geq 2$. It is however easy to exclude both $k = 2$ and $k \geq 3$ separately. There is therefore some vertex of degree 2 in G_v. In fact, by (x) and (a) we have at least two vertices of degree 2 in G_v. Thus we must have either a C_5 in $\mathcal{C}(G_v)$ or at least eight vertices with different degree in G and G_v. The latter is impossible since $|N_2(v)| = 7$. The former is impossible since we would get that at least one of w_1 and w_2 has two neighbours in a C_5-component. This would yield a cycle of length 4 through a vertex of degree 3, contradicting (a).

Case (c): $\mathrm{N}(C_4; V_4(G)) > 0$. If C is a cycle of length 4 in G containing a vertex v of degree 4 then all the vertices in C have degree 4 by (b).

Hence, $d^2(v) = 16$ and therefore $\nu(G_v) \leq \nu(G) + 2 - \mathrm{N}(C_4; N(v)) \leq 1$. Let $x \in C$ be the vertex at distance 2 from v in C. Clearly we have $d(G_v; x) = 2$. One of the two vertices in $N(x) \setminus V(C)$ has two neighbours in $N(x) \setminus V(C)$, and the other has at least one, since $G_v \cong Ch_k$ by (x). This would however give $\nu(G_v) \leq -1$.

Case (d): $\mathrm{N}(C_4; V_5(G)) > 0$. Let C be a cycle of length 4 on $\{c_1, c_2, c_3, c_4\}$ where $c_1c_3 \notin E(G)$, containing a vertex of degree 5, say c_1.

Since c_1 has no neighbours of degree 3 (by (b)) and at least two neighbours of degree at least 5 we have $d^2(v) \geq 2 \cdot 5 + 3 \cdot 4 = 22$. This gives $\nu(G_v) \leq 0$. Hence, in particular, all the vertices of C have degree 5 with three vertices of degree 4 and two vertices of degree 5 as neighbours.

Let $M_i = N(c_i) \setminus V(C)$ for $i \in [4]$. We have that $e(M_i, M_{i+2}) \geq 2$ for $i \in [2]$ since otherwise c_{i+2} would have degree 3 with second degree at least 11 in G_{c_i}. We would moreover have that $\nu(G_{c_i}) \leq 0$. Thus, inductively by (xi), we get that all components of G_{c_i} are in $\mathcal{G}$, but in all of the graphs of $\mathcal{G}$ the vertices of degree 3 all have second degree at most 10.

Each of the vertices in M_i (for $i \in [4]$) have at least one neighbour of degree 3, since otherwise there would be some vertex $m \in M_i$ such that $d^2(m) \geq 17$, which easily is seen to be impossible. Thus, a fortiori, G_{c_1} contains some vertices of degree 2, and therefore at least four vertices of degree 2. Note also that $|N_2(c_1)| = 16$ and $\mathrm{N}(C_4; N(c_1)) = 1$, since $\nu(G_{c_1}) = 0$.

All vertices of degree 3 in G_{c_1}, except possibly c_3, must belong to W_5- or $(2C_7)_{2i}$-components since otherwise there would be a cycle of length 4 through such a vertex. That vertex would then have to have degree at least 5 in G by (a)-(c). This however would yield $\mathrm{N}(C_4; N(c_1)) \geq 2$.

Hence, $\mathcal{C}(G_{c_1})$ consists of C_5's, W_5's and $(2C_7)_{2i}$'s, at least one of which is a C_5.

Now, since $d(G_{c_1}; x) = 3$ for all $x \in M_2 \cup M_4 \cup \{c_3\}$, at least one of the vertices in M_1 must have two neighbours in the C_5-component. This gives us again the contradiction $\mathrm{N}(C_4; N(c_1)) \geq 2$. $\square^{\text{(claim)}}$

Claim 4. *If $\nu(G) \leq 0$ and $G \notin \mathcal{G}$ then G_3 is 2-regular.*

Proof of claim. Firstly, $\mathrm{N}(C_4; G) = 0$ by Claim 3. One can show that G is not 3-regular (e.g. see [1, p. 202, v. 2015-07-16]). If $v \in V(G_3)$ has degree 1 in G_3 then $\nu(G_v) \leq 0$ and, by (xi), all components of G_v would be either W_5 or $(2C_7)_{2i}$. The neighbour of degree 3, u, of v would then have two neighbours of degree 4. Similarly all components of G_u are W_5 or $(2C_7)_{2i}$. This would give us that $\mathrm{N}(C_4; G) \neq 0$. Suppose now that there is a vertex v of degree 2 in G_3 with a neighbour u of degree 3. It is easily seen that $\alpha((G_v)_2) = 1$, since otherwise, by (viii), G_v would contain a C_5 with at least four vertices of degree 2 but by Lemma 1.7 at most three of these could belong to $N_2(v)$. This gives $e(N(u) \setminus \{v\}) > 0$, contradicting G being triangle-free. $\square^{\text{(claim)}}$

Claim 5. *If $\nu(G) \leq 0$ and $G \notin \mathcal{G}$ then $G_3 \cong \left(\frac{|V_3(G)|}{5}\right) \cdot C_5$.*

Proof of claim. As in the proof of Claim 4 we have $\mathrm{N}(C_4; G) = 0$ and may assume $\delta(G) = 3 \neq \Delta(G)$. By Claim 4 the induced graph G_3 consists of 2-regular components. Clearly G_3 does not contain any C_4-components. If G_3 were to contain a cycle of length 6 or more, then let v be a vertex of that cycle. We would then have $\delta(G_v) \leq 2$ with $\alpha((G_v)_2) > 1$, which is impossible for the same reason as before. $\square^{\text{(claim)}}$

Claim 6. *If $\nu(G) \leq 0$ and $G \notin \mathcal{G}$ then G is 4-regular.*

Proof of claim. By the previous claim G_3 consists of only C_5-components. If $v_1, v_2, \ldots, v_5$ are the vertices of such a component (assuming it exists) then we can show that their respective neighbours $w_1, w_2, \ldots, w_5$ that do not have degree 3 are such that the distance between w_i and w_{i+2} is exactly 2. Were the distance more than 2 then we can consider the graph $G' := G \setminus (\{v_i; i \in [5]\} \cup \{w_4\}) + w_1w_2$, for which it is easily seen that $\alpha(G') \leq \alpha(G) - 2$. Therefore we would have $\nu(G') \leq -4$. We can do the same analogously for all other pairs of w_i and w_{i+2}.

It is then easy to show that if $w_iw_{i+2} \in E(G)$ for some $i \in [5]$, then also w_jw_{j+2} for all $j \in [5]$ (taking indices modulo 5); consider e.g. G_{v_i} and $G_{v_i,v_{i+3}}$. Thus we have that the induced graph in G on $\{v_i, w_i; i \in [5]\}$ must be one of the two cases illustrated in Figure 7.

The right case in Figure 7 is easily seen to be impossible since this case would make the v_iw_i-edges redundant. The remaining case is shown to lead to a contradiction as well, as follows.

By (viii) for the G_{w_i}, $d^2(w_i) = 15$ for all $i \in [5]$. Suppose that $d^2(G_{w_1,v_2}; v_4) = 6$. Then w_5 belongs to a C_5-component, C, of G_{w_1,v_2,v_4}. By the previous, however, $V(C)$ contains at least three vertices that have degree 4 in G. Hence, $e(C, N(w_1) \cup N(v_2) \cup$

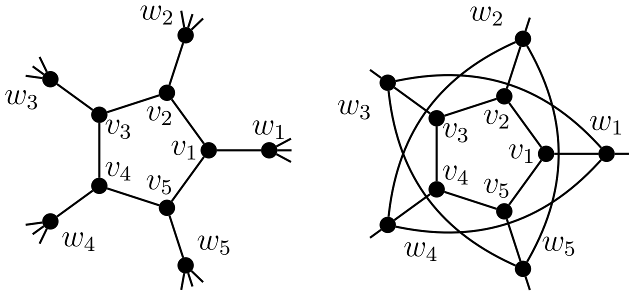

Figure 7: The two possible neighbourhoods of the C_5 from G_3 in G.

$N(v_4)\setminus\{v_1\}) \geq 8$, but $|N(w_1)\cup N(v_3)\cup N(v_4)\setminus\{v_1\}| \leq 7$, which would therefore give a cycle of length 4 in G. Hence, $d^2(G_{w_1,v_2}; v_4) \leq 5$. Since $e(w_4, N(v_2)) = 0$ we get $N(w_1)\cap N(w_4) \neq \emptyset$. Completely analogously we may show that $N(w_i)\cap N(w_{i+2}) \neq \emptyset$ for all $i \in [5]$.

This does however give us $\nu(G_{v_1,v_3}) \leq \nu(G_{v_1}) - 2 \leq \nu(G) + 3 - 2 \leq 1$, and the common neighbour of w_1 and w_3 has degree 2 in G_{v_1,v_3}, contradicting that $w_iw_{i+2} \notin E(G)$ implies $\delta(G_{v_i,v_{i+2}}) \geq 3$ for all $i \in [5]$. Thus $\delta(G) \geq 4$ and G is 4-regular by (iv) and Lemma 2.1. $\square^{(\text{claim})}$

Many of the following claims are quite close to the work of Radziszowski and Kreher in [11] and the slight modification by Backelin in [2]. Some of the proofs are just reworking their ideas to be able to use the previous properties in this article to get an analogous result. We therefore skip the details except in the parts where new ideas are used. For the detailed proofs see [9].

Claim 7. *If* $\nu(G) \leq 2$, $\delta(G) = 3$, $\mathrm{N}(C_4; G) = 0$ *then* $\delta(G_3) \geq 1$ *and* $d(w) \geq 4$ *for all* $w \in N(V_1(G_3))$.

Proof of claim. Obviously, $\delta(G_3) \geq 1$. Suppose $v \in V_1(G_3)$, then $d^2(G; v) = 11$. By (x) any vertex of degree 2 in G_v would belong to a C_5-component. However, if there were a C_5-component, C, in G_v then we would have that all five vertices of C are adjacent to some vertex in $N(v)$. This would however give us a C_4 in G. Thus there are no vertices of degree 3 at distance 2 from v in G. $\square^{(\text{claim})}$

In the remainder of this article assume that $\nu(G) \leq 0$, $G \notin \mathcal{G}$ and $H = G_v$ for some vertex $v \in V(G)$. Note that H_3 is a graph on 12 vertices since G is 4-regular and $\mathrm{N}(C_4; G) = 0$. Every vertex in H which does not have degree 3 has degree 4. The following claim follows immediately from Lemma 1.7 and that G is 4-regular.

Claim 8. *H_3 contains no cycles of length* 5.

Note that Claims 7 and 8 correspond to [11, Lemma 5.2.3]. We now define two graphs on twelve vertices, S_1 and S_2, just as in [11].

Definition 2.1. Denote the vertices of C_{12} by $\{c_0, c_1, \ldots, c_{11}\}$ so that $c_0c_1 \ldots c_{11}$ forms the cycle of length 12. The graph S_1 is formed by adding the edge c_0c_6 to C_{12} and the graph S_2 is formed by adding the two edges c_0c_6 and c_3c_9 to the C_{12}.

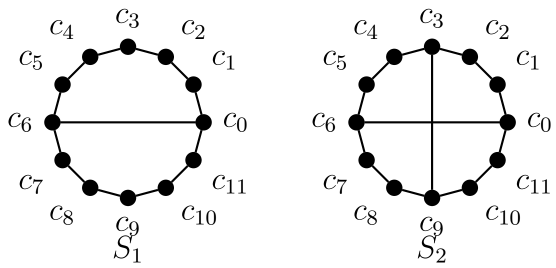

Figure 8: The two graphs S_1 and S_2.

We present a claim which consists of several statements, each being analogous to some lemma of [11]. The proofs in that article are easily adapted to the present situation using the above lemmas and properties.

Claim 9. *The induced subgraph of H on vertices of degree 3, H_3, has the following properties.*

(i) *H_3 contains no cycles of length 6.*

(ii) *Let $x, y \in V(H_3)$ have degree 2 in H_3. Furthermore suppose $N_H(x) = \{t, x_1, x_2\}$ and $N_H(y) = \{t, y_1, y_2\}$ where $t, x_1, y_1 \in V(H_3)$. Then both $x_1y_2, y_1x_2 \in E(H)$.*

(iii) *If $t \in V(H)$ has degree 3 in H_3 then it has two neighbours of degree 2 and one of degree 3 in H_3.*

(iv) *If $C \in \mathcal{C}(H_3)$, then $C \in \{K_2, C_8, C_{10}, C_{12}, S_1, S_2\}$.*

(v) *If $C \in \mathcal{C}(H_3)$, then $C \notin \{C_8, C_{10}, C_{12}\}$.*

(vi) *If $C \in \mathcal{C}(H_3)$, then $C \notin \{S_1, S_2\}$.*

Proof. (i); Analogous to [11, Lemma 5.2.4], (ii); analogous to [11, Lemma 5.2.5], (iii); analogous to [11, Lemma 5.2.6], (iv); analogous to [11, Lemma 5.2.7], (v); analogous to [11, Lemma 5.2.8], (vi); analogous to [11, Lemma 5.2.9], $\square^{\text{(claim)}}$

Note that by Claim 9 (parts (iv)-(vi)) we get that $H_3 \cong 6K_2$, since H_3 has twelve vertices. We will now prove that we must have some particular structure on the cycles of length 5 and 6 in the graph G.

Claim 10.

(i) *G contains no cycles of length 6.*

(ii) Through every pair of incident edges in G there is a cycle of length 5.

(iii) Two cycles of length 5 in G share at most one edge.

Proof. (i); Suppose that the vertices $\{c_1, c_2, \ldots, c_6\} \subseteq V(G)$ formed a cycle of length 6 (labelled cyclically in order). If $H = G_{c_1}$, then since $H_3 \cong 6K_2$ we get that $d(H_3; c_3) = 1$. However, c_5 would also be of degree 3 in H and at distance 2 from c_3, contradicting Claim 7.

(ii); Suppose otherwise and let $ux, xv \in E(G)$ be a pair of incident edges through which there is no cycle of length 5. We have $d^2(u) = 16$ and therefore $\nu(G_u) \leq \nu(G) + 2 \leq 2$. Since there is no cycle of length 5 through u, x, v we have that $e(N(u), N(v)) = 0$ and therefore $d^2(G_u; v) = 12$. This would however give $\nu(G_{u,v}) \leq \nu(G_v) - 3 \leq -1$.

(iii); Suppose otherwise, then the two cycles, C_1 and C_2, of length 5 would share exactly two consecutive edges or we would get a cycle of length 4 or less in G. Suppose therefore that the edges shared are $x_1v \in E(G)$ and $vx_2 \in E(G)$. Let x_3 and x_4 be the two remaining neighbours of v, $H = G_v$ and $X_i = N(x_i) \setminus \{v\}$. Because of (ii) there are also cycles of length 5 through the pairs of incident edges (x_2v, vx_3) and (x_2v, vx_4), whence $|E(X_2, X_3)|, |E(X_2, X_4)| \geq 1$. But since two of the vertices in X_2 belong to C_1 or C_2, and as such get paired with a vertex in X_1 we must have that there are edges from X_3 and X_4 to the same vertex in X_2, which then is of degree 2 in H_3, contradicting that $H_3 \cong 6K_2$. $\square^{\text{(claim)}}$

We are now ready to complete the proof of Lemma 2.2.

Proof of (xi). (Analogous of an argument in the proof of [2, Theorem 3]) We will prove that G contains at least one cycle of length 6, contradicting Claim 10(i).

Let $uv \in E(G)$. By Claim 10(ii)-(iii) there are two cycles C, C' of length 5 through uv which do not share any other edge than uv. Let $S = \{x_1, x_2, x_3, x_4\} = N(\{u, v\}) \cap (V(C) \cup V(C'))$ where x_1, x_2 are adjacent to u and x_3, x_4 are adjacent to v. If the only edges between the neighbourhoods of the x_i were the edges in $E(N(x_1), N(x_2))$ and $E(N(x_3), N(x_4))$ guaranteed by (ii) then we would get $\nu(G_S) \leq -3$. If $e(N(x_i), N(x_{i+1})) \geq 2$, where $i \in \{1, 3\}$, then we either get a cycle of length 4 or a cycle of length 6 through x_i and x_{i+1}.

If there are no cycles of length 6 through x_1, x_2 or x_3, x_4 we instead have an edge in $E(N(x_1) \cup N(x_2) \setminus \{u\}, N(x_3) \cup N(x_4) \setminus \{v\})$ which gives a cycle of length 6 through uv. $\square$

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