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Cycle decompositions of the Cartesian product of cycles

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Abstract

It is known that the r-dimensional hypercube Q_r can be decomposed into r-cycles and into $2r$-cycles when r is even. We generalize these results to the class of the Cartesian product of cycles. We also prove that the k-ary r-cube Q_r^k, which is the Cartesian product of r k-cycles, can be decomposed into $(tkr/2)$-cycles if t divides k and 4 divides t. Consequently, a decomposition of Q_r into $4r$-cycles for any even $r \geq 4$, is obtained.

1 Introduction

The graphs considered in this paper are finite, simple and undirected. By a k-cycle we mean a cycle of length k, denoted by C_k. A *decomposition* of a graph G is a collection $H_1, H_2, \ldots, H_r$ of edge-disjoint subgraphs of G, such that every edge of G belongs to exactly one H_i. If all the subgraphs in the decomposition of G are isomorphic to a graph H, we say that G can be decomposed into H or G has an H-decomposition. The *Cartesian product* of two graphs G_1 and G_2 is a graph $G_1 \Box G_2$ with vertex set $V(G_1) \times V(G_2)$, where vertices (u_1, u_2) and (v_1, v_2) are adjacent if and only if either $u_1 = v_1$ and u_2 is adjacent to v_2, or $u_2 = v_2$ and u_1 is adjacent to v_1.

An r-dimensional torus is the Cartesian product of r cycles. The torus $C_{k_1} \Box C_{k_2} \Box \cdots \Box C_{k_r}$ is a graph with $k_1 k_2 \ldots k_r$ vertices and $r k_1 k_2 \ldots k_r$ edges. In particular, the torus $\underbrace{C_k \Box C_k \Box \cdots \Box C_k}_{r \text{ factors}}$ is the *k-ary r-cube*, denoted by Q_r^k. The r-dimensional hypercube Q_r is the Cartesian product of r copies of the complete graph K_2. If r is even, then $Q_{r/2}^4 = Q_r$. The multidimensional tori, k-ary r-cubes and hypercubes are popular interconnection networks (see [9, 13]).

Graph decomposition has been the focus of a great deal of research. In particular, cycle decompositions of the Cartesian product of cycles have a long history. In 1973, Kotzig [12] proved that the Cartesian product of two cycles is decomposable into Hamiltonian cycles. Foregger [11] guaranteed such a decomposition for the Cartesian product of three cycles while Aubert and Schneider [2] generalised this result for the Cartesian product of a 4-regular graph and a cycle. Alspach et al. [1] extended the result further and proved that the Cartesian product of a finite number of cycles has a Hamiltonian decomposition. The existence of Hamiltonian decompositions of the hypercube Q_r, for even r, is an immediate consequence of this result. Furthermore, decompositions of the hypercubes and Cartesian product of even cycles into regular, connected, subgraphs are studied in [3, 5–8, 16]. Recently, Bogdanowicz [4] obtained some interesting results on the decomposition of the Cartesian product of directed cycles into cycles of equal lengths.

In this paper, we mainly focus on cycle decompositions of the Cartesian product of cycles.

Note that the hypercube Q_r has 2^r vertices and $r2^{r-1}$ edges. For even r, Ramras [15] proved that Q_r can be decomposed into r-cycles while Mollard and Ramras [14] obtained a decomposition of Q_r into $2r$-cycles and posed the following problem.

Problem 1.1 **([14])** *For which $k \geq 4$ dividing $r2^{r-1}$ does the hypercube Q_r have a decomposition into k-cycles?*

We consider this problem for the class of r- dimensional tori.

Problem 1.2 *For which $k \geq 4$ dividing $rk_1k_2 \dots k_r$ does the torus $C_{k_1} \square C_{k_2} \square \cdots \square C_{k_r}$ have a decomposition into k-cycles?*

El-Zanati et al. [10] proved that the graph $C_{2^{k_1}} \square C_{2^{k_2}} \square \cdots \square C_{2^{k_r}}$ can be decomposed into 2^t-cycles for any given t with $2 \leq t \leq k_1 + k_2 + \cdots + k_r$. As a consequence, they proved the existence of a cycle decomposition of Q_r into 2^t-cycles, where r is even and $2 \leq t \leq r$ and thus solved Problem 1.1 for the case $k = 2^t$.

In this paper, we obtain the following results.

Theorem 1.3 *Let $r \geq 2$, $k_i \geq 3$ for $1 \leq i \leq r$ be integers such that k_i is even for at least two values of i and let G be the Cartesian product of cycles $C_{k_1}, C_{k_2}, \dots, C_{k_r}$. Then G can be decomposed into cycles of length $2r$ and further, an edge can be selected from each of these $2r$-cycles to form a perfect matching of G.*

Theorem 1.4 *Let $r \geq 2$, $k_i \geq 3$ for $1 \leq i \leq r$ be integers such that k_i is even for at least two values of i and let G be the Cartesian product of cycles $C_{k_1}, C_{k_2}, \dots, C_{k_r}$. Let $k \in \{k_1, k_2, \dots, k_r\}$ be an even integer. Then G can be decomposed into cycles of length kr and further, $k/2$ edges can be chosen from each of these kr-cycles to form a perfect matching of G.*

For even r, $Q_r = Q^4_{r/2}$. Hence it follows that Q_r can be decomposed into r-cycles and into $2r$-cycles from Theorems 1.3 and 1.4 respectively.

Theorem 1.5 *Let t, k be positive integers such that 4 divides t and t divides k. Then the k-ary r-cube Q^k_r can be decomposed into $(tkr/2)$-cycles. Moreover, from each of these cycles, $kt/4$ edges can be selected such that they together form a perfect matching of Q^k_r.*

Corollary 1.6 *For even $r \geq 4$, the hypercube Q_r can be decomposed into $4r$-cycles.*

This solves the Problem 1.1 for the case $k = 4r$.

As the structure of the Cartesian product of cycles is recursive, use of induction is effective in proving the results for such graphs. The proofs of all our results are based on induction on the number r of cycles involved in the product. In Section 2, we prove a general induction step that is used in the proofs of all the above three theorems. In Section 3, we prove Theorems 1.3 and 1.4 while Theorem 1.5 is proved in Section 4.

2 General Induction Step

We first discuss the nature of the Cartesian product of two cycles for better understanding and set some notations which are used in the proofs.

We use the notation $[n]$ for the set $\{1, 2, \ldots, n\}$. In what follows, by product we mean the Cartesian product.

Notation 2.1 Consider the Cartesian product $C_m \square C_n$ of two cycles C_m and C_n. Label the vertices of C_m by the set $\{v_1, v_2, \ldots, v_m\}$ so that v_p is adjacent to $v_{p+1 \pmod m}$ and label the vertices of C_n by $\{1, 2, \ldots, n\}$ so that j is adjacent to $j+1 \pmod n$. So the vertex set of $C_m \square C_n$ is given by $\{(v_p, j) \colon p \in [m] \text{ and } j \in [n]\}$. Denote (v_p, j) by v^j_p for convenience; the subscripts are computed modulo m with representatives in $[m]$ and superscripts are computed modulo n with representatives in $[n]$.

By the definition of the Cartesian product of graphs, the edge set of $C_m \square C_n$ consists of n copies of the cycle C_m (say) $C^1_m, C^2_m, \ldots, C^n_m$ and the edges joining the corresonding vertices in C^j_m and C^{j+1}_m, for each $j \in [n]$. Therefore the vertex v^j_p in C^j_m is adjacent to the corresponding vertex v^{j+1}_p in C^{j+1}_m, for all $p \in [m]$. For even m, let M_1 and M_2 be the disjoint perfect matchings of C_m with $M_1 = \{v_p v_{p+1} \colon p = 1, 3, \ldots, m-1\}$ and $M_2 = \{v_p v_{p+1} \colon p = 2, 4, \ldots, m\}$. Let M^j_1 and M^j_2 be the matchings of the cycle C^j_m corresponding to M_1 and M_2, respectively. Let e^j_p be the edge $v^j_p v^j_{p+1}$ of C^j_m and f^j_p be the cross edge $v^j_p v^{j+1}_p$ for $p \in [m]$ and $j \in [n]$. (See Figure 1 for better understanding.)

We prove the general induction step followed by its illustration.

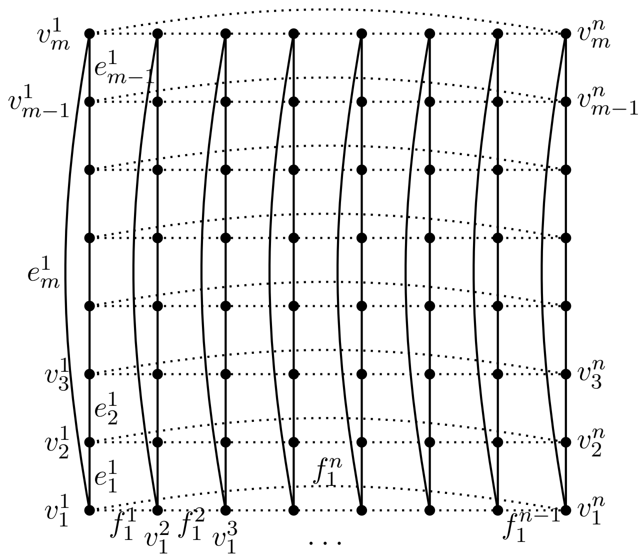


Figure 1: $C_m \Box C_n$

Theorem 2.2 (General Induction step). *Let H be a graph which has a decomposition into cycles $C_{k_1}, C_{k_2}, \ldots, C_{k_t}$ and each C_{k_i} contains a set M_i of $m_i \geq 0$ edges, such that $\bigcup_{i=1}^t M_i$ is a perfect matching of H. Let C_r be a cycle of length r. Then the graph $H \Box C_r$ has a decomposition into cycles $C^j_{k_1+2m_1}, C^j_{k_2+2m_2}, \ldots, C^j_{k_t+2m_t}$, for $1 \leq j \leq r$. Further, each $C^j_{k_i+2m_i}$ contains a set F_i^j of m_i edges such that $\bigcup_{j=1}^r \bigcup_{i=1}^t F_i^j$ is a perfect matching of $H \Box C_r$.*

Proof. Let $G = H \Box C_r$. The graph G consists of r copies $H^1, H^2, \ldots, H^r$ of H such that the vertices of H^j are adjacent to the corresponding vertices of H^{j+1} for $j \in [r]$, where the addition in the superscripts is taken modulo r. Let $C^j_{k_i}$ be the k_i-cycle in H^j corresponding to the cycle C_{k_i} of H and let M_i^j be the matching of $C^j_{k_i}$ corresponding to the matching M_i of C_{k_i}, where $i \in [t]$ and $j \in [r]$. Then M_i^j consists of m_i edges $e^j_{i1}, e^j_{i2}, \ldots, e^j_{im_i}$ of $C^j_{k_i}$. Let $M^j = \bigcup_{i=1}^t M_i^j = \{e^j_{il} \colon l \in [m_i];\ i \in [t]\}$. Then M^j is a perfect matching of H^j. Let u^j_{il} and v^j_{il} be the end vertices of the edge e^j_{il}, $l \in [m_i]$. Let f^j_{il} be the edge of G with end vertices u^j_{il} and u^{j+1}_{il} while h^j_{il} be the edge of G joining v^j_{il} to v^{j+1}_{il}.

For every i and j, we construct a $(k_i + 2m_i)$-cycle $C^j_{k_i+2m_i}$ in G from the cycle $C^j_{k_i}$ by deleting the matching M_i^j and adding the matching M_i^{j+1} of $C^{j+1}_{k_i}$ and also adding the edges f^j_{il} and h^j_{il} between these matchings. Let

$$C^j_{k_i+2m_i} = (C^j_{k_i} - M_i^j) \cup M_i^{j+1} \cup \{f^j_{il} \colon l \in [m_i]\} \cup \{h^j_{il} \colon l \in [m_i]\} \text{ (see Figure 2).}$$

Thus we have constructed $(k_i + 2m_i)$-cycles $C^j_{k_1+2m_1}, C^j_{k_2+2m_2}, \ldots, C^j_{k_t+2m_t}$ of G, for each $j \in [r]$.

We prove that these cycles are edge-disjoint. For every i, the cycles $C^1_{k_i}, C^2_{k_i}, \ldots, C^r_{k_i}$ are vertex-disjoint and so are edge-disjoint in G. Further, for every j, the cycles

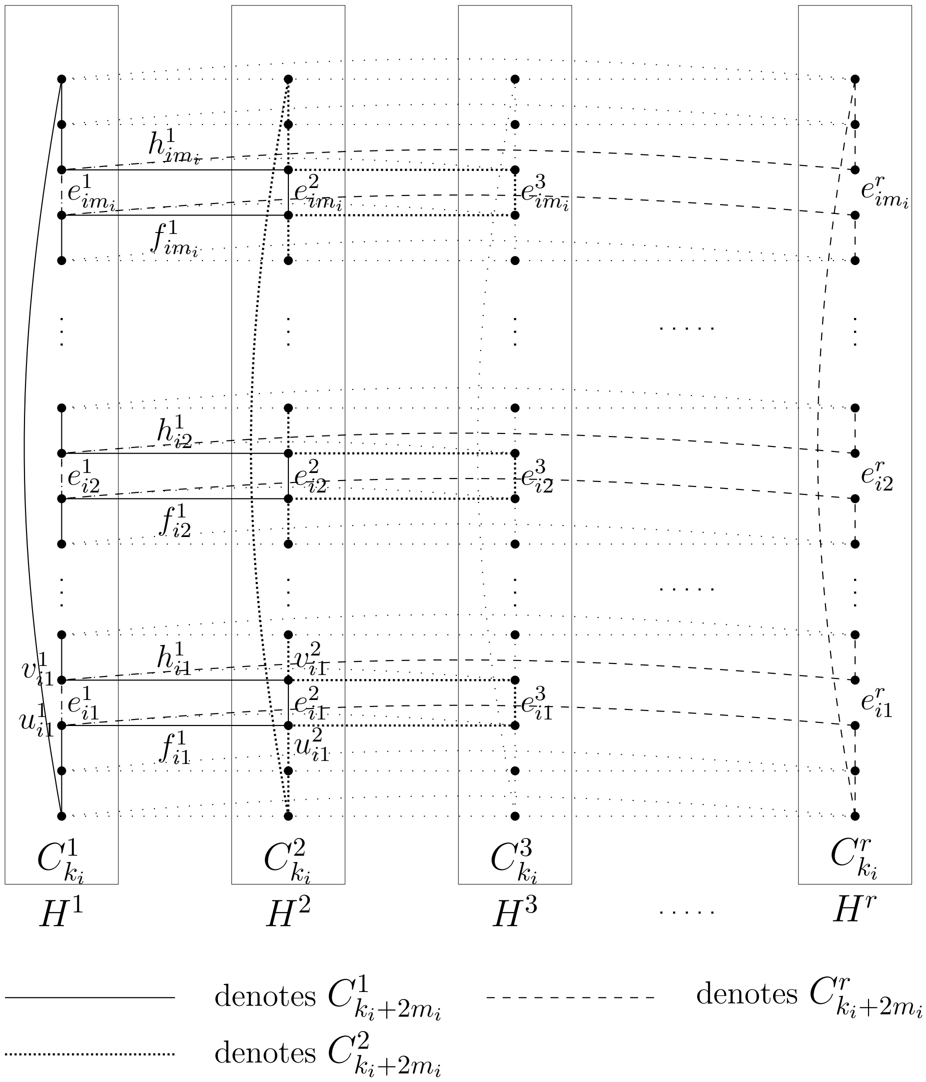


Figure 2: Construction of $(k_i + 2m_i)$-cycles in $G = H \Box C_r$

$C^j_{k_1}, C^j_{k_2}, \ldots, C^j_{k_t}$ are edge-disjoint in H^j and their matchings $M^j_1, M^j_2, \ldots, M^j_t$ are vertex-disjoint. This implies that the cycles $C^j_{k_1+2m_1}, C^j_{k_2+2m_2}, \ldots, C^j_{k_t+2m_t}$ are edge-disjoint in G. Thus the cycles $C^j_{k_i+2m_i}$, where $i \in [t]$ and $j \in [r]$, are edge-disjoint. These cycles together decompose the graph G as

$$\begin{aligned}\sum_{j=1}^{r}\sum_{i=1}^{t} |C^j_{k_i+2m_i}| &= r[(k_1 + k_2 + \cdots + k_t) + 2(m_1 + m_2 + \cdots + m_t)] \\ &= r[|E(H)| + |V(H)|] = |E(G)|.\end{aligned}$$

Now we need to select m_i edges from each cycle $C^j_{k_i+2m_i}$ that will form a perfect matching of G. Let $F^j_i = M^{j+1}_i$, for $i \in [t], j \in [r]$. Since $\bigcup_{i=1}^{t} M^j_i$ is a perfect matching of H^j and the graphs H^j are vertex-disjoint,

$$M = \bigcup_{j=1}^{r} \bigcup_{i=1}^{t} F_i^j = \bigcup_{j=1}^{r} \bigcup_{i=1}^{t} M_i^{j+1} = \bigcup_{j=1}^{r} \bigcup_{i=1}^{t} M_i^j$$

is a matching in G. In fact, M is a perfect matching of the graph G since

$$|M| = \sum_{j=1}^{r} \sum_{i=1}^{t} |M_i^j| = \sum_{j=1}^{r} \frac{|V(H)|}{2} = r\frac{|V(H)|}{2} = \frac{|V(G)|}{2}.$$

This completes the proof. □

Illustration 2.3 Let $H = C_4$. Then H trivially has a cycle decomposition. Let $\{e_1, e_2\}$ be a perfect matching of H. The graph $H \Box C_3$ consists of three copies C_4^1, C_4^2 and C_4^3 of C_4 and the edges joining the corresonding vertices in C_4^j and $C_4^{j+1 \pmod 3}$, for each $j \in [3]$. This graph decomposes into 8-cycles C_8^1, C_8^2, C_8^3 such that each C_8^j contains the matching $\{e_1^{j+1 \pmod 3}, e_2^{j+1 \pmod 3} \colon j \in [3]\}$. Clearly, the set $\{e_1^j, e_2^j \colon j \in [3]\}$ forms a perfect matching of $H \Box C_3$.

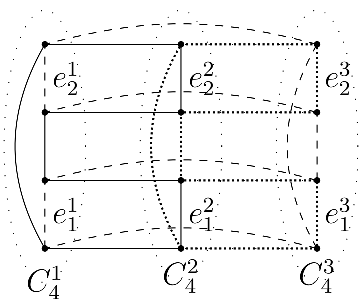


Figure 3: Decomposition of $C_4 \Box C_3$ into 8-cycles

3 Decomposition into small cycles

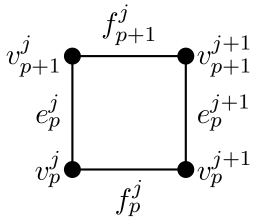
In this section, we prove Theorems 1.3 and 1.4. As the proofs are based on induction on the number of cycles involved in the product, we need to prove the basis steps for both the theorems.

The following lemma serves as the basis step for Theorem 1.3.

Lemma 3.1 *Suppose C_m and C_n are cycles of even length. Then the graph $C_m \Box C_n$ can be decomposed into 4-cycles. Moreover, from each of these 4-cycles, one edge can be selected such that the collection of these edges together forms a perfect matching of $C_m \Box C_n$.*

Proof. Recall Notation 2.1. The vertices of the graph $G = C_m \Box C_n$ are labelled by $(v_p, j) = v_p^j$, where $v_p \in V(C_m) = \{v_1, v_2, \ldots, v_m\}$ and $j \in V(C_n) = \{1, 2, \ldots, n\}$. The graph G consists of n copies $C_m^1, C_m^2, \ldots, C_m^n$ of the cycle C_m, where the corresponding vertex v_p^j in C_m^j is adjacent to v_p^{j+1} in C_m^{j+1}, for all $p \in [m]$. Let M_1 and M_2 be the disjoint perfect matchings of C_m with $M_1 = \{v_p v_{p+1} \colon p = 1, 3, \ldots, m-1\}$ and $M_2 = \{v_p v_{p+1} \colon p = 2, 4, \ldots, m\}$. Let M_1^j and M_2^j be the matchings of the cycle C_m^j corresponding to M_1 and M_2, respectively. Let e_p^j be the edge $v_p^j v_{p+1}^j$ of C_m^j and let f_p^j be the cross edge $v_p^j v_p^{j+1}$ for $p \in [m]$ and $j \in [n]$.

We construct 4-cycles which decompose the graph G by using the four edges e_p^j, e_p^{j+1}, f_p^j and f_{p+1}^j. Note that for every p and j, these four edges induce a 4-cycle in G. So in terms of the vertices, the 4-cycles are $(v_p^j, v_{p+1}^j, v_{p+1}^{j+1}, v_p^{j+1}, v_p^j)$.



For both p and j odd, denote such a 4-cycle by Z_p^j while for both p and j even, denote it by W_p^j. It follows that the cycles Z_p^j are mutually vertex-disjoint. Similarly, the cycles W_p^j are mutually vertex-disjoint. Let $Z = \{Z_p^j \colon p = 1, 3, \ldots m-1; j = 1, 3, \ldots n-1\}$ and $W = \{W_p^j \colon p = 2, 4, \ldots m; j = 2, 4, \ldots n\}$ be the collections of 4-cycles. Then $|V(Z)| = |V(W)| = 4 \times \frac{m}{2}\frac{n}{2} = mn = |V(G)|$. So the union of all cycles in Z forms a spanning subgraph of G. Similar is the case for W. Further, from the construction it is clear that each W_p^j is edge-disjoint from all the cycles Z_p^j. Therefore the collection of 4-cycles $Z \cup W$ decomposes the graph G. An illustration of such a decomposition is given in Figure 4.

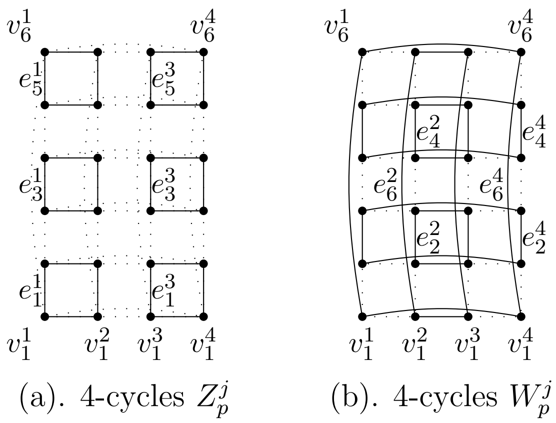


Figure 4: Decomposition of $C_6 \Box C_4$ into 4-cycles

Further, we select a set of edges

$$\mathcal{M}_1 = \{e_p^j \in E(Z_p^j) \colon p = 1, 3, \dots m-1;\ j = 1, 3, \dots n-1\}$$

from the cycles Z_p^j while from the cycles W_p^j, we select

$$\mathcal{M}_2 = \{e_p^j \in E(W_p^j) \colon p = 2, 4, \dots m;\ j = 2, 4, \dots n\}.$$

Clearly, $\mathcal{M}_1 \cup \mathcal{M}_2$ forms a matching of G. In fact,

$$|\mathcal{M}_1| + |\mathcal{M}_2| = \frac{mn}{4} + \frac{mn}{4} = \frac{mn}{2} = \frac{|V(C_m \Box C_n)|}{2}.$$

Therefore $\mathcal{M}_1 \cup \mathcal{M}_2$ forms a perfect matching of G. $\square$

We now give a formal **Proof of Theorem 1.3.**

Proof. We proceed by induction on the number of cycles r. The case $r = 2$ follows from Lemma 3.1. Suppose $r \geq 3$. Assume that the statement holds for the product of $r-1$ cycles. Write G as $G = H \Box C$, where C is a cycle and H is the product of $r-1$ cycles, at least two of which are even. H is $2(r-1)$-regular graph and so $|E(H)| = (r-1)|V(H)|$. By induction, H can be decomposed into cycles of length $2(r-1)$ (say) $Z_1, Z_2, \dots, Z_t$, where $t = \frac{|E(H)|}{2(r-1)} = \frac{|V(H)|}{2}$. Moreover, each Z_i contains an edge e_i such that $\{e_i \colon i = 1, 2, \dots, t\}$ is a perfect matching of H. By Theorem 2.2, G can be decomposed into $2r$-cycles $\Gamma_1, \Gamma_2, \dots, \Gamma_{t'}$. Then $t' = \frac{|E(G)|}{2r} = \frac{2r|V(G)|}{2}\frac{1}{2r} = \frac{|V(H)||C|}{2} = t|C|$. Moreover, every Γ_i contains an edge f_i such that $\{f_i \colon i = 1, 2, \dots, t'\}$ is a perfect matching of G. $\square$

The following lemma is the basis step for the proof of Theorem 1.4.

Lemma 3.2 *Let C_m and C_n be cycles of even lengths and let $l \in \{2m, 2n\}$. Then the graph $C_m \Box C_n$ can be decomposed into l-cycles. Moreover, from each of these l-cycles, $l/4$ edges can be selected such that the collection of these edges form a perfect matching of $C_m \Box C_n$.*

Proof. Without loss of generality, we construct a decomposition of $C_m \Box C_n$ into cycles of length $2m$.
Recall Notation 2.1. The $2m$-cycles in the decomposition of $C_m \Box C_n$ are as follows.

For *odd* j, let $Z^j = M_1^j \cup M_2^{j+1} \cup \{f_p^j \colon p \in [m]\}$
and for *even* j, let $W^j = M_1^j \cup M_2^{j+1} \cup \{f_p^j \colon p \in [m]\}$.

The $2m$-cycles Z^j (similarly, W^j) are mutually vertex-disjoint and together form a 2-regular spanning subgraph (say) F_1 (similarly, F_2) of $C_m \Box C_n$. Moreover, from the construction, it follows that F_1 and F_2 are edge-disjoint. Now

$$|E(F_1)| + |E(F_2)| = \frac{n}{2} \times 2m + \frac{n}{2} \times 2m = 2mn = |E(C_m \Box C_n)|.$$

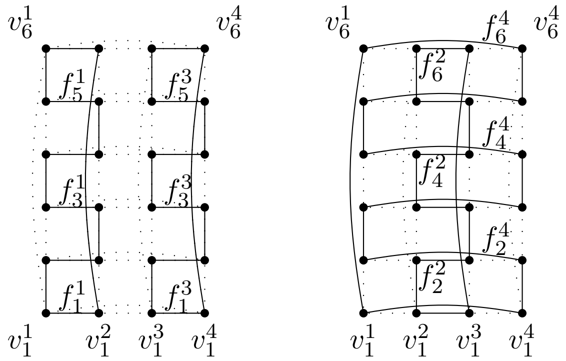


Figure 5: Decomposition of $C_6 \Box C_4$ into 12-cycles

Thus the collection $F_1 \cup F_2 = \{Z^j : j = 1, 3, \ldots n-1\} \cup \{W^j : j = 2, 4, \ldots n\}$ gives an edge decomposition of $C_m \Box C_n$ into $2m$-cycles.

Now we select the set of edges $\mathcal{M}_1 = \{f_p^j = v_p^j v_p^{j+1} : p = 1, 3, \ldots m-1\}$ from Z^j, where $j = 1, 3, \ldots n-1$ and the set of edges $\mathcal{M}_2 = \{f_p^j = v_p^j v_p^{j+1} : p = 2, 4, \ldots m\}$ from W^j, where $j = 2, 4, \ldots n$. As the cycles Z^j (similarly, W^j) are vertex-disjoint, $\mathcal{M}_1$ (similarly, $\mathcal{M}_2$) is a matching in $C_m \Box C_n$. Also, note that in $\mathcal{M}_1$, we have chosen the edges from odd levels while in $\mathcal{M}_2$, the edges are taken from even levels. So $\mathcal{M}_1 \cup \mathcal{M}_2$ forms a matching of $C_m \Box C_n$. In fact, $|\mathcal{M}_1| + |\mathcal{M}_2| = \frac{mn}{4} + \frac{mn}{4} = \frac{mn}{2} = \frac{|V(C_m \Box C_n)|}{2}$. So $\mathcal{M}_1 \cup \mathcal{M}_2$ is a perfect matching of $C_m \Box C_n$, as desired. (See Figure 5 for illustration.) $\Box$

We are all set to prove Theorem 1.4.

Proof of Theorem 1.4. The proof is by induction on the number of cycles r in the product. Without loss of generality, assume that k_1, k_2 are even. Suppose $r = 2$. Then $G = C_{k_1} \Box C_{k_2}$ and $k \in \{k_1, k_2\}$. Now the result follows by taking $l = 2k$ in the above lemma. Suppose $r \geq 3$. Let $H = C_{k_1} \Box C_{k_2} \Box \ldots \Box C_{k_{r-1}}$. Then $G = H \Box C_{k_r}$. By induction hypothesis, H can be decomposed into cycles of length $k(r-1)$ (say) $Z_1, Z_2, \ldots, Z_t$, where $t = |E(H)|/k(r-1)$. Moreover, each Z_i contains $k/2$ edges $e_{i1}, e_{i2}, \ldots, e_{ik/2}$ such that $\{e_{ij} : i \in [t], j \in [k/2]\}$ is a perfect matching of H. By Theorem 2.2, G can be decomposed into kr-cycles $\Gamma_1, \Gamma_2, \ldots, \Gamma_{t'}$ with $t' = tk_r$. Further, every Γ_i contains $k/2$ edges $f_{i1}, f_{i2}, \ldots, f_{ik/2}$ such that $\{f_{ij} : i \in [t'], j \in [k/2]\}$ is a perfect matching of G. $\Box$

4 Decomposition of Q_r^k

In this section, we prove Theorem 1.5 using induction on r. The non-trivial part of the proof is to prove the basis step. The basis step consists of two parts; the first part deals with the decomposition of $C_k \Box C_k$ into tk-cycles and the second deals with finding a perfect matching from the cycle decomposition.

It follows from the definition of the Cartesian product that the graph $C_k \Box C_k$ can be decomposed into k-cycles. In the following lemma, we construct tk-cycles, for any divisor $t \geq 2$ of k, giving a decomposition of this graph. Our construction is similar to the construction of Hamiltonian cycles giving a decomposition of $C_m \Box C_n$ due to Kotzig [12] (also see Mollard and Ramras [14]).

Lemma 4.1 *Suppose $t \geq 2$ and $k \geq 4$ be integers such that t divides k. Then the graph $C_k \Box C_k$ can be decomposed into tk-cycles.*

Proof. Let $V(C_k) = \mathbb{Z}_k$, with i adjacent to $i+1$ modulo k and let $G = C_k \Box C_k$. Then $V(G) = \mathbb{Z}_k \times \mathbb{Z}_k$. Two vertices of G are adjacent if their corresponding 2-tuples differ in exactly one component by ± 1 modulo k. We call an edge *horizontal with direction* 1, if its end vertices are $(x, y), (x+1, y)$, where $x, y \in \mathbb{Z}_k$. Similarly, we say that an edge is *vertical with direction* 2, if its end vertices are $(x, y), (x, y+1)$.

We construct a tk-cycle viz. Φ_1 as follows.

$$\Phi_1 = \langle (0,0), (1,0), \ldots, (t-1, 0),$$
$$(t-1, 1), (t, 1), \ldots, (2(t-1), 1),$$
$$(2(t-1), 2), (2(t-1)+1, 2), \ldots, (3(t-1), 2),$$
$$\vdots$$
$$((k-1)(t-1), k-1), ((k-1)(t-1)+1, k-1), \ldots, (k(t-1), k-1), (k(t-1), k) = (0,0)\rangle,$$

since $k \equiv 0 \pmod{k}$.

One can observe that Φ_1 consists of k vertical edges with direction 2, and $(t-1)k$ horizontal edges with direction 1 forming k horizontal paths each of length $t-1$. Alternately, we write $\Phi_1 = \langle (0,0), S\rangle$, where the initial vertex of Φ_1 is $(0,0)$ and its edge-direction sequence is $S = (\underbrace{1, 1, \ldots, 1}_{t-1 \text{ terms}}, 2, 1, 1, \ldots, 1, 2, \ldots, 1, 1, \ldots, 1, 2)$. Note that the string $1, 1, \ldots, 1, 2$ of length t is repeated k times in S. For convenience, we say that Φ_1 is a horizontal cycle. (See Figure 6(a).)

Similarly,

$$\Gamma_1 = \langle (0,0), (0,1), (0,2), \ldots, (0, t-1),$$
$$(1, t-1), (1, (t-1)+1), \ldots, (1, 2(t-1)),$$
$$(2, 2(t-1)), (2, 2(t-1)+1), (2, 2(t-1)+2), \ldots, (2, 3(t-1)),$$
$$\vdots$$
$$(k-1, (k-1)(t-1)), (k-1, (k-1)(t-1)+1), \ldots, (k-1, k(t-1)), (k, k(t-1)) = (0,0)\rangle$$

is a vertical cycle. The initial vertex of Γ_1 is $(0,0)$ and its edge-direction sequence is given by $S' = (\underbrace{2, 2, \ldots, 2}_{t-1 \text{ terms}}, 1, 2, 2, \ldots, 2, 1, \ldots, 2, 2, \ldots, 2, 1)$. The cycle Γ_1 consists of k horizontal edges and k vertical paths each of length $t-1$. (See Figure 6(b).)

We decompose $C_k \square C_k$ into tk-cycles using the horizontal cycles and the vertical cycles constructed from Φ_1 and Γ_1. Since t divides k, we have $k = tm$ for some $m \geq 1$.

The remainder of the proof is split into three steps as follows.

(I). We construct horizontal cycles from Φ_1 by using graph-isomorphisms.

For $i \in [m]$, let $\phi^i : V(C_k \square C_k) \to V(C_k \square C_k)$ be defined by $\phi^i((x, y)) = (x + (i-1)t, y)$. It follows that ϕ^i is a graph-isomorphism. Let $\Phi_i = \phi^i(\Phi_1)$. Then Φ_i is a tk-cycle with initial vertex $((i-1)t, 0)$ and edge-direction sequence S.

For example, the cycle Φ_2 is given by $((t, 0), S)$, where $(t, 0)$ is the initial vertex and S is the direction sequence previously defined. So

$$\begin{gathered}\Phi_2 = \langle (t,0), (t+1,0), \ldots, (2t-1,0), \\ (2t-1,1), (2t,1), \ldots, (3t-2,1), \\ (3t-2,2), (3t-1,2), \ldots, (4t-3,2), \\ \vdots \\ ((k-1)(t-1)+1, k-2), ((k-1)(t-1)+2, k-2), \ldots, (k(t-1)+1, k-2) = (1, k-2), \\ (1, k-1), (2, k-1), \ldots, (t, k-1), (t, 0)\rangle.\end{gathered}$$

Observe that $V(\Phi_1) = \{(y(t-1)+s, y) \ : \ y \in \mathbb{Z}_k, \ s \in \{0, 1, \ldots, t-1\}\}$.

Hence

$$V(\Phi_i) = \{(y(t-1) + (i-1)t + s, y) \ : \ y \in \mathbb{Z}_k, \ s \in \{0, 1, \ldots, t-1\}\}. \qquad (\star)$$

Note that the horizontal edges in the cycle Φ_i are of the form $< (x, y), (x+1, y) >$, where $x = y(t-1) + (i-1)t + s$, and the vertical edges in Φ_i are of the form $< (x, y), (x, y+1) >$, where $x = (y+1)(t-1) + (i-1)t$.

Claim: The cycles $\Phi_1, \Phi_2, \ldots, \Phi_m$ are mutually vertex-disjoint.

On the contrary assume that for $i < j$, the cycles Φ_i and Φ_j have a vertex v in common. Being a vertex in Φ_i, for some $y \in \mathbb{Z}_k$ and $s \in \{0, 1, \ldots, t-1\}$,

$$v = (y(t-1) + (i-1)t + s, \ y).$$

Similarly, as a vertex in Φ^j, for some $y' \in \mathbb{Z}_k$ and $s' \in \{0, 1, \ldots, t-1\}$,

$$v = (y'(t-1) + (j-1)t + s', \ y').$$

Therefore we get

$$y(t-1) + (i-1)t + s = y'(t-1) + (j-1)t + s'$$

in $\mathbb{Z}_k$ and $y = y'$. This implies that

$$s - s' \equiv (j-i)t \pmod{k}.$$

As $k = tm$ with $1 \leq j - i < m$, we have $s - s' \equiv 0 \pmod{t}$. But $0 \leq s, s' < t$ gives $s = s'$. Hence $(j - i)t \equiv 0 \pmod{k}$. This implies that $j - i \equiv 0 \pmod{m}$, which is a contradiction as $1 \leq j - i < m$. Thus the cycles $\Phi_1, \Phi_2, \ldots, \Phi_m$ are mutually vertex-disjoint.

Further, the subgraph $\bigcup_{i=1}^{m} \Phi_i$ has $mtk = k^2 = |V(C_k \Box C_k)|$ vertices and so it spans the graph $C_k \Box C_k$.

(II). We now construct the vertical cycles.

The map $R : V(C_k \Box C_k) \to V(C_k \Box C_k)$ defined by $R((x, y)) = (y, x)$ is a graph-isomorphism. Note that $\Gamma_1 = R(\Phi_1)$. Let $\Gamma_j = R(\Phi_j)$, for $j \in [m]$. Then Γ_j is the cycle of length tk with initial vertex $(0, (j - 1)t)$ and edge-direction sequence S'. Since the horizontal cycles Φ_i are vertex-disjoint, the vertical cycles Γ_j are also vertex-disjoint and together they too span the graph $C_k \Box C_k$

Note that as Γ_j is obtained from Φ_j just by reversing the co-ordinates of the vertices, the horizontal (similarly, vertical) edges in Γ_j are obtained from the vertical (similarly, horizontal) edges in Φ_j, just by reversing the co-ordinates of the end vertices. (See an illustration in Figure 6.)

(III). We prove that the cycles Φ_i and Γ_j are edge-disjoint for any i and j.

Without loss of generality assume that $i = 1$. Suppose if possible Φ_1 and Γ_j have an edge (say) e in common. Then the edge e is either horizontal or vertical.

Suppose e is horizontal. Then the end vertices of e are (x, y) and $(x + 1, y)$ for some $x, y \in \mathbb{Z}_k$. As e is an edge in Φ_1, from equation $(\star)$, we have $x = y(t - 1) + s$ for some $s \in \{0, 1, \ldots, t - 2\}$. Also, e is a horizontal edge in Γ_j. Therefore $y = (x + 1)(t - 1) + (j - 1)t$ in $\mathbb{Z}_k$. This gives

$$y \equiv y(t - 1)^2 + (s + 1)(t - 1) + (j - 1)t \pmod{k}.$$

As t divides k, we have

$$y \equiv y(0 - 1)^2 + (s + 1)(0 - 1) + (j - 1)0 \pmod{t}.$$

This implies that $s + 1 \equiv 0 \pmod{t}$. Therefore t divides $s + 1$, a contradiction to the fact that $0 \leq s \leq t - 2$.

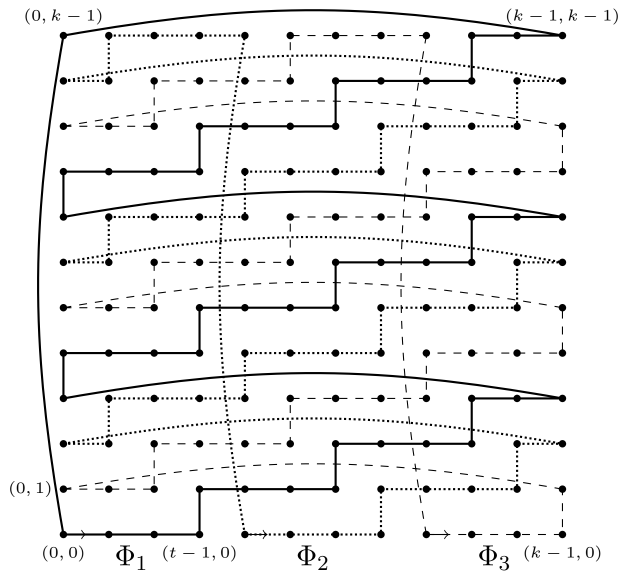
Suppose e is a vertical edge. Then the end-vertices of e are (x, y) and $(x, y + 1)$ for some $x, y \in \mathbb{Z}_k$. As e is a vertical edge in Φ_1, we have $x = (y + 1)(t - 1)$. Since $e \in E(\Gamma_j)$, $y = x(t - 1) + (j - 1)t + s$ for some $s \in \{0, 1, \ldots, t - 2\}$. Hence

$$y \equiv (y + 1)(t - 1)^2 + (j - 1)t + s \pmod{k}$$

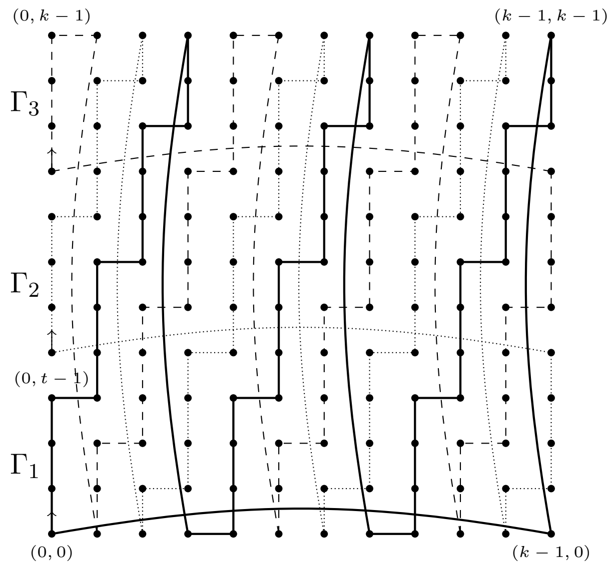
As t divides k,

$$y \equiv y + 1 + s \pmod{t},$$

giving $s + 1 \equiv 0 \pmod{t}$, a contradiction. Therefore the cycles Φ_i and Γ_j are edge-disjoint, for any i, j.



(a). Horizontal cycles Φ_1, Φ_2, Φ_3



(b). Vertical cycles $\Gamma_1, \Gamma_2, \Gamma_3$

Figure 6: Decomposition of $C_{12} \Box C_{12}$ into tk-cycles, with $t = 4, k = 12$

Thus $\bigcup_{i=1}^{m} \Phi_i$ and $\bigcup_{i=1}^{m} \Gamma_i$ are 2-regular edge-disjoint spanning subgraphs of the graph $C_k \Box C_k$. Hence the collection of tk-cycles $\{\Phi_i : i = 1, 2, \ldots, m\} \cup \{\Gamma_i : i = 1, 2, \ldots, m\}$ decompose the graph $C_k \Box C_k$. $\Box$

In the following lemma, we prove that if t is a multiple of 4, then the tk-cycles Φ_i and Γ_i that are constructed in the above lemma also satisfy an additional condition related to a perfect matching.

Lemma 4.2 *Let k, t be positive integers, t divides k and 4 divides t. Then the graph $C_k \Box C_k$ can be decomposed into tk-cycles. Further, from every tk-cycle in the decomposition, $kt/4$ edges can be chosen to form a perfect matching of $C_k \Box C_k$.*

Proof. Suppose $k = tm$ for some $m \geq 1$. Consider the horizontal tk-cycles Φ_i, $i \in [m]$, that are constructed in the proof of Lemma 4.1. Observe that for each $y \in \mathbb{Z}_k = \{0, 1, 2, \ldots, k-1\}$, every cycle Φ_i contains a unique horizontal path, $P_i^y = \langle (x, y), (x+1, y), \ldots, (x+(t-1), y) \rangle$ of length $t-1$, with $x = y(t-1) + (i-1)t$. Denote by e_{is}^y the edge of the path P_i^y with end vertices $(x+s, y)$ a nd $(x+s+1, y)$, $s = 0, 1, \ldots, t-2$. Amongst the $t-1$ edges of P_i^y, choose $t/4$ edges alternately starting from the first edge e_{i0}^y, and denote this set by M_i^y. Thus $M_i^y = \{e_{i0}^y, e_{i2}^y, e_{i4}^y, \ldots, e_{i(t/2-2)}^y\}$ is a matching. Let $M_i = \bigcup_{y=0}^{k-1} M_i^y$. Since P_i^y is vertex-disjoint with $P_i^{y'}$ for $y \neq y'$, M_i is a matching in $C_k \Box C_k$ consisting of $k|M_i^y| = kt/4 = k^2/4m$ edges of Φ_i. Let $M = \bigcup_{i=1}^{m} M_i$. Since the cycles Φ_i are mutually vertex-disjoint, M is a matching in $C_k \Box C_k$ consisting of $m|M_i| = k^2/4$ edges.

Similarly, for each $x \in \mathbb{Z}_k$, every vertical tk-cycle Γ_i contains a unique vertical path of length $t-1$ given by $P_i^x = \langle (x, y), (x, y+1), (x, y+2), \ldots, (x, y+(t-1)) \rangle$ with $y = x(t-1) + (i-1)t$. Denote by f_{is}^x the edge of the path P_i^x with end vertices $(x, y+s)$ and $(x, y+s+1)$. Select $t/4$ edges alternately from the path P_i^x, starting from the $t/2^{th}$ edge $f_{i(t/2)}^x$. Therefore the set $\{f_{i(t/2)}^x, f_{i(t/2+2)}^x, f_{i(t/2+4)}^x, \ldots, f_{i(t-2)}^x\}$, denoted by N_i^x, is a matching. Let $N_i = \bigcup_{x=0}^{k-1} N_i^x$ and let $N = \bigcup_{i=1}^{m} N_i$. Clearly, $N = \{f_{is'}^x : s' \text{ is even and } \mathrm{t}/2 \leq \mathrm{s}' \leq \mathrm{t}-2\}$ is a matching of $C_k \Box C_k$ consisting of $|N| = k^2/4$ edges from the cycles Γ_i.

Claim: $M \cup N$ is a matching of $C_k \Box C_k$.

Let $V(M)$ and $V(N)$ be the set of end vertices of the edges in M and the edges in N, respectively. Since M and N are matchings of $C_k \Box C_k$, it is sufficient to prove that $V(M) \cap V(N) = \phi$.

Assume that $V(M) \cap V(N) \neq \phi$. Let $v \in V(M) \cap V(N)$. As $v \in V(M)$, we have $v = (x+s, y)$ or $(x+s+1, y)$ with $x = y(t-1) + (j-1)t$, for some $y \in \{0, 1, \ldots, k-1\}$, $s \in \{0, 2, 4, \ldots, t/2-2\}$ and $j \in [m]$. Also $v \in V(N)$ implies that $v = (x', y'+s')$ or $(x', y'+s'+1)$ with $y' = x'(t-1) + (i-1)t$, for some $x' \in \{0, 1, \ldots, k-1\}$, $s' \in \{t/2, t/2+2, \ldots, t-2\}$ and $i \in [m]$.

Thus we get two cases as follows. Observe that because $v \in V(M) \cap V(N)$, each case involves two possible pairs for v. Also, it suffices to workout the contradiction for any one pair in each case.

Case 1. $v = (x+s, y)$ and $v = (x', y'+s')$, or $v = (x+s+1, y)$ and $v = (x', y'+s'+1)$.

Without loss of generality, assume that $v = (x + s, y)$ and $v = (x', y' + s')$. Then $x+s = x'$ and $y = y'+s'$ in $\mathbb{Z}_k$. Since $y' = x'(t-1)+(i-1)t$ and $x = y(t-1)+(j-1)t$, we have

$$\begin{aligned} y &\equiv x'(t-1) + (i-1)t + s' \pmod{k} \\ &\equiv (x+s)(t-1) + (i-1)t + s' \pmod{k} \\ &\equiv y(t-1)^2 + (j-1)t(t-1) + s(t-1) + (i-1)t + s' \pmod{k} \\ &\equiv y - s + s' \pmod{t} \ldots (\text{as } t \text{ divides } k). \end{aligned}$$

Therefore $s' \equiv s \pmod{t}$. Hence t divides $s' - s$. However, $0 \le s < s' \le t-2$ gives $0 < s' - s \le t - 2 < t$, a contradiction.

Case 2. $v = (x+s, y)$ and $v = (x', y'+s'+1)$, or $v = (x+s+1, y)$ and $v = (x', y'+s')$.

Again, without loss of generality, assume that $v = (x + s, y)$ and $v = (x', y' + s' + 1)$. Then $x + s = x'$ and $y = y' + s' + 1$ in $\mathbb{Z}_k$. Since $y' = x'(t-1) + (i-1)t$ and $x = y(t-1) + (j-1)t$, we have

$$\begin{aligned} y &\equiv x'(t-1) + (i-1)t + s' + 1 \pmod{k} \\ &\equiv (x+s)(t-1) + (i-1)t + s' + 1 \pmod{k} \\ &\equiv y(t-1)^2 + (j-1)t(t-1) + s(t-1) + (i-1)t + s' + 1 \pmod{k} \\ &\equiv y + 0 - s + 0 + s' + 1 \pmod{t} \ldots (\text{as } t \text{ divides } k). \end{aligned}$$

Therefore $s' - s + 1 \equiv 0 \pmod{t}$. Hence t divides $s' - s + 1$. But, $0 \le s < s' \le t-2$ gives $1 \le s' - s + 1 \le t - 1 < t$, a contradiction.

Thus $M \cup N$ is a matching of $C_k \Box C_k$.

The number of edges in $M \cup N$ is given by,

$$|M \cup N| = |M| + |N| = \frac{k^2}{4} + \frac{k^2}{4} = \frac{k^2}{2} = \frac{|V(C_k \Box C_k)|}{2}.$$

Hence $M \cup N$ is a perfect matching of the graph $C_k \Box C_k$.

Thus the tk-cycles Φ_i and Γ_i, $i = 1, 2, \ldots, m$, together decompose the graph $C_k \Box C_k$ and $kt/4$ edges can be selected from each of these cycles in order to form a perfect matching of $C_k \Box C_k$. □

We now prove Theorem 1.5, which is restated below for convenience.

Statement of Theorem 1.5. *Let t, k be positive integers such that t divides k and 4 divides t. Then the k-ary r-cube Q_r^k can be decomposed into $(tkr/2)$-cycles. Moreover, from each of these cycles, $kt/4$ edges can be selected such that they together form a perfect matching of Q_r^k.*

Proof. We prove the result by induction on r. The result holds for $r = 2$ by Lemmas 4.1. and 4.2. Suppose $r \geq 3$. Assume that the statement holds for Q^k_{r-1}. Now $Q^k_r = Q^k_{r-1} \Box C_k$. By induction, Q^k_{r-1} can be decomposed into cycles of lengths $tk(r-1)/2$, (say) $\Phi_1, \Phi_2, \ldots, \Phi_s$, such that each Φ_i contains a matching M_i of $kt/4$ edges so that $\bigcup_{i=1}^{s} M_i$ is a perfect matching of Q^k_{r-1}. Then, by Theorem 2.2, Q^k_r can be decomposed into cycles of length $tkr/2$, (say) $\Gamma_1, \Gamma_2, \ldots, \Gamma_{s'}$, such that each Γ_i contains a matching N_i of $tk/4$ edges such that $\bigcup_{i=1}^{s'} N_i$ is a perfect matching of Q^k_r. □

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