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Block-avoiding sequencings of points in Steiner triple systems

Donald L. Kreher

Department of Mathematical Sciences
Michigan Technological University
Houghton, Michigan, 49931
U.S.A.

Douglas R. Stinson*

David R. Cheriton School of Computer Science
University of Waterloo
Waterloo, Ontario, N2L 3G1
Canada

Abstract

Given an STS(v), we ask if there is a permutation of the points of the design such that no ℓ consecutive points in this permutation contain a block of the design. Such a permutation is called an *ℓ-good sequencing.* We prove that 3-good sequencings exist for any STS(v) with $v > 3$ and 4-good sequencings exist for any STS(v) with $v > 71$. Similar results also hold for partial STS(v). Finally, we determine the existence or nonexistence of 4-good sequencings for all the nonisomorphic STS(v) with $v = 7, 9, 13$ and 15.

1 Introduction

A *Steiner triple system of order* v is a pair $(X, \mathcal{B})$, where X is a set of v *points* and $\mathcal{B}$ is a set of 3-subsets of X (called *blocks*), such that every pair of points occur in exactly one block. We will abbreviate the phrase "Steiner triple system of order v" to STS(v).

It is well-known that an STS(v) contains exactly $v(v-1)/6$ blocks, and an STS(v) exists if and only if $v \equiv 1, 3 \bmod 6$. The definitive reference for Steiner triple systems is the book [4] by Colbourn and Rosa.

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Suppose $(X, \mathcal{B})$ is an STS(v). We ask if there is a permutation (or *sequencing*) of the points in X so that no three consecutive points in the sequencing comprise a block in $\mathcal{B}$. That is, can we find a sequencing $\pi = [x_1\ x_2\ \cdots\ x_v]$ of X such that $\{x_i, x_{i+1}, x_{i+2}\} \notin \mathcal{B}$ for all i, $1 \leq i \leq v-2$? Such a sequencing will be termed a 3-*good* sequencing for the given STS(v).

More generally, for an integer $\ell \geq 3$, we could ask if there is a sequencing of the points such that no ℓ consecutive points in the sequencing contain a block in $\mathcal{B}$. Such a sequencing will be termed ℓ-good for the given STS(v). Obviously an ℓ-good sequencing is also m-good if $3 \leq m \leq \ell$.

As an example, consider the STS(7) $(X, \mathcal{B})$, where $X = \mathbb{Z}_7$ and $\mathcal{B} = \{013, 124, 235, 346, 450, 451, 562\}$. The sequencing [0 1 2 3 4 5 6] is easily seen to be 3-good. However, it is not 4-good, as the block 013 is contained in the first four points of the sequencing. (Note that, here and elsewhere, we might write blocks $\{x, y, z\}$ as xyz if the context is clear.)

Actually, it is not difficult to see that the unique (up to isomorphism) STS(7) does not have a 4-good sequencing. By relabelling points if necessary, suppose there is a 4-good sequence for an STS(7) that begins [0 1 2 3]. There cannot be a block contained in $\{0, 1, 2, 3\}$. Hence, without loss of generality, $\{0, 1, 4\}$, $\{0, 2, 5\}$ and $\{0, 3, 6\}$ are blocks. This forces the remaining blocks to be $\{1, 2, 6\}$, $\{1, 3, 5\}$, $\{2, 3, 4\}$ and $\{4, 5, 6\}$. In particular, $\{4, 5, 6\}$ is a block, so there is no way to complete the sequence beginning [0 1 2 3] to a 4-good sequence.

A *partial Steiner triple system of order* v is a pair $(X, \mathcal{B})$, where X is a set of v *points* and $\mathcal{B}$ is a set of 3-subsets of X (called *blocks*), such that every pair of points occur in at most one block. We will abbreviate the phrase "partial Steiner triple system of order v" to partial STS(v) or PSTS(v). There are no congruential restrictions on the values v for which PSTS(v) exist. We will also consider ℓ-good sequencings of PSTS(v).

The main results we prove in this paper are that every STS(v) with $v > 3$ has a 3-good sequencing, and every STS(v) with $v > 71$ has a 4-good sequencing. Similar results are obtained for PSTS(v) as well. We also study 4-good sequencings of STS(v) for $v \leq 15$. We show that there is no 4-good sequencing of the STS(7) or STS(9), but all STS(13) and STS(15) have 4-good sequencings.

We will use the following notation. Suppose $(X, \mathcal{B})$ is an STS(v). Then, for any pair of points x, y, let $\mathsf{third}(x, y) = z$ if and only if $\{x, y, z\} \in \mathcal{B}$. The function third is well-defined because every pair of points occurs in a unique block in $\mathcal{B}$.

1.1 Background and motivation

Brian Alspach gave a talk entitled "Strongly Sequenceable Groups" at the 2018 Kliakhandler Conference, which was held at Michigan Technological University. In this talk, among other things, the notion of sequencing diffuse posets was introduced and the following research problem was posed:

> "Given a triple system of order n with $\lambda = 1$, define a poset P by letting its elements be the triples and any union of disjoint triples. This poset is not diffuse in general, but it is certainly possible that P is sequenceable."

A *sequenceable* STS(v) (or PSTS(v)) is an STS(v) in which the points can be ordered (i.e., sequenced) so that no t consecutive points can be partitioned into $t/3$ blocks, for any $t \equiv 0 \bmod 3$, $t < v$. The problem is studied in Alspach, Kreher and Pastine [1] and in Kreher and Stinson [3]. In [3], it is shown that there is a nonsequenceable STS(v) for all $v \equiv 1 \bmod 6$, $v > 7$.

One possible relaxation of the definition of sequenceable STS(v) would be to require a sequencing of the points so that no t consecutive points can be partitioned into $t/3$ blocks, for any $t \equiv 0 \bmod 3$, $t \leq w$, where $w < v$ is some specified integer. Such an STS(v) could be termed *w-semi-sequenceable*.

A 3-semi-sequenceable STS(v) has a sequencing of the points so that no three consecutive points form a block. This is identical to a "3-good sequencing." As noted above, we then generalize this notion to ℓ-good sequencings and we consider the cases $\ell = 3$ and $\ell = 4$ in detail.

Although we do not explicitly study w-semi-sequenceable STS in this paper, we note the following connection between w-semi-sequenceable STS(v) and STS(v) having ℓ-good sequencings.

Theorem 1.1. *Let $u \geq 1$ be an integer. An STS(v) that has a $(2u+1)$-good sequencing is $3u$-semi-sequenceable.*

Proof. Let π be a sequencing of the points of an STS(v) that is not $3u$-semi-sequenceable. Then, for some $t \equiv 0 \bmod 3$, there are t consecutive points in π that can be partitioned into $t/3$ blocks of the STS(v). Let these t points be denoted (in order) $x_1, \ldots, x_t$. Then

$$\{x_1, \ldots, x_t\} = \bigcup_{j=1}^{t/3} B_j,$$

where $B_1, \ldots, B_{t/3}$ are blocks in the STS(v). For $1 \leq j \leq t/3$, let

$$m_{lo}(j) = \min\{i : x_i \in B_j\}$$

and let

$$m_{hi}(j) = \max\{i : x_i \in B_j\}.$$

Clearly there is a block B_j such that $m_{lo}(j) \geq t/3$. It also holds that $m_{hi}(j) \leq t$. Therefore the block $B_j \subseteq \{x_{t/3}, \ldots, x_t\}$, which means that the sequencing π is not $(2t/3+1)$-good. ☐

2 Existence of 3-good sequencings

In this section, we show that there is a 3-good sequencing for any STS(v) with $v \geq 7$. We prove this in three ways: by a counting argument, by using a greedy algorithm, and by relabelling the points of the design in a suitable way. The counting argument and greedy algorithm can also be adapted to handle PSTS(v) with $v > 3$ ($v > 5$, respectively).

2.1 A counting argument

Let $(X, \mathcal{B})$ be an STS(v) on points $X = \{1, \dots, v\}$. For a sequencing $\pi = [x_1\ x_2\ \cdots\ x_v]$ of X, and for any i, $1 \leq i \leq v-2$, define π to be *i-forbidden* if $\{x_i, x_{i+1}, x_{i+2}\} \in \mathcal{B}$. Let $\mathsf{forbidden}(i)$ denote the set of i-forbidden sequencings. Also, define a sequencing to be *forbidden* if it is i-forbidden for at least one value of i and let $\mathsf{forbidden}$ denote the set of forbidden sequencings. Clearly, a sequencing is 3-good if and only if it is not forbidden.

Theorem 2.1. *Suppose $v > 3$ and $(X, \mathcal{B})$ is an STS(v) on points $X = \{1, \dots, v\}$. Then there is a sequencing $\pi = [x_1\ x_2\ \cdots\ x_v]$ of X that is 3-good for $(X, \mathcal{B})$.*

Proof. Clearly,

$$\mathsf{forbidden} = \bigcup_{i=1}^{v-2} \mathsf{forbidden}(i).$$

For any given value of i, it holds that $|\mathsf{forbidden}(i)| = v!/(v-2)$. This follows because, for any two points, x_i and x_{i+1}, the 3-subset $\{x_i, x_{i+1}, x_{i+2}\} \in \mathcal{B}$ if and only if $x_{i+2} = \mathsf{third}(x_i, x_{i+1})$. So given any x_i and x_{i+1}, the probability that $\{x_i, x_{i+1}, x_{i+2}\} \in \mathcal{B}$ is $1/(v-2)$.

Next, by the union bound,

$$|\mathsf{forbidden}| \leq \sum_{i=1}^{v-2} |\mathsf{forbidden}(i)| = (v-2) \times \frac{v!}{(v-2)} = v! \tag{1}$$

Equality in (1) would be obtained if and only if the sets $\mathsf{forbidden}(i)$, $1 \leq i \leq v-2$, are pairwise disjoint.

We show that equality in (1) is impossible: Consider any two intersecting blocks $\{a, b, c\}, \{c, d, e\} \in \mathcal{B}$ (here is where we use the assumption that $v > 3$). Then any sequencing in which the first five symbols are $a\ b\ c\ d\ e$ (in that order) is in $\mathsf{forbidden}(1) \cap \mathsf{forbidden}(3)$. Therefore, $|\mathsf{forbidden}| < v!$ and thus there exists a 3-good sequencing. □

Theorem 2.1 also holds for *partial* STS(v) when $v > 3$.

Theorem 2.2. *Suppose $v > 3$ and $(X, \mathcal{B})$ is a partial STS(v) on points $X = \{1, \dots, v\}$. Then there is a sequencing $\pi = [x_1\ x_2\ \cdots\ x_v]$ of X that is 3-good for $(X, \mathcal{B})$.*

Proof. If $(X, \mathcal{B})$ is an STS(v), then we are done by Theorem 2.1. Therefore, we can assume there is at least one pair $\{a, b\}$ that does not appear in any block in $\mathcal{B}$. Suppose $x_i = a$ and $x_{i+1} = b$. Then, for every possible x_{i+2}, we have $\{x_i, x_{i+1}, x_{i+2}\} \notin \mathcal{B}$. It then follows that $|\mathsf{forbidden}(i)| < v!/(v-2)$ for all i.

Now, when we apply the union bound, we have

$$|\mathsf{forbidden}| \leq \sum_{i=1}^{v-2} |\mathsf{forbidden}(i)| < (v-2) \times v!/(v-2) = v!$$

and we are done. □

2.2 A greedy algorithm

Theorem 2.1 and a slightly weaker version of Theorem 2.2 can also be proven using a greedy algorithm. First, we consider the case where $(X, \mathcal{B})$ is an STS(v), where $X = \{x_1, \ldots, x_v\}$. We begin by choosing any two distinct values for x_1 and x_2 and then we attempt to define $x_3, x_4, \ldots, x_v$ in turn, in such a way that we end up with a 3-good sequencing. Thus, our strategy is to design a greedy algorithm.

Consider any i such that $3 \leq i \leq v-1$. We require the following conditions to be satisfied:

1. $x_i \notin \{x_1, \ldots, x_{i-1}\}$, and
2. $x_i \neq \mathsf{third}(x_{i-2}, x_{i-1})$.

Thus, there are at most i values for x_i that are ruled out. Because $i \leq v-1$, there is at least one value for x_i that satisfies the two required conditions.

After choosing $x_1, x_2, \ldots, x_{v-1}$ as described above, there is only one unused value remaining for x_v. But this might not result in a 3-good sequencing, if it happens that $\{x_{v-2}, x_{v-1}, x_v\} \in \mathcal{B}$. However, in this case, it turns out that we can find a slight modification of of the sequencing $[x_1\ x_2\ \cdots\ x_v]$ that is 3-good, provided that $v > 5$.

Suppose we made sure to select x_5 such that $\{x_2, x_3, x_5\} \in \mathcal{B}$, i.e., we define $x_5 = \mathsf{third}(x_2, x_3)$. This is an allowable choice for x_5 because

- $\{x_1, x_2, x_3\} \notin \mathcal{B}$ and $\{x_2, x_3, x_4\} \notin \mathcal{B}$, which implies that

$$x_5 \notin \{x_1, x_2, x_3, x_4\},$$

  and

- $\{x_3, x_4, x_5\} \notin \mathcal{B}$, because $\{x_2, x_3, x_5\} \in \mathcal{B}$ and $x_2 \neq x_4$.

Now, suppose we have a sequencing $[x_1\ x_2\ \cdots\ x_v]$, where $\{x_2, x_3, x_5\} \in \mathcal{B}$, which fails to be 3-good only because $\{x_{v-2}, x_{v-1}, x_v\} \in \mathcal{B}$ (which is not allowed). Consider the modified sequencing $[y_1\ y_2\ \cdots\ y_v]$ obtained from $[x_1\ x_2\ \cdots\ x_v]$ by switching x_1 and x_v. In order to show that $[y_1\ y_2\ \cdots\ y_v]$ is a 3-good sequencing, we need to show that

1. $\{y_{v-2}, y_{v-1}, y_v\} = \{x_{v-2}, x_{v-1}, x_1\} \notin \mathcal{B}$, and
2. $\{y_1, y_2, y_3\} = \{x_v, x_2, x_3\} \notin \mathcal{B}$.

1. Choose a block $\{b, c, e\} \in \mathcal{B}$, let $a \neq b, c, e$ and let $d \neq a, b, c, e$.
2. Define $x_1 = a$, $x_2 = b$, $x_3 = c$, $x_4 = d$ and $x_5 = e$.
3. **For** $i = 6$ **to** $v - 1$ **do** define x_i to be any element of X that is distinct from the values $x_1, \ldots, x_{i-1}$ and $\mathsf{third}(x_{i-2}, x_{i-1})$.
4. Define x_v to be the unique value that is distinct from $x_1, \ldots, x_{v-1}$.
5. **If** $\{x_{v-2}, x_{v-1}, x_v\} \in \mathcal{B}$ **then** interchange x_1 and x_v.
6. **Return** $(\pi = [x_1\ x_2\ \cdots\ x_v])$.

Figure 1: Algorithm to find a 3-good sequencing for a partial STS(v), $(X, \mathcal{B})$

To prove 1, we observe that $\{x_{v-2}, x_{v-1}, x_1\} \notin \mathcal{B}$ because $\{x_{v-2}, x_{v-1}, x_v\} \in \mathcal{B}$ and $x_v \neq x_1$. To prove 2, we observe that $\{x_2, x_3, x_5\} \in \mathcal{B}$ and $x_v \neq x_5$ because $v > 5$. Thus the sequencing $[y_1\ y_2\ \cdots\ y_v]$ is 3-good.

The above-described process can also be carried out to find a 3-good sequencing for any partial STS(v) with $v > 5$. The resulting algorithm is presented in Figure 1.

From the discussion above, we have the following theorem.

Theorem 2.3. *Suppose that $(X, \mathcal{B})$ is a partial STS(v) with $v > 5$. Then the Algorithm presented in Figure 1 will find a sequencing π that is 3-good for $(X, \mathcal{B})$.*

2.3 Relabelling points

In this section, we give a short, elegant proof that an STS(v) with $v \geq 7$ has a 3-good sequencing. This proof was pointed out to us by Charlie Colbourn (private communication).

Given an STS(v) with $v \geq 7$, say on points $1, \ldots, v$, relabel the points so that the blocks containing 1 are

$$\{1, 2, v\},\ \{1, 3, 4\},\ \{1, 5, 6\},\ \ldots,\ \{1, v-2, v-1\}.$$

Then consider the sequencing $[1\ 2\ \cdots\ v]$. We observe the following, using the fact that $v \geq 7$:

$\{1, 2, 3\}$ is not a block because $\{1, 2, v\}$ is a block
$\{2, 3, 4\}$ is not a block because $\{1, 3, 4\}$ is a block
$\{3, 4, 5\}$ is not a block because $\{1, 3, 4\}$ is a block
$\{4, 5, 6\}$ is not a block because $\{1, 5, 6\}$ is a block
$\{5, 6, 7\}$ is not a block because $\{1, 5, 6\}$ is a block
etc.
$\{v-3, v-2, v-1\}$ is not a block because $\{1, v-2, v-1\}$ is a block
$\{v-2, v-1, v\}$ is not a block because $\{1, v-2, v-1\}$ is a block.

Therefore this sequencing is 3-good.

3 Existence of 4-good sequencings

For any integer $\ell \geq 3$, it is tempting to conjecture that all "sufficiently large" STS(v) have ℓ-good sequencings. In this section, we prove this conjecture for the case $\ell = 4$. Then, we present some results on 4-good sequencings of "small" STS(v) in Section 3.1.

We might attempt to show the existence of a 4-good sequencing using any of the three methods described in the previous section. It turned out that we were able to do this using a greedy strategy similar to the algorithm presented in Figure 1. In general, when we choose a value for x_i, it must be distinct from $x_1, \ldots, x_{i-1}$, of course. It is also required that

$$x_i \notin \{\mathsf{third}(x_{i-3}, x_{i-2}), \mathsf{third}(x_{i-3}, x_{i-1}), \mathsf{third}(x_{i-2}, x_{i-1})\}.$$

There will be a permissible choice for x_i provided that $i - 1 + 3 \leq v - 1$, which is equivalent to the condition $i \leq v - 3$. Thus we can define $x_1, x_2, \ldots, x_{v-3}$ in such a way that they satisfy the relevant conditions, and our task would be to somehow fill in the last three positions of the sequencing, after appropriate modifications, to satisfy the desired properties. We describe how to do this now, for sufficiently large values of v.

Suppose that $[x_1\ x_2\ \cdots\ x_{v-3}]$ is a 4-good partial sequencing of $X = \{1, \ldots, v\}$ (that is, there is no block contained in any four consecutive points in the sequence $[x_1\ x_2\ \cdots\ x_{v-3}]$). Let $\{\alpha_1, \alpha_2, \alpha_3\} = X \setminus \{x_1, x_2, \ldots, x_{v-3}\}$. Also, let

$$\begin{aligned}\beta_1 &= \mathsf{third}(x_{v-5}, x_{v-4}),\\ \beta_2 &= \mathsf{third}(x_{v-5}, x_{v-3}), \text{ and}\\ \beta_3 &= \mathsf{third}(x_{v-4}, x_{v-3}).\end{aligned}$$

Clearly β_1, β_2 and β_3 are distinct. Observe that x_{v-2} and x_{v-1} must be chosen so that $x_{v-2} \neq \beta_1, \beta_2, \beta_3$ and $x_{v-1} \neq \beta_3$.

By permuting $\alpha_1, \alpha_2, \alpha_3$ if necessary, we can assume the following two conditions hold:

$$\alpha_2 \neq \beta_3 \tag{2}$$

and

$$x_{v-3} \neq \mathsf{third}(\alpha_2, \alpha_3). \tag{3}$$

Now, define the following:

$$\begin{aligned}\gamma &= \mathsf{third}(\alpha_2, x_{v-3}),\\ \delta &= \mathsf{third}(\alpha_2, x_{v-4}),\\ \epsilon &= \mathsf{third}(\alpha_3, x_{v-3}), \text{and}\\ \eta &= \mathsf{third}(\alpha_2, \alpha_3).\end{aligned}$$

Next, suppose we define $x_{v-2} = \chi$, $x_{v-1} = \alpha_2$ and $x_v = \alpha_3$, where

$$\chi \notin \{x_{v-5}, x_{v-4}, x_{v-3}, \beta_1, \beta_2, \beta_3, \gamma, \delta, \epsilon, \eta\} \tag{4}$$

Table 1: Possible blocks in the last six elements of the sequencing

triple	explanation
$\{x_{v-5}, x_{v-4}, x_{v-3}\}$	greedy algorithm ensures it is not a block
$\{x_{v-5}, x_{v-4}, \chi\}$	$\{x_{v-5}, x_{v-4}, \beta_1\}$ is a block and $\chi \neq \beta_1$
$\{x_{v-5}, x_{v-3}, \chi\}$	$\{x_{v-5}, x_{v-3}, \beta_2\}$ is a block and $\chi \neq \beta_2$
$\{x_{v-4}, x_{v-3}, \chi\}$	$\{x_{v-4}, x_{v-3}, \beta_3\}$ is a block and $\chi \neq \beta_3$
$\{x_{v-4}, x_{v-3}, \alpha_2\}$	$\{x_{v-4}, x_{v-3}, \beta_3\}$ is a block and $\alpha_2 \neq \beta_3$ by (2)
$\{x_{v-4}, \chi, \alpha_2\}$	$\{x_{v-4}, \delta, \alpha_2\}$ is a block and $\chi \neq \delta$
$\{x_{v-3}, \chi, \alpha_2\}$	$\{x_{v-3}, \gamma, \alpha_2\}$ is a block and $\chi \neq \gamma$
$\{x_{v-3}, \chi, \alpha_3\}$	$\{x_{v-3}, \epsilon, \alpha_3\}$ is a block and $\chi \neq \epsilon$
$\{x_{v-3}, \alpha_2, \alpha_3\}$	this is not a block by (3)
$\{\chi, \alpha_2, \alpha_3\}$	$\{\eta, \alpha_2, \alpha_3\}$ is a block and $\chi \neq \eta$.

is to be determined. Thus, the last six elements of the sequencing will be

$$x_{v-5}\ x_{v-4}\ x_{v-3}\ \chi\ \alpha_2\ \alpha_3.$$

There should be no block in $\mathcal{B}$ contained in any four consecutive points chosen from these six points. We enumerate all the relevant triples and verify that none of them are blocks in Table 1.

Suppose $v \geq 14$. Our strategy is to define χ to be one of $x_1, x_2, \ldots, x_8$, in such a way that (4) is satisfied. Note that $v - 5 \geq 9$, so we are guaranteed that $\chi \neq x_{v-5}, x_{v-4}, x_{v-3}$. We can choose $\chi \in \{x_1, x_2, \ldots, x_8\}$ because at least one of these eight values is not in the set $\{\beta_1, \beta_2, \beta_3, \gamma, \delta, \epsilon, \eta\}$, which has size 7. Suppose we take $\chi = x_\kappa$, where $\kappa \in \{1, 2, \ldots, 8\}$. Then we redefine $x_\kappa = \alpha_1$. Another way to describe this process is to temporarily define $x_{v-2} = \alpha_1$ and then interchange x_{v-2} with x_κ.

Now, when we initially choose $x_1, x_2, x_3, \ldots$, we have no idea which value α_1 we will be interchanging with x_κ. So it is necessary to ensure that any value we "swap in" will not result in a block being contained in four successive points of the sequencing. Clearly we only have to worry about the first $8 + 3 = 11$ points, $x_1, x_2, x_3, \ldots, x_{11}$.

Define

$$Y = \big\{\mathsf{third}(x_i, x_j) : 1 \leq i < j \leq 11, |i - j| \leq 3\big\} \setminus \{x_1, \ldots, x_{11}\}.$$

(Note, in the definition of Y, that we do not care about pairs of points that are more than three positions apart.) Denote the points in Y as $y_1, \ldots, y_m$. It is not hard to verify that $m \leq 27$, because there are ten pairs x_i, x_j in $\{x_1, \ldots, x_{11}\}$ with $j - i = 1$, nine pairs with $j - i = 2$ and eight pairs with $j - i = 3$.

Having already chosen $x_1, \ldots, x_{11}$, we want to "pre-specify" some of the next points. Due to the changes that are introduced, this part of the algorithm will be referred to as the "modified greedy approach." To be specific, we define $x_{14} = y_1$,

$x_{16} = y_2, \ldots, x_{2m+12} = y_m$. Note that no three of the y_i's are contained in four consecutive points of the sequencing, from x_{12} to x_{2m+12}.

The following diagram might be helpful in the subsequent discussion:

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	x_{12}	x_{13}

y_1	x_{15}	y_2	x_{17}	$\cdots$	x_{2m+7}	y_{m-2}	x_{2m+9}	y_{m-1}	x_{2m+11}	y_m

In this diagram, the red values have been defined and we need to determine the black values. Let us consider how the greedy algorithm must be modified in order to accomplish this. We have the following additional restrictions "looking ahead" when choosing values for $x_{12}, x_{13}, x_{15}, \ldots, x_{2m+11}$:

- each of $x_{12}, x_{13}, x_{15}, \ldots, x_{2m+11}$ must be distinct from $y_1, \ldots, y_m$;
- we require that $\{x_{11}, x_{12}, y_1\} \notin \mathcal{B}$, so we must define
$$x_{12} \neq \mathsf{third}(x_{11}, y_1);$$
- we require that
$$\{x_{11}, x_{13}, y_1\}, \{x_{12}, x_{13}, y_1\}, \{x_{13}, y_1, y_2\} \notin \mathcal{B},$$
so we must define
$$x_{13} \neq \mathsf{third}(x_{11}, y_1), \mathsf{third}(x_{12}, y_1), \mathsf{third}(y_1, y_2);$$
- in the "general" case, for $i = 2, \ldots, m-1$, we require that
$$\{x_{2i+9}, x_{2i+11}, y_i\}, \{y_{i-1}, x_{2i+11}, y_i\}, \{x_{2i+11}, y_i, y_{i+1}\} \notin \mathcal{B},$$
so we must define
$$x_{2i+11} \neq \mathsf{third}(x_{2i+9}, y_i), \mathsf{third}(y_{i-1}, y_i), \mathsf{third}(y_i, y_{i+1});$$
- finally, we require that
$$\{x_{2m+9}, x_{2m+11}, y_m\}, \{y_{m-1}, x_{2m+11}, y_m\} \notin \mathcal{B},$$
so we must define
$$x_{2m+11} \neq \mathsf{third}(x_{2m+9}, y_m), \mathsf{third}(y_{m-1}, y_m).$$

Of course, we need to ensure that the greedy algorithm can choose values for all these x_i's.

Now consider what happens when we swap x_κ with α_1. The value $\alpha_1 \notin Y$, so α_1 cannot form a block with any two of the points $x_1, \ldots, x_{11}$. Because $\kappa \leq 8$, there are no blocks contained in any four consecutive points chosen from the first 11

1. Determine $x_1, \ldots, x_{11}$ using the greedy approach.
2. Fill in the values $y_1, \ldots, y_m$ and the determine the remaining values $x_{12}, \ldots, x_{2m+11}$ using the modified greedy approach.
3. Determine $x_{2m+13}, \ldots, x_{v-3}$ using the greedy approach.
4. Define the values $x_{v-2} = \alpha_1, x_{v-1} = \alpha_2, x_v = \alpha_3$ as described in the text, and then swap x_{v-2} with x_κ.
5. **Return** $(\pi = [x_1\ x_2\ \cdots\ x_v])$.

Figure 2: Algorithm to find a 4-good sequencing for an STS(v), $(X, \mathcal{B})$

points of the sequencing. At the opposite end, we have guaranteed that there are no blocks contained in any four consecutive points chosen from the last six points of the sequencing, because of the way that x_κ was chosen.

Summarizing, the resulting algorithm has the high-level structure described in Figure 2.

All the above steps can be carried out if we ensure that the first $2m+12$ elements of the sequencing do not overlap with the last six elements of the sequencing. Because $m \leq 27$, this condition is guaranteed to hold if $v - 5 \geq 2 \times 27 + 12 + 1$, or $v \geq 72$. So we have proven the following.

Theorem 3.1. *Suppose $v > 71$ and $(X, \mathcal{B})$ is an STS(v) on points $X = \{1, \ldots, v\}$. Then there is a sequencing $\pi = [x_1\ x_2\ \cdots\ x_v]$ of X that is 4-good for $(X, \mathcal{B})$.*

A similar result can also be proven for PSTS(v) using this technique.

3.1 Results on 4-good sequencings of STS(v) for $v \leq 15$

We have shown in Section 1 that there is no 4-good sequencing for the unique STS(7). Here, we use a counting method to establish the same result, as well as an analogous result for the unique STS(9).

Suppose we take the points of an STS(v) to be $1, \ldots, v$. Suppose, by relabelling points if necessary, that $[1\ 2\ 3\ 4\ \cdots\ v]$ a 4-good sequencing of an STS(v). We say that a block B is of *type* i if $|B \cap \{1, 2, 3, 4\}| = i$. Clearly, we must have $i \in \{0, 1, 2\}$.

For $i = 0, 1, 2$, let b_i denote the number of blocks of type i. Simple counting allows us to determine the values b_0, b_1 and b_2. First, because the sequencing is 4-good, we have $b_2 = \binom{4}{2} = 6$. Next, because each point appears in $(v-1)/2$ blocks, we have $b_1 = 4((v-1)/2 - 3)) = 2v - 14$. Finally, because the total number of blocks is $v(v-1)/6$, we have $b_0 = v(v-1)/6 - (2v - 14) - 6 = v(v-1)/6 - 2v + 8$.

Now, if $v = 7$, we obtain $b_2 = 6$, $b_1 = 0$ and $b_0 = 1$. The block of type 0 must be $\{5, 6, 7\}$. Since these are the last three points of the sequencing, the sequencing is not even 3-good. Therefore there is no 4-good sequencing of the STS(7).

If $v = 9$, we obtain $b_2 = 6$, $b_1 = 4$ and $b_0 = 2$. If the sequencing is 4-good, then any block B of type 0 must contain both 5 and 9 (if not, then $B \subseteq \{5, 6, 7, 8\}$ or

$B \subseteq \{6, 7, 8, 9\}$, neither of which can occur if the sequencing is 4-good). But there is at most one block in the STS(v) that contains the pair $\{5, 9\}$, so $b_0 = 2$ is impossible. Therefore there is no 4-good sequencing of the STS(9).

However, for $v = 13, 15$, we quickly found 4-good sequencings of all the nonisomorphic STS(v) by computer, by using a simple backtracking algorithm. We have found such sequencings for the two nonisomorphic STS(13) and the 80 nonisomorphic STS(15); these are presented in the Appendix.

4 Conclusion

For any integer $\ell \geq 3$, let $n(\ell)$ denote the smallest integer such that the following property is satisfied:

$$\text{any STS}(v) \text{ with } v > n(\ell) \text{ has an } \ell\text{-good sequencing.} \tag{5}$$

Also, define $n(\ell) = \infty$ if no integer satisfying (5) exists.

We conjecture that $n(\ell)$ is finite for every integer $\ell \geq 3$. Further, based on the computational results from Section 3.1, we conjecture that $n(4) = 9$.

Acknowledgement

We thank Charlie Colbourn for providing the construction presented in Section 2.3.

Appendix: 4-good sequencings for STS(13) and STS(15)

We present 4-good sequencings for the two nonisomorphic STS(13) and the 80 nonisomorphic STS(15). These designs are listed in the same order as in the *Handbook of Combinatorial Designs* [2, Table 1.27 and 1.28].

Table 2: 4-good sequencings for the STS(13)

1:	[0 1 3 6 2 4 7 8 5 9 b a c]	2:	[0 1 3 6 2 4 7 8 5 9 a b c]

Table 3: 4-good sequencings for the STS(15)

1:	[0 1 3 6 7 4 2 8 a 5 9 d c b e]	2:	[0 1 3 6 7 4 2 8 a 5 9 d b c e]
3:	[0 1 3 6 7 4 2 8 a 5 9 d b c e]	4:	[0 1 3 6 7 4 2 8 a 5 9 d b c e]
5:	[0 1 3 6 7 4 2 8 a 5 9 b c d e]	6:	[0 1 3 6 7 4 2 8 a 5 9 c d b e]
7:	[0 1 3 6 7 4 2 8 a 5 9 c b d e]	8:	[0 1 3 6 7 4 2 8 9 5 a d b c e]
9:	[0 1 3 6 7 4 2 8 9 5 a d c b e]	10:	[0 1 3 6 7 4 2 8 9 5 a c b d e]
11:	[0 1 3 6 7 4 2 8 9 a b 5 d c e]	12:	[0 1 3 6 7 4 2 8 9 5 a b c d e]
13:	[0 1 3 6 7 4 2 8 9 a b 5 d c e]	14:	[0 1 3 6 7 4 2 8 a 5 9 d b c e]
15:	[0 1 3 6 7 4 2 8 9 5 a b c d e]	16:	[0 1 3 6 7 4 2 8 a 5 9 d c b e]
17:	[0 1 3 6 7 4 2 8 9 a b 5 d c e]	18:	[0 1 3 6 7 4 2 8 9 5 a d b c e]
19:	[0 1 3 6 7 4 2 8 a 5 9 c e b d]	20:	[0 1 3 6 7 4 2 8 9 5 a b c d e]
21:	[0 1 3 6 7 4 2 8 9 a b 5 d c e]	22:	[0 1 3 6 7 4 2 8 9 5 a c d b e]

23: [0 1 3 6 7 4 5 8 2 9 *a b c d e*]
24: [0 1 3 6 7 4 5 8 2 9 *a b c d e*]
25: [0 1 3 6 7 4 5 8 2 9 *a b c d e*]
26: [0 1 3 6 7 4 5 8 2 *a b* 9 *c d e*]
27: [0 1 3 6 7 4 5 8 2 9 *a b c d e*]
28: [0 1 3 6 7 4 5 8 2 9 *a b c d e*]
29: [0 1 3 6 7 4 5 8 2 9 *a b c d e*]
30: [0 1 3 6 7 4 5 8 2 9 *a b c d e*]
31: [0 1 3 6 7 4 5 8 2 9 *a b c d e*]
32: [0 1 3 6 7 4 5 8 2 9 *a b c d e*]
33: [0 1 3 6 7 4 5 8 2 9 *a b c d e*]
34: [0 1 3 6 7 4 5 8 2 9 *a b c d e*]
35: [0 1 3 6 2 4 5 7 8 *a* 9 *b c d e*]
36: [0 1 3 6 2 4 5 7 8 *a* 9 *b c d e*]
37: [0 1 3 6 2 4 5 7 8 9 *a b c d e*]
38: [0 1 3 6 2 4 5 7 8 9 *a c b d e*]
39: [0 1 3 6 2 4 5 7 8 9 *a c b d e*]
40: [0 1 3 6 2 4 5 7 8 *a* 9 *b c d e*]
41: [0 1 3 6 2 4 5 7 8 9 *a b c d e*]
42: [0 1 3 6 2 4 5 7 8 9 *a b c d e*]
43: [0 1 3 6 2 4 5 7 8 9 *a c b d e*]
44: [0 1 3 6 2 4 5 7 8 *a* 9 *b c d e*]
45: [0 1 3 6 2 4 5 7 8 9 *a b c d e*]
46: [0 1 3 6 2 4 5 7 8 9 *a c b d e*]
47: [0 1 3 6 2 4 5 7 8 9 *a c b d e*]
48: [0 1 3 6 2 4 5 7 8 9 *a c b d e*]
49: [0 1 3 6 2 4 5 7 8 9 *a b c d e*]
50: [0 1 3 6 2 4 5 7 8 9 *a b c d e*]
51: [0 1 3 6 2 4 5 7 9 8 *a b c d e*]
52: [0 1 3 6 2 4 5 7 9 8 *a b c d e*]
53: [0 1 3 6 2 4 5 7 9 8 *a b c d e*]
54: [0 1 3 6 2 4 5 7 8 9 *a b c d e*]
55: [0 1 3 6 2 4 5 7 8 9 *a b c d e*]
56: [0 1 3 6 2 4 5 7 9 8 *a b c d e*]
57: [0 1 3 6 2 4 5 7 9 8 *a b c d e*]
58: [0 1 3 6 2 4 5 7 9 8 *a b c d e*]
59: [0 1 3 6 2 4 5 7 8 9 *a b c d e*]
60: [0 1 3 6 2 4 5 7 8 9 *a b c d e*]
61: [0 1 3 6 7 4 2 8 9 *a b* 5 *d c e*]
62: [0 1 3 6 7 4 5 8 2 9 *a b c d e*]
63: [0 1 3 6 7 4 5 8 2 9 *a b c d e*]
64: [0 1 3 6 7 4 5 8 2 *a* 9 *c b d e*]
65: [0 1 3 6 2 4 5 7 8 9 *a b c d e*]
66: [0 1 3 6 2 4 5 7 9 8 *a b c d e*]
67: [0 1 3 6 2 4 5 7 8 9 *a b c d e*]
68: [0 1 3 6 2 4 5 7 8 9 *a b c d e*]
69: [0 1 3 6 2 4 5 7 8 9 *a b c d e*]
70: [0 1 3 6 2 4 5 7 8 9 *a b c d e*]
71: [0 1 3 6 2 4 5 7 8 9 *a b c d e*]
72: [0 1 3 6 2 4 5 7 8 9 *a b c d e*]
73: [0 1 3 6 2 4 5 7 8 9 *a b c d e*]
74: [0 1 3 6 2 4 5 7 8 9 *a b c d e*]
75: [0 1 3 6 2 4 5 7 8 9 *a b c d e*]
76: [0 1 3 6 2 4 5 7 8 9 *a b c d e*]
77: [0 1 3 6 2 4 5 7 8 9 *a b c d e*]
78: [0 1 3 6 2 4 5 7 8 9 *a b c d e*]
79: [0 1 3 6 2 4 5 7 9 8 *a b c d e*]
80: [0 1 3 6 2 5 7 4 8 *a d* 9 *b c e*]

References

[1] B. Alspach, D. L. Kreher and A. Pastine, Sequenceable triple systems, Preprint.

[2] C. J. Colbourn and J. H. Dinitz, *Handbook of Combinatorial Designs, Second Edition*, Chapman & Hall/CRC, 2006.

[3] D. L. Kreher and D. R. Stinson, Nonsequenceable Steiner triple systems, *Bull. ICA* **86** (2019), 64–68.

[4] C J. Colbourn and A. Rosa, *Triple Systems*, Oxford University Press, 1999.

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