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Covering relations of k-Grassmannian permutations of type B

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Abstract

The main result of this work is the characterization of the covering relations of the Bruhat order of the maximal parabolic quotients of type B. Our approach is mainly combinatorial and is based in the pattern of the corresponding permutations also called signed k-Grassmannians permutations. We obtain that a covering relation can be classified in four different pairs of permutations. This answers a question raised by Ikeda and Matsumura providing a nice combinatorial model for maximal parabolic quotients of type B.

1 Introduction

This work focuses on the study of the Bruhat order of the maximal parabolic quotients of type B. Some main papers providing a combinatorial approach to this subject are Deodhar [4] which gives reduced decompositions while Stanley [12] characterizes the pattern of the permutations for elements of these quotients. We may also obtain these patterns in the work of Papi [9].

We are interested in obtaining the covering relations in this context. Notice that a covering relation is the occurrence of pairs which are comparable by the Bruhat order

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and have length difference equals to one. We know that parabolic quotients inherit the Bruhat order of the Weyl group by projection (for details, see Bjorner-Brenti [2] and also Stembridge [13]). Furthermore, the covering relations for all classical Weyl groups may be found in the work of Federico [5] which in particular includes the type B. However, it is not clear the behavior of these relations in the quotients.

To answer this question, we use the well-known characterization of these elements in terms of signed k-Grassmannian permutations. Once the Weyl group reflections are explicitly given, we investigate how they act over the permutations case-by-case. We conclude that a covering relation can be sorted in four different classes of pair of permutations. It is worth noticing that it is possible because we have a nice formula to compute the length of these permutations.

Our motivation is geometric by the fact that such quotients parametrize the Schubert varieties that equip the isotropic Grassmannians (real and complex) with a cellular structure. In the particular context of real Grassmannians, the non-zero coefficients for the boundary map of the cellular (co-)homology occur for these covering pairs. This issue will be approached in a subsequent paper by the same authors. Furthermore, our combinatorial description may be used to illustrate the classical Chevalley rule as a particular case of the Pieri rule for the complex Isotropic Grassmannians.

According to Buch-Kresch-Tamvakis [3], there is a bijective correspondence between k-Grassmannian permutations and the so called k-strict partitions which also can be depicted as diagrams. This approach has been very useful for several computations in the algebraic geometric setting (for instance, [3]). However, we present a slightly modified version in terms of half-shifted Young diagrams (HSYD's) whose definition goes back very closely to the model developed by Pragacz-Ratajski [10]. We provide the corresponding description of the covering relations in terms of these "new" diagrams.

Finally, the work of Ikeda-Matsumura [6] describes the covering relations according to the weak Bruhat order while it leaves open the characterization of these relations for the Bruhat order. In this former work, there is a hint indicating that the model of Maya diagrams could be useful to approach this problem. So, we also present a version of our result in terms of this class of diagrams which indeed gives some symmetry and provides a good picture of the pairs of covering relations.

This work is organized as follows: In Section 2 we introduce the main ingredients about the Grassmannian permutations. In Section 3, we state and prove our main result in the term of the four types of covering pairs. The remaining two sections are devoted to representing the permutations by diagrams. In Section 4, we define the half-Shifted Young diagrams establishing their correspondence with the types of pairs. In Section 5, we show how the Maya diagrams are useful to provide a nice picture of the covering relations.

2 Grassmannian permutations

We let $\mathbb{N} = \{1, 2, 3, \dots\}$ and $\mathbb{Z}$ be the set of integers. For $n, m \in \mathbb{Z}$, where $n \leqslant m$, denote the set $[n, m] = \{n, n+1, \dots, m\}$. For $n \in \mathbb{N}$, denote $[n] = [1, n]$.

The Weyl group $\mathcal{W}_n$ of type B, also called hyperoctahedral group, is generated by $\Sigma = \{s_i \colon i = 0, 1, \dots, n-1\}$ for which we have the following relations

$$\begin{aligned} s_i^2 &= 1\,,\ i \geq 0\,; \\ s_0 s_1 s_0 s_1 &= s_1 s_0 s_1 s_0\,; \\ s_{i+1} s_i s_{i+1} &= s_i s_{i+1} s_i\,,\ 1 \leqslant i < n-1\,; \\ s_i s_j &= s_j s_i\,,\ |i-j| \geqslant 2. \end{aligned}$$

The length $\ell(w)$ of $w \in \mathcal{W}_n$ is the minimal number of s_i's in a decomposition of w in terms of the generators. In this case, we say that this is a reduced decomposition of w. Each s_i, $i \geq 0$, is called a simple reflection. The other reflections are those conjugate to some s_i.

There is a partial order in $\mathcal{W}_n$ called the Bruhat-Chevalley order. We say that $w' \leq w$ if given a reduced decomposition $w = s_{i_1} \cdots s_{i_{\ell(w)}}$ then $w' = s_{i_{j_1}} \cdots s_{i_{j_k}}$ for some indices $1 \leq j_1 < \cdots < j_k \leq \ell(w)$ (this is called the "Subword Property"). It is known that $\mathcal{W}_n$ has a maximum element w_0 which is an involution, i.e, $w_0^2 = 1$.

Let $w, w' \in \mathcal{W}_n$ with $\ell(w) = \ell(w') + 1$, i.e., if $w = s_{i_1} \cdots s_{i_{\ell(w)}}$ is reduced decomposition then $w' = s_{i_1} \cdots \widehat{s_{i_j}} \cdots s_{i_{\ell(w)}}$ is a reduced decomposition as well. By the Exchange Property of Coxeter groups, there is a reflection $t \in \mathcal{W}_n$ (not necessarily simple) such that $w' = wt$ (see [2], Theorem 2.2.2). In this case, we say that it is a covering relation where w covers w'.

Consider the set of all barred permutations w of the form

$$\overline{n}, \overline{n-1}, \dots, \overline{1}, 0, 1, \dots, n-1, n$$

using the bar to denote a negative sign, and we take the natural order on them, as above. The hyperoctahedral group $\mathcal{W}_n$ is composed by the barred permutations subject to the relation $\overline{w(i)} = w(\overline{i})$, for all i. Then, it is usual to denote w in one-line notation by the sequence $w(1) \cdots w(n)$ of positive positions. However, we also could use the full description of w with the corresponding negative reflections, if it requires so.

The simple reflections are

$$\begin{aligned} s_0 &= (\overline{1}, 1)\,; \\ s_i &= (\overline{i+1}, \overline{i})(i, i+1)\,,\ \text{ for all } i \geq 1. \end{aligned}$$

If we think the elements of $\mathcal{W}_n$ acting at right over the permutations, we have that s_0 changes the sign in the first position and s_i changes the entries in the positions i and $i+1$, i.e., in the one-line notation

$$w(1)w(2)\cdots w(n) \cdot s_0 = \overline{w(1)}w(2)\cdots w(n)\,; \tag{2.1}$$

$$w(1)\cdots w(i)w(i+1)\cdots w(n) \cdot s_i = w(1)\cdots w(i+1)w(i)\cdots w(n). \tag{2.2}$$

By Proposition 8.1.5 of [2], the set of reflections is determined by

$$\{(i,j)(\bar{i},\bar{j})\colon 1 \le i < |j| \le n\} \bigcup \{(i,\bar{i})\colon 1 \le i \le n\}. \tag{2.3}$$

Hence, the action of a reflection $(i,j)(\bar{i},\bar{j})$, for some $1 \le i < |j| \le n$, will permute the entries in positions i and j. The reflection $(i,\bar{i})$, for some $1 \le i \le n$, changes the sign at i-th position.

The length of $w \in \mathcal{W}_n$ is given by the following formula (see [2], Eq. (8.3))

$$\ell(w) = \mathrm{inv}(w(1), \ldots, w(n)) - \sum_{\{j \mid w(j)<0\}} w(j) \tag{2.4}$$

where

$$\mathrm{inv}(w(1), \ldots, w(n)) = \#\{(i,j)\colon 1 \le i < j \le n, w(i) > w(j)\}.$$

We now state the main result about the covering relations of $\mathcal{W}_n$ according to [5]. We need some preliminary notation. Given $w \in \mathcal{W}_n$, a rise of w is a pair $(i,j) \in [\pm n]$ such that $i < j$ and $w(i) < w(j)$. A rise (i,j) is said to be free if there is no $k \in [\pm n]$ such that $i < k < j$ and $w(i) < w(k) < w(j)$. Furthermore, a rise (i,j) of w is central if

$$(0,0) \in [i,j] \times [w(i), w(j)].$$

A central rise (i,j) of w is symmetric if $j = -i$.

Theorem 2.1 ([5], Theorem 5.5, [1] Lemmas 1,2)**.** *Let $w', w \in \mathcal{W}_n$. Then, the w covers w' if, and only if, either*

1. *$w = w' \cdot (i,j)(-i,-j)$, where (i,j) is a non-central free rise of w', or*
2. *$w = w' \cdot (i,j)$, where (i,j) is a central symmetric free rise of w'.*

For each $0 \le k \le n-1$, define the set $(k) = \Sigma - \{s_k\}$ of simple reflections without s_k. The corresponding (parabolic) subgroup $\mathcal{W}_{(k)}$ is generated by s_i, with $i \ne k$. Notice that $\mathcal{W}_{(k)} \cong \mathcal{W}_k \times S_{n-k}$, where $\mathcal{W}_k$ is the subgroup generated by s_i, $0 \le i \le k$. For any $w = w(1) \cdots w(n)$, it follows that its coset $w\mathcal{W}_{(k)}$ is composed by permutations with its first k entries permuted with signs changed – corresponding to the $\mathcal{W}_k$ part – together with the permutations with the remaining $(n-k)$ permuted – but without change of signs corresponding to the S_{n-k} part.

The set of the minimum-length coset representatives for $\mathcal{W}_n/\mathcal{W}_{(k)}$ is defined by

$$\mathcal{W}_n^{(k)} = \{w \in \mathcal{W}_n \colon \ell(w) < \ell(ws_i), \forall i \ge 0, i \ne k\}.$$

Indeed, there exists a unique minimal length element in each coset $w\mathcal{W}_{(k)}$.

Now, we can give an explicit description of these representatives according to the above description of the cosets $w\mathcal{W}_{(k)}$. In fact, by Equation (2.4), we must seek inside each coset the elements with minimal number of inversions and negative numbers. In the first k entries, we always may take only positive elements ordered since it gives the least contribution to the length while for the remaining $n-k$ entries we only order them since we cannot avoid negative entries. So, we can conclude that the one-line notation of w can be identified by the form

$$w = w_{u,\lambda} = u_1 \cdots u_k | \overline{\lambda_r} \cdots \overline{\lambda_1}\, v_1 \cdots v_{n-k-r} \tag{2.5}$$

where

$$\begin{array}{rl} 0 < u_1 < \cdots < u_k\ , & u_i = w(i),\ \text{for } 1 \leq i \leq k\,; \\ 0 < \lambda_1 < \cdots < \lambda_r\ , & \overline{\lambda_i} = w(k+r-i+1),\ \text{for } 1 \leq i \leq r\,; \\ 0 < v_1 < \cdots < v_{n-k-r}\ , & v_i = w(k+r+i),\ \text{for } 1 \leq i \leq n-k-r. \end{array} \tag{2.6}$$

They are called k-Grassmannian permutations.

Remark 2.2. Theorem 2.1 still holds for elements of $\mathcal{W}_n^{(k)}$ since $\mathcal{W}_n^{(k)} \subset \mathcal{W}_n$ and the projection $\pi \colon \mathcal{W}_n \to \mathcal{W}_n^{(k)}$ preserves the Bruhat order (see [13], Prop. 1.1).

We now define a pair of partitions α and λ associated with each k-Grassmannian permutation w given by Equation (2.5). The negative part of w provides us a strict partition λ which satisfies $n \geqslant \lambda_r > \cdots > \lambda_1 > 0$. For each i, $0 < i \leqslant k$, we define

$$\alpha_i = u_i - i + d_i\,, \ \text{ with } d_i = \#\{\lambda_j \colon \lambda_j > u_i\}. \tag{2.7}$$

We claim that α is a partition, i.e., $n-k \geq \alpha_k \geq \alpha_{k-1} \geq \cdots \geq \alpha_1 \geq 0$. Indeed, for each i, we may collect all indexes that are greater than u_i that appear in the permutation (2.5) accordingly to the position they occupy by the following formula

$$[n] - [u_i] = \{u_j \colon u_j > u_i\} \cup \{\lambda_j \colon \lambda_j > u_i\} \cup \{v_j \colon v_j > u_i\}$$

where the cardinality is given by

$$u_i = n - \#\{u_j \colon u_j > u_i\} - \#\{\lambda_j \colon \lambda_j > u_i\} - \#\{v_j \colon v_j > u_i\}. \tag{2.8}$$

Now, we observe that $\#\{u_j \colon u_j > u_i\} = k - i$. It follows that Equation (2.8) is equivalent to

$$u_i = n - k + i - d_i - \mu_i. \tag{2.9}$$

where $\mu_i = \#\{v_j \colon v_j > u_i\}$. Then, Equation (2.7) may be rewritten as

$$\alpha_i = n - k - \mu_i. \tag{2.10}$$

By Equation (2.10), it is now clear that $n-k \geq \alpha_k \geq \alpha_{k-1} \geq \cdots \geq \alpha_1 \geq 0$. Denote $|\alpha| = \sum_{i=1}^{k} \alpha_i$ and $|\lambda| = \sum_{i=1}^{r} \lambda_i$. Observe that $\mu_i = \mu_i(w)$ also depends on the choice of w.

Lemma 2.3. *Let $w \in \mathcal{W}_n^{(k)}$. The length $\ell(w)$ of w is given by the sum of entries of the pair α, λ, i.e.,*

$$\ell(w) = |\alpha| + |\lambda|.$$

Proof. The sum of λ's corresponds to the sum $-\sum_{\{j|w(j)<0\}} w(j)$ in Equation (2.4). It remains to show that the inversions of $(w(1),\ldots,w(n))$ are given by the sum of the α's. Since there is a unique descent in position k, all inversions correspond to inversions among the u_i's with all λ's and some of v_j's. Then, for each $1 \leqslant i \leqslant k$, we have that the number of inversions related to u_i is $n-k-\#\{v_j : u_i < v_j\} = \alpha_i$, by Equation (2.10). Hence, $\mathrm{inv}(w(1),\ldots,w(n)) = |\alpha|$. □

3 Bruhat order of Grassmannian permutations

Let w and w' be permutations in $\mathcal{W}_n^{(k)}$ written in one-line notation as

$$w = u_1 \cdots u_k|\overline{\lambda_r} \cdots \overline{\lambda_1}\, v_1 \cdots v_{n-k-r},$$
$$w' = u'_1 \cdots u'_k|\overline{\lambda'_{r'}} \cdots \overline{\lambda'_1}\, v'_1 \cdots v'_{n-k-r'}.$$

Denote by α, λ and α', λ' the pair of partitions associated with w and w', respectively. Also denote $\mu_i = \mu_i(w)$ and $\mu'_i = \mu_i(w')$.

We call:

- w, w' a pair of *type B1* if

$$w = \cdots \mid \cdots \overline{1} \cdots \qquad \text{and} \qquad w' = \cdots \mid \cdots 1 \cdots .$$

In other words, we choose w such that $\lambda_1 = 1$, and w' is obtained from w by removing the negative sign from $\overline{1}$.

- w, w' a pair of *type B2* if

$$w = \cdots \mid \cdots \overline{a} \cdots (a-1) \cdots \qquad \text{and} \qquad w' = \cdots \mid \cdots \overline{a-1} \cdots a \cdots ,$$

where $a > 0$. In other words, there are $t \in [r]$ and $q \in [n-k-r]$ such that $\lambda_t = a$ and $v_q = a-1$, and w' is obtained from w by switching v_q and λ_t.

- w, w' a pair of *type B3* if

$$w = \cdots a \cdots \mid \cdots (a-x) \cdots \qquad \text{and} \qquad w' = \cdots (a-x) \cdots \mid \cdots a \cdots ,$$

where $a > x > 0$. In other words, there are $p \in [k]$ and $q \in [n-k-r]$ such that $u_p = a$ and $v_q = a-x$. The permutation w' is obtained from w by switching u_p and v_q.

- w, w' a pair of *type B4* if

$$w = \cdots (a-x) \cdots \mid \cdots \overline{a} \cdots \qquad \text{and} \qquad w' = \cdots a \cdots \mid \cdots \overline{a-x} \cdots ,$$

where $a > x > 0$. In other words, there are $p \in [k]$ and $t \in [r]$ such that $u_p = a-x$ and $\lambda_t = a$. The permutation w' is obtained from w by switching u_p and λ_t.

The following lemma states a property for pairs of type B3 and B4.

Lemma 3.1. *Let* $w, w' \in \mathcal{W}_n^{(k)}$.

1. *If* w, w' *is a pair of type B3 then* $[v_q + 1, u_p - 1] \subset \{\lambda_i \mid i \in [r]\}$;
2. *If* w, w' *is a pair of type B4 then* $[u_p + 1, \lambda_t - 1] \subset \{v_i \mid i \in [n-k-r]\}$;

Proof. Let w, w' be a pair of type B3. Suppose that $z \in [v_q + 1, u_p - 1]$ such that $z = u_i$ for some $i < p$. Then $u'_i = z > v_q = u'_p$, which is impossible since w' is a k-Grassmannian permutation. If $z = v_i$ for some $i > q$ then $v'_i = z < u_p = v'_q$, which is also impossible. The proof of second statement is analogous. $\square$

To define the previous four types of pairs, we only require that both w and w' belong to $\mathcal{W}_n^{(k)}$. In principle, it is not clear the relationship between them. This is the content of our main theorem.

Theorem 3.2. *Let* $w, w' \in \mathcal{W}_n^{(k)}$. *Then* w *covers* w' *if, and only if,* w, w' *is a pair of type B1, B2, B3 or B4.*

Proof. Consider the sets of positive integers $I_1 = [k]$, $I_2 = [k+1, r]$ and $I_3 = [k+r+1, n]$, and their respective sets of negative integers $I_{\overline{1}} = [\overline{k}, \overline{1}]$, $I_{\overline{2}} = [\overline{r}, \overline{k+1}]$ and $I_{\overline{3}} = [\overline{n}, \overline{k+r+1}]$. Notice that each set represent a block of positions in w as shown below

$$w = \underbrace{\overline{v_{n-k-r}} \cdots \overline{v_1}}_{I_{\overline{3}}} \; \underbrace{\lambda_1 \cdots \lambda_r}_{I_{\overline{2}}} \,|\, \underbrace{\overline{u_k} \cdots \overline{u_1}}_{I_{\overline{1}}} \; 0 \; \underbrace{u_1 \cdots u_k}_{I_1} \,|\, \underbrace{\overline{\lambda_r} \cdots \overline{\lambda_1}}_{I_2} \; \underbrace{v_1 \cdots v_{n-k-r}}_{I_3}.$$

Explicitly,

$$w(i) = \begin{cases} \overline{v_{i-k-r}} & \text{for } i \in I_3\,; \\ \overline{\lambda_{k+r+1-i}} & \text{for } i \in I_2\,; \\ u_i & \text{for } i \in I_1\,; \\ 0 & \text{for } i = 0\,; \\ \overline{u_{-i}} & \text{for } i \in I_{\overline{1}}\,; \\ \lambda_{k+r+1+i} & \text{for } i \in I_{\overline{2}}\,; \\ \overline{v_{-(i+k+r)}} & \text{for } i \in I_{\overline{3}}. \end{cases} \tag{3.1}$$

Suppose that w covers w', i.e., $w' \leqslant w$ and $\ell(w) = \ell(w') + 1$. Hence, by Theorem 2.1, there are $\overline{n} \leqslant i < j \leqslant n$ such that (i) either $w(i) > w(j)$ and $w' = w \cdot (i,j)(-i,-j)$, if $|i| \neq |j|$, when (i,j) is a non-central free rise of w', or (ii) $w(i) > w(j)$ and $w' = w \cdot (i,j)$, if $|i| = |j|$, when (i,j) is a central symmetric free rise of w'.

First of all, $i = 0$ if, and only if, $j = 0$ since the symmetry of $\mathcal{W}_n^{(k)}$ implies that $w(0) = 0$. Then, we will always consider non-zero i and j.

The set $[-n, n]$ is the disjoint union $I_{\overline{3}} \cup I_{\overline{2}} \cup I_{\overline{1}} \cup \{0\} \cup I_1 \cup I_2 \cup I_3$. We will prove the theorem by checking all possible combinations of $i < j$ such that $i \in I_m$ and $j \in I_l$, for $m, l \in \{\overline{3}, \overline{2}, \overline{1}, 1, 2, 3\}$. Table 1 encloses such information for i and j, where the rows denote I_m and the column denote I_l.

Table 1: Possible choices for $i < j$ such that $w(i) > w(j)$.

$i \diagdown j$	I_3	I_2	I_1	$I_{\overline{1}}$	$I_{\overline{2}}$	$I_{\overline{3}}$
I_3	✗					
I_2	✗	✗				
I_1	✓	✓	✗			
$I_{\overline{1}}$	✗	✓	✗	✗		
$I_{\overline{2}}$	✓	✓	✓	✓	✗	
$I_{\overline{3}}$	✗	✓	✗	✓	✗	✗

There are some choices of $i < j$ for which the relation $w(i) > w(j)$ is not satisfied. For instance, for every $i, j \in I_3$ such that $i < j$, we have $w(i) = v_{i-k-r} < v_{j-k-r} = w(j)$. Empty cells in Table 1 means $i > j$. A cell marked with ✗ means that for all $i < j$ in the respective set, we have $w(i) < w(j)$. Cells marked with ✓ are the ones such that we could have $i < j$ satisfying $w(i) > w(j)$.

When $|i| \neq |j|$, we know w' is obtained by swapping the values $w(i)$ and $w(j)$, and also swapping the values $w(-i)$ and $w(-j)$. Then, there is a symmetry in the choice of i, j. For instance, choosing $i < j$ such that $i \in I_{\overline{2}}$ and $j \in I_{\overline{1}}$ is equivalent to choose $-j < -i$ such that $-j \in I_1$ and $-i \in I_2$. It is enough to verify the case where i and j belongs to I_1 and I_2, respectively.

Therefore, we only have to check the five possibilities in Table 1 represented by the shaded cells marked with ✓.

1st case: Suppose that $i \in I_1$ and $j \in I_3$ such that $w(i) > w(j)$. Let $a > x > 0$ be integers such that $w(i) = u_i = a$ and $w(j) = v_{j-k-r} = a - x$. The permutation w' is obtained from w by swapping $w(i) = a$ and $w(j) = a - x$, and swapping the respective negatives $w(-i) = \overline{a}$ and $w(-j) = \overline{a-x}$. In short

$$\begin{aligned} w &= u_1 \cdots a \cdots u_k | \overline{\lambda_r} \cdots \overline{\lambda_1}\, v_1 \cdots (a-x) \cdots v_{n-k-r}, \\ w' &= u_1 \cdots (a-x) \cdots u_k | \overline{\lambda_r} \cdots \overline{\lambda_1}\, v_1 \cdots a \cdots v_{n-k-r}. \end{aligned}$$

Let us compare the lengths of w and w'. Lemma 3.1(i) says that all integers $a - x + 1, \ldots, a - 1$ belongs to the λ's. Then

$$\begin{aligned} \{v'_q \colon v'_q > u'_i\} &= \{v'_q \colon v'_q > a - x\} = \{v'_{j-k-r}, v'_{j-k-r+1}, \ldots, v'_{n-k-r}\} \\ &= \{a\} \cup \{v_{j-k-r+1}, \ldots, v_{n-k-r}\} = \{a\} \cup \{v_q \colon v_q > u_i\} \end{aligned}$$

and

$$\{v_q \colon v'_q > u'_t\} = \{v_q \colon v_q > u_i\}, \text{ for } t \in [k], t \neq i.$$

Hence,

$$\mu'_t = \begin{cases} \mu_i + 1 & \text{if } t = i\,; \\ \mu_t & \text{if } t \neq i. \end{cases}$$

It follows from Equation (2.10) that

$$\alpha'_t = \begin{cases} \alpha_i - 1 & \text{if } t = i\,; \\ \alpha_t & \text{if } t \neq i. \end{cases} \tag{3.2}$$

This equation implies that $|\alpha| = |\alpha'| + 1$. Since $|\lambda| = |\lambda'|$, it is clear that $\ell(w) = \ell(w') + 1$. Therefore, w, w' is a pair of type B3.

2nd case: Suppose that $i \in I_1$ and $j \in I_2$ such that $w(i) > w(j)$. Observe that if we swap $w(i) = u_i$ and $w(j) = \overline{\lambda_{j-k-r}}$, it would put a negative entry in the first k positions of w', which is not allowed by Equations (2.6). Therefore, this is not a valid case.

3rd case: Suppose that $i \in I_{\overline{1}}$ and $j \in I_2$ such that $w(i) > w(j)$. Let $a > x > 0$ be integers such that $w(j) = \overline{\lambda_{k+r+1-j}} = \overline{a}$ and $w(i) = \overline{u_{-i}} = \overline{a-x}$. The permutation w' is obtained from w by swapping $w(i) = \overline{a-x}$ and $w(j) = \overline{a}$ and the respective negatives $w(-i) = a - x$ and $w(-j) = a$. In short

$$\begin{aligned} w &= u_1 \,\cdots\, (a-x) \,\cdots\, u_k|\overline{\lambda_r} \,\cdots\, \overline{a} \cdots \overline{\lambda_1}\, v_1 \,\cdots\, v_{n-k-r}, \\ w' &= u_1 \,\cdots\, a \,\cdots\, u_k|\overline{\lambda_r} \,\cdots\, \overline{a-x} \cdots \overline{\lambda_1}\, v_1 \,\cdots\, v_{n-k-r}. \end{aligned}$$

Let us compare the lengths of w and w'. Lemma 3.1 says that all integers $a - x + 1, \ldots, a - 1$ are in the v's. Then

$$\begin{aligned} \{v_q \colon v_q > u_{-i}\} &= \{v_q \colon v_q > a - x\} = \{a - x + 1, \ldots, a - 1\} \cup \{v_q \colon v_q > a\} \\ &= \{a - x + 1, \ldots, a - 1\} \cup \{v'_q \colon v'_q > u'_{-i}\} \end{aligned}$$

and

$$\{v_q \colon v'_q > u'_t\} = \{v_q \colon v_q > u_i\}, \text{ for } t \in [k], t \neq -i.$$

Hence,

$$\mu_t = \begin{cases} \mu'_{-i} + (x-1) & \text{if } t = -i\,; \\ \mu'_t & \text{if } t \neq -i. \end{cases}$$

Then, it follows from Equation (2.10) that

$$\alpha'_t = \begin{cases} \alpha_{-i} + (x-1) & \text{if } t = -i\,; \\ \alpha_t & \text{if } t \neq -i. \end{cases} \tag{3.3}$$

This equation implies that $|\alpha| = |\alpha'| - x + 1$. Since $|\lambda| = |\lambda'| + x$, it is clear that $\ell(w) = \ell(w') + 1$. Therefore, w, w' is a pair of type B4.

4th case: Suppose that $i \in I_{\overline{2}}$ and $j \in I_3$ such that $w(i) > w(j)$. Let $a > x > 0$ be integers such that $w(i) = \lambda_{k+r+1+i} = a$ and $w(j) = v_{j-k-r} = a - x$. The permutation w' is obtained from w by swapping $w(i) = a$ and $w(j) = a - x$, and swapping the respective negatives $w(-i) = \overline{a}$ and $w(-j) = \overline{a-x}$. In short

$$\begin{aligned} w &= u_1 \,\cdots\, u_k|\overline{\lambda_r} \cdots \overline{a} \cdots \overline{\lambda_1}\, v_1 \cdots (a-x) \cdots v_{n-k-r}, \\ w' &= u_1 \,\cdots\, u_k|\overline{\lambda_r} \cdots \overline{a-x} \cdots \overline{\lambda_1}\, v_1 \cdots a \cdots v_{n-k-r}. \end{aligned}$$

Let us compare the lengths of w and w'. Since $w, w' \in \mathcal{W}_n^{(k)}$, all integers $a-x+1, \dots, a-1$ should be in the u's (this can be proved likewise in Lemma 3.1). Denote by $p \in [0,k]$ the largest integer such that $u_p < a-x$ (if required, take $u_0 = 0$). Clearly, $u_{p+1} = a-x+1, \dots, u_{p+x-1} = a-1$. Then

$$\begin{aligned}
&\{v'_q\colon v'_q > u'_t\} = \{v_q\colon v_q > u_i\}, \text{ for every } t \in [p]\,;\\
&\{v'_q\colon v'_q > u'_t\} = \{v_q\colon v_q > u_i\} \cup \{a\}, \text{ for every } t \in [p+1, p+x-1]\,;\\
&\{v'_q\colon v'_q > u'_t\} = \{v_q\colon v_q > u_i\}, \text{ for every } t \in [p+x, k].
\end{aligned}$$

and, hence,

$$\mu'_t = \begin{cases} \mu_t + 1 & \text{if } p < t \leqslant p+x-1\,;\\ \mu_t & \text{otherwise.}\end{cases}$$

By Equation (2.10),

$$\alpha_t = \begin{cases} \alpha'_t + 1 & \text{if } p < t \leqslant p+x-1\,;\\ \alpha'_t & \text{otherwise.}\end{cases} \tag{3.4}$$

and $|\alpha| = |\alpha'| + (x-1)$. Clearly, $|\lambda| = |\lambda'| + x$, which implies that $\ell(w) = \ell(w') + 2x - 1$. By hypothesis, $2x-1$ should be equal to 1, which lead us to conclude that $x = 1$. Therefore, w, w' is a pair of type B2.

5th case: Suppose that $i \in I_{\overline{2}}$ and $j \in I_2$ such that $w(i) > w(j)$. First of all, assume $-i < j$. Let $a > x > 0$ be integers such that $w(i) = \lambda_{k+r+1+i} = a$ and $w(j) = \overline{\lambda_{k+r-j}} = \overline{a-x}$. The permutation w' is obtained from w by swapping $w(i) = a$ and $w(j) = \overline{a-x}$, and the respective negatives $w(-i) = \overline{a}$ and $w(-j) = a-x$. In short

$$\begin{aligned}
w &= u_1 \cdots u_k | \overline{\lambda_r} \cdots \overline{a} \cdots \overline{a-x} \cdots \overline{\lambda_1}\, v_1 \cdots v_{n-k-r},\\
w' &= u_1 \cdots u_k | \overline{\lambda_r} \cdots (a-x) \cdots a \cdots \overline{\lambda_1}\, v_1 \cdots v_{n-k-r}.
\end{aligned}$$

Notice that if there is some $\overline{\lambda_m}$ to the right of $w'(-i) = a-x$ or $a > v_1$ then it would not satisfy Equations (2.6). Hence, we have $-i = k+r-1$, $j = k+r$ and $a < v_1$, and both $a-x$ and a of w' should be added to the v's. Moreover, all integers $a-x+1, \dots, a-1$ should be in the u's. Denote by $p \in [0,k]$ the largest integer such that $u_p < a-x$. Clearly, $u_{p+1} = a-x+1, \dots, u_{p+x-1} = a-1$.

Let us compare the lengths of w and w'. We have that

$$\begin{aligned}
&\{v'_q\colon v'_q > u'_t\} = \{v_q\colon v_q > u_t\} \cup \{a-x, a\}, \text{ for every } t \in [p]\,;\\
&\{v'_q\colon v'_q > u'_t\} = \{v_q\colon v_q > u_t\} \cup \{a\}, \text{ for every } t \in [p+1, p+x-1]\,;\\
&\{v'_q\colon v'_q > u'_t\} = \{v_q\colon v_q > u_t\}, \text{ for every } t \in [p+x, k],
\end{aligned}$$

and, hence,

$$\mu'_t = \begin{cases} \mu_t + 2 & \text{if } t \leqslant p\,;\\ \mu_t + 1 & \text{if } p < t \leqslant p+x-1\,;\\ \mu_t & \text{if } p+x-1 < t \leqslant k.\end{cases}$$

By Equation (2.10),

$$\alpha_t = \begin{cases} \alpha'_t + 2 & \text{if } t \leqslant p\,; \\ \alpha'_t + 1 & \text{if } p < t \leqslant p + x - 1\,; \\ \alpha'_t & \text{if } p + x - 1 < t \leqslant k. \end{cases}$$

and $|\alpha| = |\alpha'| + 2p + (x - 1)$. Clearly, $|\lambda| = |\lambda'| + (a - x) + a$, which implies that $\ell(w) = \ell(w') + 2a + 2p - 1$. By hypothesis, $2a + 2p - 1$ should be equal to 1. This implies that $a = 1$, $p = 0$, and x is an integer such that $1 > x > 0$. Therefore, this is not a valid case.

If $-i > j$ then we can proceed as above to show that this is also not a valid case.

Finally, suppose that $-i = j$. Let $a > 0$ be an integer such that $w(i) = \lambda_{k+r+1+i} = a$ and $w(j) = \overline{\lambda_{k+r-j}} = \overline{a}$. The permutation w' is obtained from w by swapping $w(i) = a$ and $w(j) = \overline{a}$. In short

$$\begin{aligned} w &= u_1 \cdots u_k | \overline{\lambda_r} \cdots \overline{a} \cdots \overline{\lambda_1}\, v_1 \cdots v_{n-k-r}, \\ w' &= u_1 \cdots u_k | \overline{\lambda_r} \cdots a \cdots \overline{\lambda_1}\, v_1 \cdots v_{n-k-r}. \end{aligned}$$

By Equations (2.6), w' lies in $\mathcal{W}_n^{(k)}$ if, and only if, $-i = j = k + r$, and $a < v_1$. Then, a should be added to the v's. Denote by $p \in [0, k]$ the largest integer such that $u_p < a$. Let us compare the length of w and w'. We have

$$\begin{aligned} \{v'_q \colon v'_q > u'_t\} &= \{v_q \colon v_q > u_t\} \cup \{a\}, \text{ for every } t \in [p]\,; \\ \{v'_q \colon v'_q > u'_t\} &= \{v_q \colon v_q > u_t\}, \text{ for every } t \in [p + 1, k]. \end{aligned}$$

and, hence,

$$\mu'_t = \begin{cases} \mu_t + 1 & \text{if } t \leqslant p\,; \\ \mu_t & \text{if } p < t \leqslant k. \end{cases}$$

By Equation (2.10),

$$\alpha_t = \begin{cases} \alpha'_t + 1 & \text{if } t \leqslant p\,; \\ \alpha'_t & \text{if } p < t \leqslant k. \end{cases} \tag{3.5}$$

and $|\alpha| = |\alpha'| + p$. Clearly, $|\lambda| = |\lambda'| + a$, which implies that $\ell(w) = \ell(w') + a + p$. By hypothesis, $a + p$ should be equal to 1, which lead us to conclude that $a = 1$ and $p = 0$. Therefore, w, w' is a pair of type B1.

Clearly, we implicitly proved the reciprocal. □

As consequence of Theorem 3.2, if one starts with any permutation $w \in \mathcal{W}_n^{(k)}$, we establish certain conditions to determine all possible w' of $\mathcal{W}_n^{(k)}$ covered by w.

Corollary 3.3 (Length-decreasing). *Let $w \in \mathcal{W}_n^{(k)}$. All possible $w' \in \mathcal{W}_n^{(k)}$ covered by w are described below:*

1. *If $w = \cdots | \cdots \overline{1} \cdots$ then w covers $w' = \cdots | \cdots 1 \cdots$;*

2. *If* $w = \cdots \mid \cdots \overline{a} \cdots (a-1) \cdots$ *then* w *covers* $w' = \cdots \mid \cdots \overline{a-1} \cdots a \cdots$*;*

3. *If* $w = \cdots a \cdots \mid \cdots b \cdots$ *where* $a > b$ *and all positive integers* $b+1, b+2, \ldots, a-1$ *belong to the* λ*'s then* w *covers* $w' = \cdots b \cdots \mid \cdots a \cdots$*;*

4. *If* $w = \cdots b \cdots \mid \cdots \overline{a} \cdots$ *where* $a > b$ *and all positive integers* $b+1, b+2, \ldots, a-1$ *belong to the* v*'s then* w *covers* $w' = \cdots a \cdots \mid \cdots \overline{b} \cdots$*.*

Proof. For statements (1) and (2), clearly w' belongs to $\mathcal{W}_n^{(k)}$, which implies that w, w' are pairs of type B1 or B2, respectively.

For (3), the condition of $b+1, b+2, \ldots, a-1$ belong to the λ's guarantees that $w' \in \mathcal{W}_n^{(k)}$. Then, w, w' is a pair of type B3. The same argument holds for (4), concluding that w, w' is a pair of type B4. $\square$

In some sense, Corollary 3.3 combines the results of Theorem 3.2 and Lemma 3.1.

Example 3.4. Consider $w = 2\,6|\overline{7}\,\overline{5}\,\overline{1}\,3\,4$ where $n = 7$ and $k = 2$. Let us determine $w' \in \mathcal{W}_7^{(2)}$ such that w covers w' following the Corollary 3.3. It is immediate that w covers $2\,6|\overline{7}\,\overline{5}\,1\,3\,4$ which is a pair of type B1.

To get a pair of type B2, we should pick an entry $\overline{a}$ in the λ's and an entry $(a-1)$ in the v's. The only possible choice is the pair of entries $\overline{5}$ and 4 which gives that w covers $2\,6|\overline{7}\,\overline{4}\,\overline{1}\,3\,5$.

To get a pair of type B3, we should pick an entry a in the u's and an entry b in the v's such that $a > b$ and all positive integers $b+1, b+2, \ldots, a-1$ belong to the λ's. Choosing 6 and 4 gives a covering since the only integer between $b = 4$ and $a = 6$ is contained in the λ's. Hence, w covers $2\,4|\overline{7}\,\overline{5}\,\overline{1}\,3\,6$. Choosing 6 and 3 does not give a covering since 4 is a number between $b = 3$ and $a = 6$, but it does not belong to the λ's.

Finally, to get a pair of type B4, we should pick an entry b in the u's and an entry $\overline{a}$ in the λ's such that $a > b$ and all positive integers $b+1, b+2, \ldots, a-1$ belong to the v's. We only have two pairs of entries that satisfy such conditions: $b = 6$ and $\overline{a} = \overline{7}$, which does not have integers between them; and $b = 2$ and $\overline{a} = \overline{5}$ since the integers 3 and 4 between b and a are in the v's. It gives that w covers $2\,7|\overline{6}\,\overline{5}\,\overline{1}\,3\,4$ and $5\,6|\overline{7}\,\overline{2}\,\overline{1}\,3\,4$, respectively.

Putting these cases together we have the following five covering pairs:

$$\begin{gathered}
w = 2\,6|\overline{7}\,\overline{5}\,\overline{\mathbf{1}}\,3\,4 \text{ and } w_1' = 2\,6|\overline{7}\,\overline{5}\,\mathbf{1}\,3\,4 \text{ of type } B1;\\
w = 2\,6|\overline{7}\,\overline{\mathbf{5}}\,\overline{1}\,3\,\mathbf{4} \text{ and } w_2' = 2\,6|\overline{7}\,\overline{\mathbf{4}}\,\overline{1}\,3\,\mathbf{5} \text{ of type } B2;\\
w = 2\,\mathbf{6}|\overline{7}\,\overline{5}\,\overline{1}\,3\,\mathbf{4} \text{ and } w_3' = 2\,\mathbf{4}|\overline{7}\,\overline{5}\,\overline{1}\,3\,\mathbf{6} \text{ of type } B3;\\
w = 2\,\mathbf{6}|\overline{\mathbf{7}}\,\overline{5}\,\overline{1}\,3\,4 \text{ and } w_4' = 2\,\mathbf{7}|\overline{\mathbf{6}}\,\overline{5}\,\overline{1}\,3\,4 \text{ of type } B4;\\
w = \mathbf{2}\,6|\overline{7}\,\overline{\mathbf{5}}\,\overline{1}\,3\,4 \text{ and } w_5' = \mathbf{5}\,6|\overline{7}\,\overline{\mathbf{2}}\,\overline{1}\,3\,4 \text{ of type } B4.
\end{gathered}$$

We also have a similar version of Corollary 3.3 where we start with any permutation $w' \in \mathcal{W}_n^{(k)}$ and we want to determine all possible $w \in \mathcal{W}_n^{(k)}$ that cover w'.

Corollary 3.5 (Length-increasing). *Let $w' \in \mathcal{W}_n^{(k)}$. All possible w of $\mathcal{W}_n^{(k)}$ that cover w' are described below:*

1. *If $w' = \cdots \mid \cdots 1 \cdots$ then $w = \cdots \mid \cdots \overline{1} \cdots$ covers w';*
2. *If $w' = \cdots \mid \cdots \overline{a-1} \cdots a \cdots$ then $w = \cdots \mid \cdots \overline{a} \cdots (a-1) \cdots$ covers w';*
3. *If $w' = \cdots b \cdots \mid \cdots a \cdots$ where $a > b$ and all positive values $b+1, b+2, \ldots, a-1$ belong to the λ's then $w = \cdots a \cdots \mid \cdots b \cdots$ covers w';*
4. *If $w' = \cdots a \cdots \mid \cdots \overline{b} \cdots$ where $a > b$ and all positive values $b+1, b+2, \ldots, a-1$ belong to the v's then $w = \cdots b \cdots \mid \cdots \overline{a} \cdots$ covers w'.*

4 Half-Shifted Young Diagrams

In this section we present how we can associate a Young Diagram model to each k-Grassmannian permutation such that the Bruhat order may be characterized by some patterns in those diagrams. We adapt constructions of [3], [10], [7] and [14].

Given n, k integers such that $0 \leqslant k < n$, consider a pair $\Lambda = \alpha, \lambda$ of partitions, where $\alpha = (0 \leqslant \alpha_1 \leq \cdots \leq \alpha_k \leqslant n-k)$ is a partition and $\lambda = (0 < \lambda_1 < \cdots < \lambda_r \leqslant n)$ is a strict partition.[1] We say that Λ is a double partition.

The partition α may be represented by a Young diagram defined by

$$\mathcal{D}_\alpha = \{(i,j) \in \mathbb{Z}^2 \colon 1 \leq i \leq k\,,\ 1 \leq j \leq \alpha_i\} \tag{4.1}$$

while the strict partition λ by a shifted Young diagram defined by

$$\mathcal{SD}_\lambda = \{(i,j) \in \mathbb{Z}^2 \colon 1 \leq i \leq r\ \ ,\ i \leq j \leq i-1+\lambda_{r+1-i}\}. \tag{4.2}$$

The diagram $\mathcal{D}_\alpha$ of a partition α fits into a rectangle of dimensions $k \times (n-k)$. The shifted Young diagram $\mathcal{SD}_\lambda$ of a strict partition λ is commonly represented inside a stair shaped triangle with n lines. Let us denote by $\mathcal{D}_{k,n}$ the set of all partitions whose respective Young diagrams are inside a rectangle $k \times (n-k)$ and by $\mathcal{SD}_n$ the set of all strict partitions whose respective shifted Young diagrams are inside a stair shaped diagram of length n. Finally, we define $\mathcal{P}(k,n)$ as the set of the pairs α, λ with $\alpha \in \mathcal{D}_{k,n}$ and $\lambda \in \mathcal{SD}_n$ satisfying $\ell(\lambda) \leq \alpha_1$, i.e.,

$$\begin{aligned} 0 \leq \alpha_1 \leq \cdots \leq \alpha_k \leq n-k\,; \\ 0 < \lambda_1 < \cdots < \lambda_r \leqslant n\,; \\ \ell(\lambda) \leq \alpha_1. \end{aligned} \tag{4.3}$$

A half-shifted Young Diagram (HSYD) of the pair $\Lambda = \alpha, \lambda \in \mathcal{P}(k,n)$ is obtained by juxtaposition of the diagrams $\mathcal{D}_\alpha$ and $\mathcal{SD}_\lambda$ such that $\mathcal{D}_\alpha$ is above $\mathcal{SD}_\lambda$. We say

[1] If $k = 0$ then α is represented as an empty set. An empty λ is represented by $r = 0$.

that $\mathcal{D}_\alpha$ is the top diagram and that $\mathcal{SD}_\lambda$ is the bottom diagram.[2] The condition $\alpha_1 \geq \ell(\lambda)$ is equivalent to say that the number of lines of the bottom diagram is at most the number of boxes in the first line of the top diagram. The HSYD for the pair $\alpha = (3, 5)$, $\lambda = (1, 5, 7)$ is shown in Figure 1.

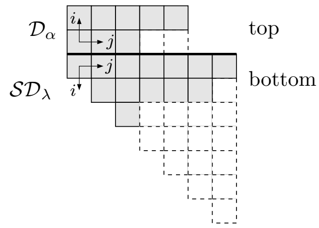

Figure 1: The HSYD build up fitting the Young diagram of $\alpha = (3, 5)$ together with the shifted Young diagram of $\lambda = (1, 5, 7)$.

The k-Grassmannian permutations are parametrized by the HSYD's.

Lemma 4.1. *There is an order preserving bijection between $\mathcal{W}_n^{(k)}$ and $\mathcal{P}(k, n)$.*

We can describe this bijection explicitly as follows. For a given $w \in \mathcal{W}_n^{(k)}$, consider α and λ as defined in Section 1. The first two statements of Equation (4.3) means that α a partition and λ a strict partition. For the third statement, notice that

$$\ell(\lambda) = \#\{\lambda_j \colon \lambda_j < u_1\} + \#\{\lambda_j \colon \lambda_j > u_1\} = \#\{\lambda_j \colon \lambda_j < u_1\} + d_1$$

Since $\#\{\lambda_j \colon \lambda_j < u_1\} \leq u_1 - 1$, it follows by Equation (2.7) that $\ell(\lambda) \leq \alpha_1$.

For the inverse, there is a nice geometric interpretation. We start with the bottom partition and construct the corresponding partition as a permutation of $[\pm n]$ by symmetrizing the diagram with respect to the diagonal (see, Section 6.2 of [7]). Consider the north-east to south-west path along the corresponding diagram and number each step with all integers from $\overline{n}$ to n. We take the sequence of the vertical numbers to obtain a permutation $\widetilde{w}$ in which the negative entries occur firstly ($\widetilde{w}$ is indeed a 0-Grassmannian permutation). In order to get the descent at position k, we insert the permutation $\widetilde{w}$ through the south-east to north-west path determined by the partition α inside the $k \times (n - k)$ rectangle (see, Section 3.3 of [7]). As we collect the vertical sequence, we obtain the numbers $u_1, u_2, \dots, u_k$. The horizontal ones correspond to the remaining part of w so that we get a k-Grassmmannian permutation. It is illustrated by Figure 2 for $\alpha = (3, 5)$ and $\lambda = (1, 5, 7)$.

There is also another way to provide the bijection through a method introduced by [3]. For each $i \in [k]$, we say that the $(\alpha_i + i)$-th column of the bottom diagram

[2] An alternative approach is that given by the k-strict partitions according to [3] where the corresponding diagram is obtained by transposing the top diagram given by the α partition and putting the bottom diagram given by the λ partition together at the right of the transposed top diagram (see definition 1.1, [3]). Our model is also distinguished with respect to that of Tamvakis [14] where both the top and bottom diagrams are simultaneously either fixed at left or shifted (see Figure 3, [14]).

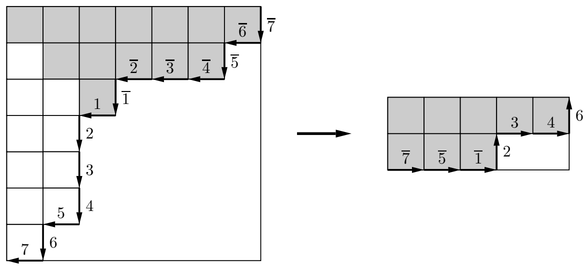

Figure 2: The diagram of $\lambda = (1, 5, 7)$ with its reflection that determines $\widetilde{w} = \overline{7}\,\overline{5}\,\overline{1}\,2\,3\,4\,6$ by collecting the vertical numbers obtained from the NE to SW path. Numbering the SW to NE path in the diagram of α determines $w = 2\,6\,\overline{7}\,\overline{5}\,\overline{1}\,3\,4$ by taking firstly the two vertical numbers and gathering the remaining ones.

is h-related. Geometrically, it means that we can draw a diagonal line from the last box of the respective line of the top digram until the first box of the corresponding column in the bottom diagram. The remaining columns are said to be v-related [3]. Then the number of empty boxes in each h-related column corresponds to the first k entries $u_1, u_2, \ldots, u_k$ of w. The number of empty boxes in the remaining v-related columns corresponds to the sequence $v_1, v_2, \ldots, v_{n-k-r}$. For instance, Figure 3 shows us the two h-related columns for the double partition $\alpha = (3, 5)$, $\lambda = (1, 5, 7)$.

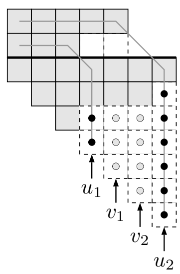

Figure 3: The gray lines represent the h-related columns for $\alpha = (3, 5)$ and $\lambda = (1, 5, 7)$. We have that $u_1 = 2$ and $u_2 = 6$ is the number of empty boxes (black dots) in the h-related columns, and $v_1 = 3$ and $v_2 = 4$ is the number of empty boxes (gray dots).

A remarkable consequence of the Theorem 3.2 is the following proposition.

Proposition 4.2. *Let $w, w' \in \mathcal{W}_n^{(k)}$ with α, λ and α', λ' being the corresponding partitions. Then, w, w' is a pair of*

1. *type B1 if, and only if, for every $i \in [k]$ and $j \in [r-1]$ we have $\alpha'_i = \alpha_i$, and $\lambda'_j = \lambda_{j+1}$.*

[3]The h-related resembles a relationship with the horizontal lines of the top diagram while v-related with the vertical lines.

2. *type B2 if, and only if, for every* $i \in [k]$ *and* $j \in [r]$ *we have*

$$\alpha_i' = \alpha_i \qquad \textit{and} \qquad \lambda_j' = \begin{cases} \lambda_j - 1 & \textit{if } j = t \\ \lambda_j & \textit{if } j \neq t \end{cases},$$

   for some $t \in [r]$.

3. *type B3 if, and only if, for every* $i \in [k]$ *and* $j \in [r]$ *we have*

$$\alpha_i' = \begin{cases} \alpha_i - 1 & \textit{if } i = p \\ \alpha_i & \textit{if } i \neq p \end{cases} \qquad \textit{and} \qquad \lambda_j' = \lambda_j,$$

   for some $p \in [k]$.

4. *type B4 if, and only if, for every* $i \in [k]$ *and* $j \in [r]$ *we have*

$$\alpha_i' = \begin{cases} \alpha_i + x - 1 & \textit{if } i = p \\ \alpha_i & \textit{if } i \neq p \end{cases} \qquad \textit{and} \qquad \lambda_j' = \begin{cases} \lambda_j - x & \textit{if } j = t \\ \lambda_j & \textit{if } j \neq t \end{cases},$$

   for some $p \in [k]$ *and* $t \in [r]$.

Proof. The partition λ' is obtained straightforward from the definition. The computation of the α's has already been done in the middle of the proof of the Theorem 3.2. If w, w' is a pair of type B1, the conclusion of the proof of the 5th case is that $a = 1$ and $p = 0$. In particular, it follows that $\alpha_i = \alpha_i'$ for all $i \in [k]$, by Equation (3.5). If w, w' is a pair of type B2, the conclusion of the proof of the 4th case is that $x = 1$. In particular, it follows that $\alpha_i = \alpha_i'$ for all $i \in [k]$, by Equation (3.4). If w, w' is a pair either of type B3 or B4, the result follows directly by Equations (3.2) and (3.3), respectively. □

The results of the Proposition 4.2 may be understood in terms of the operation of removing boxes of the HSYD. There are two types of boxes that can be removed. The first one, which we will call a *corner*, is a box of the diagram when removed produces a new diagram without any further operation. The second one, which we will call a *middle bottom box*, is a box (which is not a corner) of the bottom diagram when removed produces a new diagram after a further movement of boxes. A middle bottom box is neither a corner nor a diagonal; it lies in a h-related column; and all boxes to the right of it should belong to a v-related column (for more details, see [8]).

If w, w' is a pair of type B1, by Proposition 4.2[i], the diagram of Λ' is obtained by removing a corner in the diagonal of the bottom diagram: the top partitions α's are the same and the bottom partitions λ's are also the same but re-enumerated.

If w, w' is a pair of type B2, by Proposition 4.2 [ii], the diagram of Λ' is obtained from the diagram of Λ by removing a corner of the bottom diagram that belongs to a v-related column: the top partitions α's are the same and the λ's are the same except for one which is removed.

If w, w' is a pair of type B3, by Proposition 4.2 [iii], the diagram of Λ' is obtained from the diagram of Λ by removing a corner of the top diagram: the bottom partitions λ's are the same and the α's are the same except for one which is removed.

If w, w' is a pair of type B4, by Proposition 4.2 [iv], the diagram of Λ' is obtained from the diagram of Λ by removing a box of the bottom diagram that belongs to an h-related column which is either a corner or a middle box, respectively, when $x = 1$ or $x \neq 1$. If $x = 1$ then the bottom partitions λ's are the same and the top partitions α's are the same except for one which is removed. If $x \neq 1$ then both partitions are changed which is a consequence of the movement of the $(x-1)$ boxes at the right of the removed box in the bottom diagram to the top diagram.

In Figure 4, we illustrate these operations for $w = 2\,6|\overline{7}\,\overline{5}\,\overline{1}\,3\,4$ as in Example 3.4. We label the removed box with the corresponding type of the pair it determines.

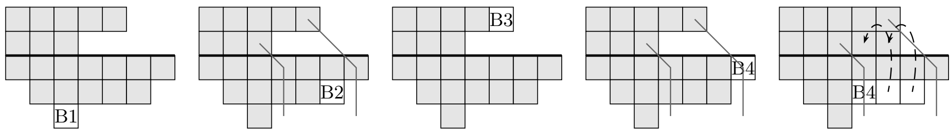

Figure 4: The five coverings of $w = 2\,6|\overline{7}\,\overline{5}\,\overline{1}\,3\,4$ obtained by removing boxes in its HSYD according to the type of the pair.

Remark 4.3. The k-Grassmannians permutations parametrize the Schubert varieties that equip the isotropic Grassmannians (real and complex) with a cellular structure. Furthermore, the HSYD's contain information about the associated β-sequences (see [7], Proposition 3) from which is possible to compute the boundary map of the cellular homology, with the aid of the Theorem 3.2. Thus, we were able to generalize the results of [11] which computes the homology coefficients for the real maximal isotropic Grassmannians. The results will appear in a subsequent paper.

Remark 4.4. Another geometric motivation for the appearance of this combinatorial description is the Pieri Rule for the (complex) Isotropic Grassmannians $\mathrm{IG}(n-k, 2n)$ for n sufficiently large. Indeed, we have the classical Chevalley rule (see Equation (5.2), Wilson [15]) which gives the product in the level of the cohomology ring of a Schubert class σ_Λ with the divisor class σ_1 as a combination of Schubert classes $\sigma_{\Lambda'}$, where Λ' is a diagram obtained from Λ by adding a box and Λ, Λ' is a pair of diagrams as in Proposition 4.2. The Pieri coefficients for the pairs of types B1,B2,B3, and B4 are respectively $2, 2, 1, 1$ according to the description made by [15] since the coefficient is either 2 when the added box belongs to a v-related column or 1 otherwise (see also [1], Corollary 11, [3], Theorem 2.1).

5 Maya diagram and dual permutations

The Maya diagram of a permutation w in $\mathcal{W}_n^{(k)}$ is a row of n boxes where each box is marked with a symbol $\circ$, $\bullet$, or $\times$ as follows: the integers $u_1, \ldots, u_k$ are the positions of the boxes with $\circ$, $\lambda_1, \ldots, \lambda_r$ are the positions of the boxes with $\bullet$, and $v_1, \ldots, v_{n-k-r}$ are the positions of the vacant boxes which will be marked with $\times$. For instance, the permutation $w = 2\,6|\overline{7}\,\overline{5}\,\overline{1}\,3\,4$ in $\mathcal{W}_7^{(2)}$ is denoted as

$$\begin{array}{|c|c|c|c|c|c|c|}\hline \bullet & \circ & \times & \times & \bullet & \circ & \bullet \\ \hline \end{array}$$

The longest element w_0^k of $\mathcal{W}_n^{(k)}$ is the k-Grassmannian permutation given by

$$w_0^k = 1\,2\,\cdots\,k|\overline{n}\,\overline{n-1}\,\cdots\,\overline{k+1}.$$

The next lemma states some properties of Maya diagrams followed by the definition.

Lemma 5.1.

1. *The identity permutation e in $\mathcal{W}_n^{(k)}$ is represented as*

$$\begin{array}{cccccc} 1 & & k & k{+}1 & & n \\ \hline \multicolumn{1}{|c|}{\circ} & \multicolumn{1}{c|}{\cdots} & \multicolumn{1}{c|}{\circ} & \multicolumn{1}{c|}{\times} & \multicolumn{1}{c|}{\cdots} & \multicolumn{1}{c|}{\times} \\ \hline \end{array}$$

2. *The longest element w_0^k in $\mathcal{W}_n^{(k)}$ is represented as*

$$\begin{array}{cccccc} 1 & & k & k{+}1 & & n \\ \hline \multicolumn{1}{|c|}{\circ} & \multicolumn{1}{c|}{\cdots} & \multicolumn{1}{c|}{\circ} & \multicolumn{1}{c|}{\bullet} & \multicolumn{1}{c|}{\cdots} & \multicolumn{1}{c|}{\bullet} \\ \hline \end{array}$$

3. *For every permutation w of $\mathcal{W}_n^{(k)}$, the Maya diagram of w contains exactly k boxes marked with $\circ$;*

4. *Let $w \in \mathcal{W}_n^{(k)}$. Given $i \in [k]$, the integer μ_i is the number of vacant boxes $\times$ to the right of the i-th box marked with $\circ$.*

We can use assertion (4) of Lemma 5.1 to compute the length of w in its Maya Diagram. By Lemma 2.3, it is the sum of α's and λ's. The λ's correspond to the sum of the positions of the $\bullet$'s. The computation of the α's follows by Equation (2.10) which says that $\alpha_i = n - k - \mu_i$. For instance, if $w = 2\,6|\overline{7}\,\overline{5}\,\overline{1}\,3\,4$ then we have that

$$\begin{array}{lcl} i=1: & \begin{array}{|c|c|c|c|c|c|c|}\hline \bullet & \circ & \times & \times & \bullet & \circ & \bullet \\ \hline \end{array} & \to \quad \mu_1 = 2; \\ i=2: & \begin{array}{|c|c|c|c|c|c|c|}\hline \bullet & \circ & \times & \times & \bullet & \circ & \bullet \\ \hline \end{array} & \to \quad \mu_2 = 0. \end{array}$$

Hence, $\alpha_1 = 5 - 2 = 3$, $\alpha_2 = 5$, $\lambda_1 = 1, \lambda_2 = 5$, and $\lambda_3 = 7$ such that $\ell(w) = 21$.

Given $w, w' \in \mathcal{W}_n^{(k)}$ such that w covers w', we also can denote the four types of pairs using the Maya diagram as follows:

- Type B1: w should contain $\bullet$ in the first position, while w' contains $\times$ in the first position. This pair can be represented as

$$w = \begin{array}{cc} 1 & \\ \hline \multicolumn{1}{|c|}{\bullet} & \multicolumn{1}{c|}{\cdots} \\ \hline \end{array} \quad \text{and} \quad w' = \begin{array}{cc} 1 & \\ \hline \multicolumn{1}{|c|}{\times} & \multicolumn{1}{c|}{\cdots} \\ \hline \end{array}$$

- Type B2: w should contain $\bullet$ in position a and $\times$ in position $a-1$, while w' contains $\times$ in position a and $\bullet$ in position $a-1$. This pair can be represented as

$$w = \begin{array}{|c|c|c|c|} \multicolumn{1}{c}{} & \multicolumn{1}{c}{\scriptstyle a-1} & \multicolumn{1}{c}{\scriptstyle a} & \multicolumn{1}{c}{} \\ \hline \cdots & \times & \bullet & \cdots \\ \hline \end{array} \quad \text{and} \quad w' = \begin{array}{|c|c|c|c|} \multicolumn{1}{c}{} & \multicolumn{1}{c}{\scriptstyle a-1} & \multicolumn{1}{c}{\scriptstyle a} & \multicolumn{1}{c}{} \\ \hline \cdots & \bullet & \times & \cdots \\ \hline \end{array}$$

- Type B3: w should contain $\circ$ in position a, $\times$ in position $a-x$, while w' contains $\times$ in position a, $\circ$ in position $a-x$. Moreover, by Lemma 3.1, both contain $\bullet$ in all positions between $a-x$ and a. This pair can be represented as

$$w = \begin{array}{|c|c|c|c|c|c|c|} \multicolumn{1}{c}{} & \multicolumn{1}{c}{\scriptstyle a-x} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{\scriptstyle a} & \multicolumn{1}{c}{} \\ \hline \cdots & \times & \bullet & \cdots & \bullet & \circ & \cdots \\ \hline \end{array} \quad \text{and} \quad w' = \begin{array}{|c|c|c|c|c|c|c|} \multicolumn{1}{c}{} & \multicolumn{1}{c}{\scriptstyle a-x} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{\scriptstyle a} & \multicolumn{1}{c}{} \\ \hline \cdots & \circ & \bullet & \cdots & \bullet & \times & \cdots \\ \hline \end{array}$$

- Type B4: w should contain $\bullet$ in position a, $\circ$ in position $a-x$, while w' contains $\circ$ in position a, $\bullet$ in position $a-x$. Moreover, by Lemma 3.1, both contain $\times$ in all positions between $a-x$ and a. This pair can be represented as

$$w = \begin{array}{|c|c|c|c|c|c|c|} \multicolumn{1}{c}{} & \multicolumn{1}{c}{\scriptstyle a-x} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{\scriptstyle a} & \multicolumn{1}{c}{} \\ \hline \cdots & \circ & \times & \cdots & \times & \bullet & \cdots \\ \hline \end{array} \quad \text{and} \quad w' = \begin{array}{|c|c|c|c|c|c|c|} \multicolumn{1}{c}{} & \multicolumn{1}{c}{\scriptstyle a-x} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{\scriptstyle a} & \multicolumn{1}{c}{} \\ \hline \cdots & \bullet & \times & \cdots & \times & \circ & \cdots \\ \hline \end{array}$$

Using Maya diagrams to represent these pairs also give us an easier way to identify each type of pairs. In fact, if w is any permutation in $\mathcal{W}_n^{(k)}$ then we can find all permutations w' covered by w merely looking for the above patterns in the Maya diagram of w. In other words, we could easily rewrite Corollary 3.3 in terms of Maya diagrams.

For instance, consider $w = 2\,6|\overline{7}\,\overline{5}\,\overline{1}\,3\,4 \in \mathcal{W}_7^{(2)}$ of Example 3.4. We can obtain the same five covering pairs using only the above patterns of w as it follows:

$$\begin{array}{lll}
B1: & w = \begin{array}{|c|c|c|c|c|c|c|} \hline \bullet & \circ & \times & \times & \bullet & \circ & \bullet \\ \hline \end{array} & \text{and} \quad w'_1 = \begin{array}{|c|c|c|c|c|c|c|} \hline \times & \circ & \times & \times & \bullet & \circ & \bullet \\ \hline \end{array} \\
B2: & w = \begin{array}{|c|c|c|c|c|c|c|} \hline \bullet & \circ & \times & \times & \bullet & \circ & \bullet \\ \hline \end{array} & \text{and} \quad w'_2 = \begin{array}{|c|c|c|c|c|c|c|} \hline \bullet & \circ & \times & \bullet & \times & \circ & \bullet \\ \hline \end{array} \\
B3: & w = \begin{array}{|c|c|c|c|c|c|c|} \hline \times & \circ & \times & \times & \bullet & \circ & \bullet \\ \hline \end{array} & \text{and} \quad w'_3 = \begin{array}{|c|c|c|c|c|c|c|} \hline \bullet & \circ & \times & \circ & \bullet & \times & \bullet \\ \hline \end{array} \\
B4: & w = \begin{array}{|c|c|c|c|c|c|c|} \hline \bullet & \circ & \times & \times & \bullet & \circ & \bullet \\ \hline \end{array} & \text{and} \quad w'_4 = \begin{array}{|c|c|c|c|c|c|c|} \hline \bullet & \circ & \times & \times & \bullet & \bullet & \circ \\ \hline \end{array} \\
B4: & w = \begin{array}{|c|c|c|c|c|c|c|} \hline \bullet & \circ & \times & \times & \bullet & \circ & \bullet \\ \hline \end{array} & \text{and} \quad w'_5 = \begin{array}{|c|c|c|c|c|c|c|} \hline \bullet & \bullet & \times & \times & \circ & \circ & \bullet \\ \hline \end{array}
\end{array}$$

Given $w \in \mathcal{W}_n^{(k)}$, define $w^\vee = w w_0^k$ the *dual permutation of* w. Notice that the action of w_0^k on w will reverse position and sign of the last $(n-k)$ positions of w. In other words, if w is written as in Equation (2.5), the one-line notation of the dual permutation of w is

$$w^\vee = u_1 \cdots u_k|\overline{v_{n-k-r}} \cdots \overline{v_1}\,\lambda_1 \cdots \lambda_r.$$

Clearly, we have $w^\vee \in \mathcal{W}_n^{(k)}$. The Maya diagram of $w^\vee$ is given by replacing all $\bullet$'s of w by $\times$'s, and replacing all $\times$'s of w by $\bullet$'s. For instance, the dual of $w = 2\,6|\overline{7}\,\overline{5}\,\overline{1}\,3\,4$ is the permutation $w^\vee = 2\,6|\overline{4}\,\overline{3}\,1\,5\,7$ and the Maya diagram is

$$\begin{array}{|c|c|c|c|c|c|c|} \hline \times & \circ & \bullet & \bullet & \times & \circ & \times \\ \hline \end{array}$$

Let us compute the length of $w^\vee$.

Lemma 5.2. *The length of $w^\vee$ is $\ell(w^\vee) = \ell(w_0^k) - \ell(w)$.*

Proof. First of all, let us compute the length of w_0. Using Equation (2.4) we can easily show that $\ell(w_0^k) = \frac{1}{2}(n+3k+1)(n-k)$ since $\mathrm{inv}(w_0(1), \dots, w_0(n)) = k(n-k)$ and

$$-\sum_{\{j|w(j)<0\}} w(j) = \frac{1}{2}(n+k+1)(n-k).$$

Notice that the partitions associated with $w^\vee$ are $\alpha_i^\vee = n-k-\mu_i(w^\vee) = n-k-d_i$ for $i \in [k]$ and $\lambda_i^\vee = v_i$ for $i \in [n-k-r]$. Using Equation (2.9), we have

$$\begin{aligned}
\ell(w) + \ell(w^\vee) &= \sum_{i=1}^{k} \alpha_i + \sum_{i=1}^{r} \lambda_i + \sum_{i=1}^{k} \alpha_i^\vee + \sum_{i=1}^{n-k-r} \lambda_i^\vee \\
&= \sum_{i=1}^{k} (n-k) + \sum_{i=1}^{r} \lambda_i + \sum_{i=1}^{k} (n-k-d_i-\mu_i) + \sum_{i=1}^{n-k-r} v_i \\
&= k(n-k) + \sum_{i=1}^{r} \lambda_i + \sum_{i=1}^{k} (u_i - i) + \sum_{i=1}^{n-k-r} v_i \\
&= k(n-k) + \left(\sum_{i=1}^{r} \lambda_i + \sum_{i=1}^{k} u_i + \sum_{i=1}^{n-k-r} v_i \right) - \sum_{i=1}^{k} i \\
&= k(n-k) + \sum_{i=1}^{n} i - \sum_{i=1}^{k} i = k(n-k) + \sum_{i=k+1}^{n} i \\
&= k(n-k) + \frac{1}{2}(n+k+1)(n-k) = \frac{1}{2}(n+3k+1)(n-k).
\end{aligned}$$

Hence, $\ell(w) + \ell(w^\vee) = \ell(w_0^k)$. □

Corollary 5.3. *$w' \leqslant w$ if, and only if, $w^\vee \leqslant (w')^\vee$. Moreover, $\ell(w) - \ell(w') = \ell((w')^\vee) - \ell(w^\vee)$. In particular, w covers w' if, and only if, $(w')^\vee$ covers $w^\vee$.*

Then, the duality of a permutation also implies a duality over the covering pairs. The next proposition states a duality among the type of pairs.

Proposition 5.4. *Let w, w be permutations in $\mathcal{W}_n^{(k)}$ such that w covers w'. Then*

1. *w, w' is a pair of type B1 if, and only if, $(w')^\vee, w^\vee$ is a pair of type B1;*
2. *w, w' is a pair of type B2 if, and only if, $(w')^\vee, w^\vee$ is a pair of type B2;*
3. *w, w' is a pair of type B3 if, and only if, $(w')^\vee, w^\vee$ is a pair of type B4;*

Proof. This result can be easily obtained using the pairs w, w' and $(w')^\vee, w^\vee$ represented in the Maya diagrams. □

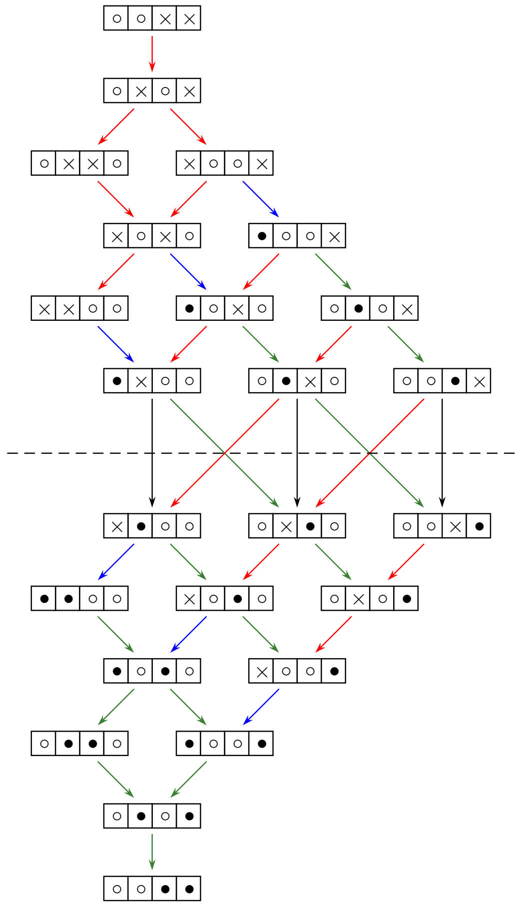

Figure 5: Bruhat graph for $n = 4$ and $k = 2$.

For instance, $w = 2\,\mathbf{6}|\overline{7}\,\overline{5}\,\overline{1}\,3\,\mathbf{4}$ and $w'_3 = 2\,\mathbf{4}|\overline{7}\,\overline{5}\,\overline{1}\,3\,\mathbf{6}$ from Example 3.4 is a pair of type B3, whereas $(w'_3)^\vee = 2\,\mathbf{4}|\overline{\mathbf{6}}\,\overline{3}\,1\,5\,7$ and $w^\vee = 2\,\mathbf{6}|\overline{\mathbf{4}}\,\overline{3}\,1\,5\,7$ is a pair of type B4.

Let $w, w' \in \mathcal{W}_n^{(k)}$. We write $w' \to w$ if w covers w'. The Bruhat graph is the graph such that the set of vertices is $\mathcal{W}_n^{(k)}$ and the (oriented) arrows are the covering relation for the Bruhat order.

Example 5.5. Let $n = 4$ and $k = 2$. Figure 5 represent the Bruhat graph for $\mathcal{W}_4^{(2)}$. The type of each covering is denoted using different colors: blue arrows are pairs of type B1; black arrows are pairs of type B2; red arrows are pairs of type B3; and green arrows are pairs of type B4. The duality of permutations and pairs can be seen as a symmetry through the horizontal dashed line, where symmetric vertices and arrows are, respectively, dual permutations and coverings.

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