AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **77(2)** (2020), Pages 144–156

Closed-neighborhood union-closed graphs

Kelly Guest

Tuskegee University
Tuskegee, AL, U.S.A.

Sarah Holliday

Southern Polytechnic State University
Marietta, GA, U.S.A.

Peter Johnson

Auburn University
Auburn, AL, U.S.A.

Dieter Rautenbach

Universität Ulm
Ulm, Baden-Württemberg, Germany

Matthew Walsh

Purdue University at Fort Wayne
Fort Wayne, IN, U.S.A.

Abstract

A CNUC graph is one in which the union of any two closed neighborhoods of single vertices is a closed neighborhood of a single vertex. We show that every (finite, simple) graph is an induced subgraph of a CNUC graph. This paper provides the definition of the *cnuc* number of a graph, the minimum number of vertices that must be added in order to embed the graph as an induced subgraph of a CNUC graph. This number is determined or estimated for several classes of graphs.

1 Introduction

All graphs in this paper will be finite and simple. Our notation will be conventional: a graph G has vertex set $V(G)$ and edge set $E(G)$; an edge with end-vertices u, v may be denoted uv or vu; if $v \in V(G)$, $N_G(v) = \{u \in V(G) \mid uv \in E(G)\}$, the open

ISSN: 2202-3518

neighborhood of v in G, and $N_G[v] = \{v\} \cup N_G(v)$, the closed neighborhood of v in G; if $S \subseteq V(G)$, $N_G(S) = \bigcup_{v \in S} N_G(v)$ and $N_G[S] = N_G(S) \cup S$; P_n, C_n, and K_n will denote respectively the path, cycle, and complete graph on n vertices; $\chi(G)$ denotes the chromatic number of a graph G, $\gamma(G) = \min[|S|; S \subseteq V(G) \text{ and } N_G[S] = V(G)]$ the domination number of G, and $\alpha(G)$ denotes the vertex independence number of G.

In most papers in graph theory, the subscript G is usually omitted from N_G. In this paper, the subscript will never be omitted, for we will be embedding smaller graphs into larger graphs, and it will be important to distinguish between the neighborhoods of vertices in different graphs. For instance, we have the following, which can be taken as a definition of "induced subgraph" if one has never heard of the concept before, or as pointing out an easily verified defining property of induced subgraphs, otherwise.

Lemma 1.1. *If G is a subgraph of H, then G is an induced subgraph of H if and only if, for each $S \subseteq V(G)$, $N_G[S] = N_H[S] \cap V(G)$.*

An indexed family $\mathcal{A} = [A_i; i \in I]$ of sets is *union-closed* if and only if for each pair $i, j \in I$ there exists $k \in I$ such that $A_i \cup A_j = A_k$. By induction on m, one sees that $\mathcal{A}$ is union-closed if and only if for any positive integer m and any $i_1, \ldots, i_m \in I$, there exists $k \in I$ such that $A_{i_1} \cup \cdots \cup A_{i_m} = A_k$.

A graph G is *closed-neighborhood union-closed*, CNUC for short, if and only if the family $[N_G[v]; v \in V(G)]$ is union-closed. For instances, K_n is CNUC for all n, C_n is CNUC only for $n = 3$, and P_n is CNUC if and only if $n \leq 3$.

Lemma 1.2. *If G is a CNUC graph, then some $w \in V(G)$ is adjacent to every other vertex of G.*

Proof. By previous remarks, since $[N_G[v]; v \in V(G)]$ is union-closed, and $V(G)$ is finite, $V(G) = \bigcup_{v \in V(G)} N_G[v] = N_G[w]$ for some $w \in V(G)$. □

Corollary 1.1. *If G is a CNUC graph, then G is connected.*

With the aid of Lemma 1.2 it can be seen that there are almost no CNUC graphs among the "usual suspects." For instance, by Lemma 1.2 the only bipartite graphs that might possibly be CNUC, with more than one vertex (K_1 is CNUC), are the stars, $K_{1,a}$, and if $a > 2$, then $K_{1,a}$ is clearly not CNUC. Therefore $K_{1,1} = K_2$ and $K_{1,2} = P_3$ are, with K_1, the only CNUC bipartite graphs. We have already ruled out the cycles C_n, $n > 3$, and now Lemma 1.2 rules them out again, as it does the Petersen graph. By Lemma 1.2 any complete r-partite graph, $r \geq 3$, which is CNUC must have at least one part of size 1; if it has a part of size ≥ 3 then it cannot be CNUC, because the union of the closed neighborhoods of two vertices of such a larger partite set is not a closed neighborhood of any single vertex in G. So a CNUC complete r-partite graph, $r \geq 3$, must have all parts of sizes 1 or 2, with at least one of size 1. Conversely, every such graph is CNUC.

Other background issues:

1. What of open-neighborhood union-closed graphs? The "open-neighborhood union-closed graphs" would contain only $\overline{K}_n$, $n = 1, 2, \ldots$; verification is left to the reader.

2. This study was inspired by a famous problem of Peter Frankl [2]: "Given a finite union-closed family $\mathcal{F}$ of sets, is there necessarily an element that appears in at least one-half of the sets in $\mathcal{F}$?" In Frankl's problem the sets are distinct, and $\emptyset \in \mathcal{F}$ is permitted; nonetheless, in the case of either $[N_G[v]; v \in V(G)]$ or $\{N_G[v] \mid v \in V(G)\}$, when G is CNUC, the answer to Frankl's question is trivially yes, by Lemma 1.2: there is some $w \in V(G)$ such that $w \in N_G[v]$ for all $v \in V(G)$. Obviously, not all union-closed families are obtained by closed neighborhoods of graphs.

3. If $N_G[u] = N_G[v]$ and $u \neq v$, let us call u and v *closed neighborhood (c-n) clones* in G. If $u, v \in V(G)$ are such clones then G is CNUC if and only if the graph $\tilde{G}$ obtained by replacing u and v by a new vertex x, with $N_{\tilde{G}}(x) = N_G(u)\backslash\{v\} = N_G(v)\backslash\{u\}$, is CNUC. Thus *c-n* clones can be merged, or single vertices split into *c-n* clones, without affecting CNUC-ness. This means that in looking for CNUC graphs, we could (but we will not) confine our attention to graphs G such that the sets $N_G[v]$, $v \in V(G)$, are distinct.

2 The main result

Let $\mathcal{F}$ be a finite collection of non-empty finite sets, and G a graph. A function $\phi : V(G) \to \mathcal{F}$ such that $u, v \in V(G)$, $u \neq v$, are adjacent in G if and only if $\phi(u) \cap \phi(v) \neq \emptyset$, is called a *non-empty-finite-set-intersection-graph representation* of G. Notice that ϕ is not required to be an injection, and that if ϕ is a non-empty-finite-set-intersection-graph representation of G, and $\phi(u) = \phi(v)$, then $N_G[u] = N_G[v]$.

For short, we shall call such a function ϕ a *committee representation* of G, and, if G has such a representation, for some choice of $\mathcal{F}$, we shall call G a *committee graph.* The following is well-known (see [3]).

Lemma 2.1. *Every finite simple graph G is a committee graph.*

Proof. Let C be the collection of all maximal cliques in G, and let $\mathcal{F} = 2^C\backslash\{\emptyset\}$, the set of all non-empty subsets of C. Define $\phi : V(G) \to \mathcal{F}$ by $\phi(v) = \{K \in C \mid v \in V(K)\}$, the set of all maximal cliques in G containing v. Clearly $u, v \in V(G)$, $u \neq v$, are adjacent in G if and only if $\phi(u) \cap \phi(v) \neq \emptyset$. □

The *committee number* of a graph, G, denoted $cn(G)$, is the minimum value of $\left|\bigcup_{F\in\mathcal{F}} F\right|$ such that $\phi : V(G) \to \mathcal{F}$ is a committee representation of G. By sharpening the proof preceding, we see that

$$cn(G) \leq \min[|C|; C \text{ is a collection of cliques in } G \text{ which cover both } V(G) \text{ and } E(G)].$$

For all n, $cn(K_n) = 1$ and $cn(\overline{K}_n) = n$. It will be useful to note that

$$cn(G) \;=\; \min_{\phi} \left| \bigcup_{v \in V(G)} \phi(v) \right|,$$

where the minimum is taken over all committee representations of G.

Theorem 2.1. *Every finite simple graph is an induced subgraph of a CNUC graph.*

Proof. Suppose that G is a finite simple graph. If G is CNUC then G is an induced subgraph of a CNUC graph, namely, itself. So suppose that G is not CNUC.

Let $\phi : V(G) \to \mathcal{F}$ be a committee representation of G. We may as well suppose that ϕ is surjective. Let $\hat{\mathcal{F}}$ denote the collection of all unions of sets in $\mathcal{F}$, except for the empty union; then $\hat{\mathcal{F}}$, like $\mathcal{F}$, is a finite collection of non-empty finite sets.

For $S \in \hat{\mathcal{F}} \backslash \mathcal{F}$, let w_S be a "new" vertex, not in $V(G)$, nor equal to w_{S_1} for any $S_1 \in \hat{\mathcal{F}} \backslash \mathcal{F}$, $S \neq S_1$. Let $\hat{V} = V(G) \cup \{w_S \mid S \in \hat{\mathcal{F}} \backslash \mathcal{F}\}$, and let $\hat{\phi} : \hat{V} \to \hat{\mathcal{F}}$ be the extension of ϕ defined on $\hat{V} \backslash V(G)$ by $\hat{\phi}(w_S) = S$. Then let $\hat{G} = \left(\hat{V}, \hat{E}\right)$ be the graph of which $\hat{\phi}$ is a committee representation; if $u, v \in \hat{V}$, $u \neq v$, then $uv \in \hat{E}$ if and only if $\hat{\phi}(u) \cap \hat{\phi}(v) \neq \emptyset$.

Since $\hat{\phi}$ is an extension of ϕ, and ϕ is a committee representation of G, it follows that G is an induced subgraph of $\hat{G}$. To see that $\hat{G}$ is CNUC: for $S \in \mathcal{F}$, let w_S denote any of the vertices $\{u \in V(G) \mid \phi(u) = S\}$. (If there is more than one of these vertices, note that they are all *c-n* clones in $\hat{G}$). With this notational convenience, it is straightforward to see that for $u, v \in \hat{V}$, $N_{\hat{G}}[u] \cup N_{\hat{G}}[v] = N_{\hat{G}}\left[w_{\hat{\phi}(u) \cup \hat{\phi}(v)}\right]$. □

We will give another proof of Theorem 2.1 that does not utilize committee representation in Section 4, in the proof of Theorem 4.1.

Let d_G denote the degree function on $V(G)$, and let $\Delta(G)$, $\delta(G)$ denote the maximum and minimum degree, respectively, in G. The following result shows that every CNUC graph other than K_1 has a proper CNUC induced subgraph.

Theorem 2.2. *Suppose that G is CNUC, $v \in V(G)$, and $d_G(v) = \delta(G)$. Then $G - v$ is CNUC.*

Proof. If $G - v$ is not CNUC then there exist $u, w \in V(G) \backslash \{v\}$ such that $N_{G-v}[u] \cup N_{G-v}[w]$ is not the closed neighbor set of any single vertex in $G - v$.

Therefore, neither of $N_{G-v}[u]$, $N_{G-v}[w]$ is contained in the other. Therefore, $|N_{G-v}[u] \cup N_{G-v}[w]| \geq \delta(G)+1$. Therefore, the $x \in V(G)$ such that $N_G[u] \cup N_G[w] = N_G[x]$ is not v. But then $N_{G-v}[u] \cup N_{G-v}[w] = N_{G-v}[x]$ and $x \in V(G - v)$, contrary to supposition. □

3 The *cnuc* number

Definition 3.1. The *cnuc* number of G, denoted $cnuc(G)$, is

$$cnuc(G) = \min[|V(H)| - |V(G)| \quad ; \quad H \text{ is a } CNUC \text{ graph containing } G \text{ as an induced subgraph}]$$

Let $z(n) = 2^n - n - 1$. From the proof of Theorem 2.1 we have the following.

Theorem 3.1. *Suppose that G is not CNUC, and $\phi : V(G) \to 2^F \backslash \{\emptyset\}$ is a committee representation of G. If $\eta(\phi) = |\{\phi(v) \mid v \in V(G)$ and $|\phi(v)| \geq 2\}|$ then $cnuc(G) \leq z(|F|) - \eta(\phi)$.*

Proof. Note that $z(|F|) = |\{S \subseteq F \mid |S| \geq 2\}|$. Let $\mathcal{F}$ denote the range of ϕ, and let $\hat{\mathcal{F}}$ be as in the proof of Theorem 2.1. If $S \in \hat{\mathcal{F}} \backslash \mathcal{F}$, then S is the union of two or more different sets in $\mathcal{F}$, and so $|S| \geq 2$. Therefore, by the proof of Theorem 2.1,

$$\begin{aligned} cnuc(G) &\leq |\hat{\mathcal{F}} \backslash \mathcal{F}| \\ &\leq |\{S \subseteq F \mid |S| \geq 2\}| - |\{S \in \mathcal{F} \mid |S| \geq 2\}| \\ &= z\left(|F|\right) - \eta\left(\phi\right). \end{aligned}$$

□

Corollary 3.1. *For any graph G, $cnuc(G) \leq z(cn(G))$.*

These inequalities raise extremal questions: For which G do there exist F and ϕ satisfying the hypothesis of Theorem 3.1 and $cnuc(G) = z(|F|) - \eta(\phi)$? For which G is $cnuc(G) = z(cn(G))$? This latter problem we will solve in this section (Theorem 3.4); the former is beyond our reach at present.

Lemma 3.1. *Suppose that (i) $\emptyset \neq S \subseteq V(G)$; (ii) for all $U \subseteq S$, $|U| \geq 2$, $N_G[U]$ is not the closed neighbor set in G of any single vertex in G; and (iii) the function $U \to N_G[U]$ from $\{U \subseteq S \mid |U| \geq 2\}$ into $2^{V(G)}$ is injective. Then $cnuc(G) \geq z(|S|)$.*

Proof. Suppose that H is CNUC and contains G as an induced subgraph. If $U \subseteq S$ and $|U| \geq 2$, then any vertex $w = w(U)$ in H such that $N_H[w] = N_H[U]$ cannot be a vertex of G, for, if it were, then by Lemma 1.1, $N_H[w] \cap V(G) = N_G[w] = N_H[U] \cap V(G) = N_G[U]$, contradicting (ii). If $U_1, U_2 \subseteq S$, $|U_1|, |U_2| \geq 2$, and $U_1 \neq U_2$, and if $w_i \in V(H)$ satisfies $N_H[w_i] = N_H[U_i]$, $i = 1, 2$, then, again invoking Lemma 1.1 as just above, $N_H[w_i] \cap V(G) = N_G[U_i]$, $i = 1, 2$. Since $N_G[U_1] \neq N_G[U_2]$, by (iii), it follows that $w_1 \neq w_2$. Putting it all together, $|V(H) \backslash V(G)| = |V(H)| - |V(G)| \geq |\{U \subseteq S \mid |U| \geq 2\}| = z(|S|)$. Therefore, $cnuc(G) \geq z(|S|)$. □

Theorem 3.2. *For any graph G, $cnuc(G) \geq z(\gamma(G))$.*

Proof. If $\gamma(G) = 1$ then $z(\gamma(G)) = 0$. So we may suppose that $\gamma(G) \geq 2$. Let $S \subseteq V(G)$ be a dominating set in G satisfying $|S| = \gamma(G)$. The desired conclusion will be supplied by Lemma 3.1 if S satisfies (ii) and (iii) in the hypothesis of that lemma.

(ii): If, for some $U \subseteq S$, $|U| \geq 2$, there were some $w \in V(G)$ such that $N_G[w] = N_G[U]$, then $(S \backslash U) \cup \{w\}$ would be a dominating set in G with fewer vertices than S. This is a contradiction, which establishes that S satisfies (ii) in Lemma 3.1.

(iii): If, for some $U_1, U_2 \subseteq S$, $|U_1|, |U_2| \geq 2$, $U_1 \neq U_2$, we have $N_G[U_1] = N_G[U_2]$, then $(U_1 \backslash U_2) \cup (U_2 \backslash U_1)$ is non-empty, and removing any vertex of that set from S would result in a smaller dominating set in G—again, impossible; so S satisfies (iii) in Lemma 3.1.

□

Theorem 3.3. *If G is a graph on n vertices, then $cnuc(G) \geq 2^{\alpha(G)} - n - 1 = z(\alpha(G)) - (n - \alpha(G))$.*

Proof. Suppose that H is a CNUC graph containing G as an induced subgraph, and let $S \subseteq V(G)$ be independent in G satisfying $|S| = \alpha(G)$. If $U_1, U_2 \subseteq S$ with $U_1 \neq U_2$, then $N_G[U_1] \neq N_G[U_2]$ (because S is independent), and thus $N_H[U_1] \neq N_H[U_2]$, by Lemma 1.1 again, because G is an induced subgraph of H.

Therefore, if we choose, for each $U \subseteq S$, $|U| \geq 2$, a vertex $w(U) \in V(H)$ such that $N_H[w(U)] = N_H[U]$, then the map $U \to w(U)$ is injective. Further, because $U \subseteq S \cap N_H[w(U)]$ and S is independent in H, $w(U) \notin S$, so we have an injection from $\{U \subseteq S \mid |U| \geq 2\}$ into $V(H) \backslash S$. Therefore $2^{|S|} - |S| - 1 = z(\alpha(G)) \leq |V(H)| - |S| = |V(H)| - n + n - \alpha(G)$. Choosing H so that $|V(H)| - n = |V(H)| - |V(G)| = cnuc(G)$, the claimed inequality follows. □

Theorem 3.4. *$cnuc(G) = z(cn(G))$ if and only if G is a disjoint union of cliques.*

Proof. Suppose $G = H_1 + \cdots + H_m$, with $+$ denoting disjoint union, and $H_1, \ldots, H_m$ are cliques. If $\phi : V(G) \to 2^F \backslash \{\emptyset\}$ is a committee representation of G, then subsets of F assigned by ϕ to vertices in different H_i are non-empty and disjoint; therefore, $|F| \geq m$. Therefore, $cn(G) \geq m$. On the other hand, assigning $\{i\}$ to each vertex of H_i, $i = 1, \ldots, m$, gives a committee representation of G. Therefore $cn(G) = m$. By Corollary 3.1, $cnuc(G) \leq z(m)$, and by Theorem 3.2, $cnuc(G) \geq z(\gamma(G)) = z(m)$, so $cnuc(G) = z(cn(G))$.

Now suppose that $cnuc(G) = z(cn(G))$. Let $|F| = cn(G)$ and let $\phi : V(G) \to 2^F \backslash \{\emptyset\}$ be a committee representation of G. With $\eta(\phi)$ defined as in Theorem 3.1, we conclude, from Theorem 3.1, that $\eta(\phi) = 0$. Therefore, ϕ assigns only singletons to vertices of G. Therefore, G is a disjoint union of cliques. □

A disjoint union G of m cliques, not all single vertices, is obtainable from $\overline{K_m}$ by splitting single vertices into closed-neighborhood clones. A CNUC extension $\hat{H}$ of G with $|V(\hat{H})| - |V(G)| = 2^m - m - 1$ is obtainable from a CNUC extension H of $\overline{K_m}$ with $|V(H)| - m = 2^m - m - 1$ by exploding each vertex of $\overline{K_m}$ into a clique of closed-neighborhood clones of that vertex *in* H. Here is how H is formed. For each $U \subseteq V(\overline{K_m})$, $|U| \geq 2$, let $w(U) \in V(H)$ be adjacent to every vertex of U, and to no vertex of $V(\overline{K_m}) \backslash U$. If $U_1, U_2 \subseteq V(\overline{K_m})$, $|U_1|, |U_2| \geq 2$, and $U_1 \neq U_2$, then $w(U_1)$, $w(U_2)$ are adjacent in H if and only if $U_1 \cap U_2 \neq \emptyset$. This construction of H on $\overline{K_m}$ suggests a construction on other graphs, which we will consider in the next section.

4 Another proof of Theorem 2.1 and an exact value for $cnuc(G)$

Lemma 4.1. *Suppose that $U_1, U_2 \subseteq V(G)$, $S_i = N_G[U_i]$, $i = 1, 2$, and $S_1 \cap U_2 \neq \emptyset$. Then for any sets $U_1', U_2' \subseteq V(G)$ such that $S_i = N_G[U_i']$, $i = 1, 2$, $S_1 \cap U_2' \neq \emptyset$ and $S_2 \cap U_1' \neq \emptyset$.*

Proof. We show that $S_2 \cap U_1 \neq \emptyset$. Since U_1 could be any subset of $V(G)$ such that $S_1 = N_G[U_1]$, this will finish the proof.

Since $S_1 \cap U_2 \neq \emptyset$, some $u_2 \in U_2$ is either a neighbor of some $u_1 \in U_1$, or actually is in U_1 itself. Either way, since $S_2 = N_G[U_2]$, $U_1 \cap S_2 \neq \emptyset$. □

For any graph G, let

$$CVT(G) = \{N_G[U] \mid U \subseteq V(G), |U| \geq 2, \text{ and } N_G[U] \text{ is not the closed neighborhood in } G \text{ of any single vertex}\},$$

the set of "CNUC-violating territories" in G. Note that an element of $CVT(G)$ can be the closed neighbor set, in G, of different sets $U \subseteq V(G)$. For example, $CVT(P_4)$ has one element, the entire vertex set of P_4, which is the closed neighbor set of 9 different subsets of $V(P_4)$.

The following theorem is a much sharper version of Theorem 2.1, and could be proven without assuming the result of Theorem 2.1. But it will be convenient to assume that result.

Theorem 4.1. *For any graph G, $cnuc(G) = |CVT(G)|$. Further, there is exactly one (up to isomorphism) CNUC graph containing G as an induced subgraph on $|V(G)| + cnuc(G)$ vertices.*

Proof. For $S \in CVT(G)$ let $U(S)$ be the union of sets $U' \subseteq V(G)$ such that $S = N_G[U']$. Then $S = N_G[U(S)]$. (Note: $U(S) = \{v \in S \mid N_G[v] \subseteq S\}$.)

Any CNUC graph H containing G as an induced subgraph must contain vertices $w(S)$, $S \in CVT(G)$, such that $N_H[w(S)] = N_H[U(S)]$. By Lemma 1.1, $N_H[w(S)] \cap V(G) = N_H[U(S)] \cap V(G) = N_G[U(S)] = S$.

Since $S \in CVT(G)$, S is not the closed neighbor set in G of any single vertex in G. If $w(S) \in V(G)$ then we would have $S = N_H[w(S)] \cap V(G) = N_G[w(S)]$ (Lemma 1.1). Therefore $w(S) \notin V(G)$.

Suppose that $S_1, S_2 \in CVT(G)$ and $S_1 \neq S_2$. Since $S_i = N_H[w(S_i)] \cap V(G)$, $i = 1, 2$, it follows that $w(S_1) \neq w(S_2)$.

By the conclusions in the two paragraphs just above, the map $S \to w(S)$ carries $CVT(G)$ injectively into $V(H) \backslash V(G)$. Therefore, since H was an arbitrary CNUC graph containing G as an induced subgraph, it follows that $cnuc(G) \geq |CVT(G)|$.

Now we will see that there is exactly one CNUC graph H on $|V(G)| + |CVT(G)|$ vertices containing G as an induced subgraph; this will finish the proof. For each $S \in CVT(G)$, let $w(S)$ be a "new" vertex, not in $V(G)$. Let $V(H) = V(G) \cup \{w(S) \mid S \in CVT(G)\}$. The adjacencies in H among vertices of G will be those in G, so G

will be an induced subgraph of H. For each $S \in CVT(G)$, $N_H[w(S)] \cap V(G) = S$. It remains to specify the adjacencies among the $w(S)$: if $S_1, S_2 \in CVT(G)$, $S_1 \neq S_2$, then $w(S_1)$ and $w(S_2)$ are adjacent in H if and only if $S_1 \cap U(S_2) \neq \emptyset$. By Lemma 4.1, this is equivalent to $S_2 \cap U(S_1) \neq \emptyset$.

Before showing that H is CNUC, we will prove the uniqueness claim. It is shown above that if H is any CNUC graph containing G as an induced subgraph then there must exist $|CVT(G)|$ vertices $w(S) \in V(H) \backslash V(G)$, $S \in CVT(G)$, satisfying $N_H[w(S)] = N_H[U(S)]$. It is further shown that for each $S \in CVT(G)$, $N_H[w(S)] \cap V(G) = S$.

If $|V(H)| = |V(G)| + |CVT(G)|$, then $V(H)\backslash V(G) = \{w(S) | S \in CVT(G)\}$. Uniqueness of H will be proved if we show that adjacencies among the $w(S)$ must be as defined above: for $S_1, S_2 \in CVT(G)$, $S_1 \neq S_2$, $w(S_1)$ and $w(S_2)$ are adjacent in H if and only if $S_1 \cap U(S_2) \neq \emptyset$, which is equivalent to $S_2 \cap U(S_1) \neq \emptyset$.

Suppose that $S_1, S_2 \in CVT(G)$, and $S_1 \neq S_2$. If $S_1 \cap U(S_2) \neq \emptyset$ then $w(S_1)$ is adjacent to some vertex of $U(S_2)$, because $w(S_1)$ is adjacent to every vertex of S_1. Therefore, $w(S_1) \in N_H[U(S_2)] = N_H[w(S_2)]$, whence $w(S_1)$ and $w(S_2)$ are adjacent in H.

On the other hand, if $S_1 \cap U(S_2) = \emptyset$ then, since $w(S_1)$ is adjacent to no vertex of $V(G)$ outside of S_1, $w(S_1) \notin N_H[U(S_2)] = N_H[w(S_2)]$, so $w(S_1)$ and $w(S_2)$ are not adjacent in H. The uniqueness of H is proved.

Finally, we prove that H is CNUC. Suppose that $u, v \in V(H)$, and $u \neq v$. We aim to show that for some $y \in V(H)$, $N_H[y] = N_H[u] \cup N_H[v]$.

<u>Case 1:</u> $u, v \in V(G)$ and $N_G[u] \cup N_G[v] = N_G[x]$ for some $x \in V(G)$. In this case we will see that $N_H[u] \cup N_H[v] = N_H[x]$.

Suppose that, for some $S \subseteq CVT(G)$, $w(S) \in N_H[u]$. Then $u \in S = N_G[U(S)]$. Therefore some $z \in U(S)$ is adjacent to u or equal to u; in either case, $z \in N_G[u] \subseteq N_G[x]$. Therefore $x \in N_G[z] \subseteq N_G[U(S)] = S$, so x is adjacent to $w(S)$, in H.

This shows that $N_H[u] \cup N_H[v] \subseteq N_H[x]$. Now suppose that $S \in CVT(G)$ and $w(S) \in N_H[x]$. Then $x \in S = N_G[U(S)]$. Consequently, some $z \in U(S)$ is in $N_G[x] = N_G[u] \cup N_G[v]$; say $z \in N_G[u]$. Then $u \in N_G[z] \subseteq N_G[U(S)] = S$ so $w(S) \in N_H[u]$. This shows that $N_H[x] \subseteq N_H[u] \cup N_H[v]$. Therefore, $N_H[x] = N_H[u] \cup N_H[v]$.

<u>Case 2:</u> $u, v \in V(G)$ and $T = N_G[u] \cup N_G[v]$ is not the closed neighborhood set, in G, of a single vertex. Then $T \in CVT(G)$, and $u, v \in U(T)$. We aim to show that $N_H[u] \cup N_H[v] = N_H[w(T)]$. It suffices to show that for $S \in CVT(G)\backslash\{T\}$, $w(S) \in N_H[u] \cup N_H[v]$ if and only if $w(S) \in N_H[w(T)]$.

Suppose, for such S, that $w(S) \in N_H[u]$. Then $u \in S \cap U(T)$, so $w(S)$ and $w(T)$ are adjacent in H. Thus $w(S) \in N_H[w(T)]$. Now suppose that $w(S) \in N_H[w(T)]$. Then $S \cap U(T) \neq \emptyset$, so, by Lemma 4.1, since $T = N_G[\{u, v\}]$, $S \cap \{u, v\} \neq \emptyset$. Therefore, $w(S)$ is adjacent (in H) to one of u, v. Therefore, $w(S) \in N_H[u] \cup N_H[v]$.

<u>Case 3:</u> $u \in V(G)$, $v = w(S)$ for some $S \in CVT(G)$.
<u>Subcase 3.1:</u> $\hat{S} = N_G[u] \cup S = N_G[\{u\} \cup U(S)] \in CVT(G)$. In this subcase we will see that $N_H[u] \cup N_H[v] = N_H[w(\hat{S})]$. Since $N_H[w(\hat{S})] \cap V(G) = \hat{S} = N_G[u] \cup (N_H[w(S)] \cap V(G))$, to show $N_H[w(\hat{S})] \subseteq N_H[u] \cup N_H[w(S)]$ it suffices to show that for $\tilde{S} \in CVT(G)$, if $w(\tilde{S}) \in N_H[w(\hat{S})]$ then $w(\tilde{S}) \in N_H[u] \cup N_H[w(S)]$.

Clearly $w(\hat{S}) \in N_H[u]$. Suppose that $\tilde{S} \in CVT(G)\backslash\{\hat{S}\}$ and $w(\tilde{S})$, $w(\hat{S})$ are adjacent. Then $\tilde{S} \cap U(\hat{S}) \neq \emptyset$. By Lemma 4.1, since $\hat{S} = N_G[\{u\} \cup U(S)]$, $\tilde{S} \cap [\{u\} \cup U(S)] \neq \emptyset$. Therefore $w(\tilde{S})$ is adjacent either to u or to something in $U(S)$; if to u then $w(\tilde{S}) \in N_H[u]$. If $w(\tilde{S})$ is adjacent to a vertex of $U(S)$, then $\tilde{S} \cap U(S) \neq \emptyset$, whence $w(\tilde{S}) \in N_H[w(S)]$.

To prove the inclusion in the other direction, it again suffices to consider vertices $w(\tilde{S})$, $\tilde{S} \in CVT(G)\backslash\{\hat{S}\}$. This time we suppose that $w(\tilde{S}) \in N_H[u] \cup N_H[w(S)]$, and aim to show that $w(\tilde{S}) \in N_H[w(\hat{S})]$.

If $w(\tilde{S}) \in N_H[u]$ then $u \in \tilde{S}$, so $\emptyset \neq \tilde{S} \cap \{u\} \subseteq \tilde{S} \cap U(\hat{S})$, whence $w(\tilde{S}) \in N_H[w(\hat{S})]$. If $w(\tilde{S}) \in N_H[w(S)]$ then either $\tilde{S} = S$, in which case $w(\tilde{S}) = w(S) \in N_H[w(\hat{S})]$ because $U(S) \subseteq U(\hat{S})$ implies $\emptyset \neq U(S) \subseteq S \cap U(\hat{S})$, or $w(\tilde{S})$ and $w(S)$ are adjacent. In this case, $\emptyset \neq \tilde{S} \cap U(S) \subseteq \tilde{S} \cap U(\hat{S})$, so $w(\tilde{S}) \in N_H[w(\hat{S})]$.

<u>Subcase 3.2:</u> $N_G[u] \cup S = N_G[\{u\} \cup U(S)] \notin CVT(G)$. Then $N_G[u] \cup S = N_G[x]$ for some $x \in V(G)$. In this subcase we will see that $N_H[u] \cup N_H[v] = N_H[x]$. Again, since the intersections of the two sides of this equation with $V(G)$ are equal to $N_G[u] \cup S$ (by Lemma 1.1 and the fact that $N_H[w(S)] \cap V(G) = S$), it suffices to consider vertices $w(\tilde{S})$, $\tilde{S} \in CVT(G)$, and to show that $w(\tilde{S}) \in N_H[u] \cup N_H[v]$ if and only if $w(\tilde{S}) \in N_H[x]$.

Suppose that $w(\tilde{S}) \in N_H[u]$. Then $u \in \tilde{S} = N_G[U(\tilde{S})]$ so $u \in N_G[z]$ for some $z \in U(\tilde{S})$. Then $z \in N_G[u] \subseteq N_G[x]$, so $x \in N_G[z] \subseteq N_G[U(\tilde{S})] = \tilde{S}$. Therefore $w(\tilde{S})$ is adjacent to x; thus $w(\tilde{S}) \in N_H[x]$.

Suppose that $w(\tilde{S}) \in N_H[v] = N_H[w(S)]$. Then $\emptyset \neq U(\tilde{S}) \cap S \subseteq U(\tilde{S}) \cap N_G[x]$. Therefore, $x \in N_G[U(\tilde{S})] = \tilde{S}$, so x and $w(\tilde{S})$ are adjacent in H; that is, $w(\tilde{S}) \in N_H[x]$.

Finally (in the proof of this subcase), suppose that $w(\tilde{S}) \in N_H[x]$. Then $x \in \tilde{S} = N_G[U(\tilde{S})]$. Therefore $\tilde{S} \cap \{x\} \neq \emptyset$. Since $N_G[U(\tilde{S})] = \tilde{S}$ and $N_G[x] = N_G[\{u\} \cup U(S)]$, by Lemma 4.1 it follows that $\tilde{S} \cap [\{u\} \cup U(S)] \neq \emptyset$. Therefore, $w(\tilde{S})$ is adjacent to some vertex in $\{u\} \cup U(S)$. Consequently, either $w(\tilde{S}) \in N_H[u]$ or $\tilde{S} \cap U(S) \neq \emptyset$. If the latter, then $w(\tilde{S}) \in N_H[w(S)]$. In any case, $w(\tilde{S}) \in N_H[x]$ implies $w(\tilde{S}) \in N_H[u] \cup N_H[w(S)]$.

<u>Case 4:</u> $u = w(S_1)$, $v = w(S_2)$ for some $S_1, S_2 \in CVT(G)$, $S_1 \neq S_2$. Let $S = S_1 \cup S_2$.
<u>Subcase 4.1:</u> $S \in CVT(G)$. In this case we will see that $N_H[w(S_1)] \cup N_H[w(S_2)] = N_H[w(S)]$. Since the intersection of the sets on each side of this equation with $V(G)$ is $S = S_1 \cup S_2$, it suffices to consider vertices $w(\tilde{S})$, $\tilde{S} \in CVT(G)$.

Suppose that $w(\tilde{S}) \in N_H[w(S_1)]$. Then, whether $w(\tilde{S})$ and $w(S_1)$ are equal or adjacent, $\tilde{S} \cap U(S_1) \neq \emptyset$. Since $U(S_1) \cup U(S_2) \subseteq U(S)$, it follows that $\tilde{S} \cap U(S) \neq \emptyset$ and, therefore, $w(\tilde{S}) \in N_H[w(S)]$.

Now suppose that $w(\tilde{S}) \in N_H[w(S)]$. Then $\tilde{S} \cap U(S) \neq \emptyset$. By Lemma 4.1, since $S = N_G[U(S)] = N_G[U(S_1) \cup U(S_2)]$, and $\tilde{S}$ is a closed neighbor set in G, $\tilde{S} \cap (U(S_1) \cup U(S_2)) \neq \emptyset$. Thus either $\tilde{S} \cap U(S_1) \neq \emptyset$ or $\tilde{S} \cap U(S_2) \neq \emptyset$. Therefore, $w(\tilde{S}) \in N_H[w(S_1)] \cup N_H[w(S_2)]$.

<u>Subcase 4.2:</u> $S \notin CVT(G)$. Then $S = N_G[x]$ for some $x \in V(G)$. In this case we will see that $N_H[w(S_1)] \cup N_H[w(S_2)] = N_H[x]$. As in the previous subcase, it suffices to consider vertices $w(\tilde{S})$, $\tilde{S} \in CVT(G)$.

Suppose that $w(\tilde{S}) \in N_H[w(S_1)]$. Then $\emptyset \neq \tilde{S} \cap U(S_1) \subseteq \tilde{S} \cap (U(S_1) \cup U(S_2))$, so, by Lemma 4.1, since $\tilde{S}$ is a closed neighbor set in G and $N_G[x] = N_G[U(S_1) \cup U(S_2)]$, it follows that $\emptyset \neq \tilde{S} \cap \{x\}$, which implies that $x \in \tilde{S}$. Since $x \in \tilde{S}$, x and $w(\tilde{S})$ are adjacent, whence $w(\tilde{S}) \in N_H[x]$.

Now suppose that $w(\tilde{S}) \in N_H[x]$. Then $x \in \tilde{S} = N_G[U(\tilde{S})]$, so $\tilde{S} \cap \{x\} \neq \emptyset$. Since $\tilde{S}$ is a closed neighbor set in G, and $N_G[x] = N_G[U(S_1) \cup U(S_2)]$, by Lemma 4.1 it follows that $\tilde{S} \cap [U(S_1) \cup U(S_2)] \neq \emptyset$. Therefore, $w(\tilde{S}) \in N_H[w(S_1)] \cup N_H[w(S_2)]$. □

Corollary 4.1. *If G' is obtained from G by splitting a vertex into two closed-neighborhood clones, or by collapsing two closed-neighborhood clones into a single vertex, then $cnuc(G) = cnuc(G')$.*

Proof. It is straightforward to see that there is a one-to-one correspondence between $CVT(G)$ and $CVT(G')$. □

5 *cnuc* numbers of some well known graphs

Theorem 5.1. *Suppose that $r \geq 2$, $n_1, \ldots, n_r$ are positive integers, $I = \{i \mid n_i = 1\}$ and $J = \{j \mid n_j \geq 3\}$. Let $G = K_{n_1,\ldots,n_r}$.*
If $I = \emptyset$ then $cnuc(G) = 1 + \sum_{j\in J}(z(n_j) - 1)$.
If $I \neq \emptyset$ then $cnuc(G) = \sum_{j\in J}(z(n_j) - 1)$.

Proof. Let the parts of G be $V_1, \ldots, V_r$ with $|V_i| = n_i$, $i = 1, \ldots, r$. If $I = \emptyset$, then $V(G) \in CVT(G)$, and $V(G) = N_G[U]$ for all $U \subseteq V(G)$ containing vertices from different parts, and also for $U = V_i$, $i = 1, \ldots, r$.

Therefore, the only other possible sets $S \in CVT(G)$ are $S = N_G[U]$, $U \in V_j$, $j \in J$, $2 \leq |U| \leq n_j - 1$, and, clearly such a set of vertices is in $CVT(G)$. Since, for such a U, $S = N_G[U] = V(G)\backslash(V_j\backslash U)$, $S = N_G[U]$ for only one set U.

Therefore, $cnuc(G) = |CVT(G)|$
$= \sum_{j\in J} |\{U \subseteq V_j \mid 2 \leq |U| \leq n_j - 1\}| + 1 = 1 + \sum_{j\in J}(z(n_j) - 1)$.
When $I \neq \emptyset$ the argument is similar. The difference is that $V(G) \notin CVT(G)$. □

The Fibonacci numbers are, as usual, defined by $f_0 = 0$, $f_1 = 1$, and, for $n > 1$, $f_n = f_{n-1} + f_{n-2}$.

Theorem 5.2. *For $n \geq 3$, $cnuc(P_n) = 2f_n - n - 1$.*

Proof. For $n \geq 3$, the closed neighbor sets of the n single vertices of P_n are distinct, To count $CVT(P_n)$, we shall count all the sets $N[U]$, $U \subseteq V(P_n)$, and then subtract $n+1$, for the cases $U = \emptyset$ and $|U| = 1$. It will turn out that $|\{N[U] \mid U \subseteq V(P_n)\}| = 2f_n$, for $n \geq 3$. For reference, let $a_n = |\{N[U] \mid U \subseteq V(P_n)\}|$.

We replace subsets of $V(P_n)$ by their indicator functions, expressed as binary words. For instance, 11011 stands for the subset of $V(P_5)$ consisting of the 1^{st}, 2^{nd}, 4^{th}, and 5^{th} vertices along the path, scanning left to right. Clearly the binary words representing closed neighbor sets in P_n are composed of blocks of 1's of lengths ≥ 2, separated by blocks of zeros, with length 2 allowed only for 1-blocks at either end of the word. So, for instance, when $n = 5$ the binary words representing closed neighbor sets (including $N[\emptyset]$ and $N[v]$ for single vertices v) are

$$00000, 11000, 11100, 01110, 00111, 00011, 11110, 01111, 11011, \text{ and } 11111.$$

So $a_5 = 10 = 2f_5$

Using the language of regular expressions [4] we can give a *method of formation* for all the binary words in which we are interested:

$$\mathcal{L} = (\lambda \cup 111^*)(00^*1111^*)^*(\lambda \cup 00^*(\lambda \cup 11))$$

Some explanation:

1. λ stands for the empty word;

2. $(w)^*$ means: we can insert here any finite number, including zero, of concatenations of the word w with itself—so, 111^* stands for 11 followed by any finite number of 1's, while $(111)^*$ stands for any block of 1's with length divisible by 3; but $(00^*1111^*)^*$ stands for any finite concatenation of finite words, each of which consists of a non-empty block of 0's followed by a block of 3 or more 1's;

3. $\cup$ means "or";

4. juxtaposed brackets mean that we concatenate any products of the brackets, in the order indicated;

5. $\mathcal{L}$ stands for the set of all words that can be formed by the method indicated on the other side of the equation. Here is an informal description of the formation of one of those words, reading left-to-right.

   - First we might, or might not, have a block of two or more 1's.
   - Then there are concatenated some finite number—zero or more—of words each of which starts with one or more 0's, followed by three or more 1's.
   - Finally, from the last of the three major brackets, we end either with nothing (λ), meaning we were done with whatever came from the second bracket, or with 0^k, $k \geq 1$, or with 0^k11, $k \geq 1$.

Here are the first few shortest words in $\mathcal{L}$, expressed as concatenations of words produced by the 3 main brackets in the generating expression:
$\lambda = \lambda\lambda\lambda$, $0 = \lambda\lambda 0$, $11 = (11)\lambda\lambda$, $00 = \lambda\lambda(00)$, $111 = (111)\lambda\lambda$, $110 = (11)\lambda 0$, $011 = \lambda\lambda(011)$, $000 = \lambda\lambda(000)$, $1111 = (1111)\lambda\lambda$, $1110 = (111)\lambda 0$,
$1100 = (11)\lambda(00)$, $0111 = \lambda(0111)\lambda$, $0011 = \lambda\lambda(0011)$, $0000 = \lambda\lambda(0000)$.
If the reader needs exercise, we recommend verifying that the words of length 5 in $\mathcal{L}$ are exactly the 10 words representing closed neighbor sets in P_5.

A more important exercise: prove to yourself that each word $w \in \mathcal{L}$ is *uniquely* a concatenation, $w = xyz$, in which $x \in (\lambda \cup 111^*)$, $y \in (00^*1111^*)$, and $z \in (\lambda \cup 00^*(\lambda \cup 11))$.

The generating function of $\mathcal{L}$ is the power series $\sum_{n=0}^{\infty} u_n x^n$ (x here is just a variable symbol) in which u_n is the number of words of $\mathcal{L}$ of length n. It is clear that $u_n = a_n$ except, debatably, when $n = 0$, and when $n = 1$, because the word 1 is not in $\mathcal{L}$.

Because of the way power series are multiplied, and because of the uniqueness of the representation of words in $\mathcal{L}$ as concatenations of words from the 3 main brackets in the generating expression for $\mathcal{L}$, the generating function for $\mathcal{L}$ is the product of the generating functions for $\lambda \cup 111^*$, $(00^*1111^*)^*$, and $\lambda \cup 00^*(\lambda \cup 11)$. Thus

$$\begin{aligned}\sum_{n=0}^{\infty} u_n x^n &= \left(1 + \frac{x^2}{1-x}\right) \frac{1}{1 - \frac{x}{1-x}\frac{x^3}{1-x}} \left(1 + \frac{x}{1-x}(1+x^2)\right) \\ &= \frac{(1-x+x^2)(1+x^3)}{1-2x+x^2-x^4} \\ &= 1 - x + \frac{2x}{1-x-x^2}.\end{aligned}$$

Multiplying both sides of the equation above by $1 - x - x^2$ and rearranging, one obtains

$$\begin{aligned}&u_0 + (u_1 - u_0)x + (u_2 - u_1 - u_0)x^2 + (u_3 - u_2 - u_1)x^3 + \sum_{n=4}^{\infty}(u_n - u_{n-1} - u_{n-2})x^n \\ &= 1 + x^3\end{aligned}$$

from which we conclude $u_0 = u_1 = 1$, $u_2 = 2 = 2f_2$, $u_3 = 4 = 2f_3$, and for $n \geq 4$, $u_n = u_{n-1} + u_{n-2}$. Thus $a_n = u_n = 2f_n$ for $n \geq 2$, and the result (for $n \geq 3$) follows. □

Theorem 5.3. *For $n \geq 4$,*

$$cnuc(C_n) = \sum_{i=0}^{\lfloor \frac{n}{4} \rfloor} \frac{2n}{n-2i}\binom{n-2i}{2i} - (n+1).$$

Proof. As in the proof of Theorem 5.2, we will represent subsets of the set of vertices of the graph of interest by their indicator (characteristic) functions, displayed in the shape of the graph. That is, we want to count *cyclic* binary strings of length n in which each maximal block of 1's is of length at least 3, for these are the indicator functions of sets $N[U]$, $U \subseteq V(C_n)$. Having seen that the number of such binary circles is

$$\sum_{i=0}^{\lfloor \frac{n}{4} \rfloor} \frac{2n}{n-2i}\binom{n-2i}{2i},$$

we subtract $n+1$ to get $cnuc(C_n) = |CVT(n)|$ because for $n \geq 4$ the $n+1$ sets $N[\emptyset]$, $N[U]$, $|U| = 1$, are distinct.

Let A_n be the set of binary circular strings (read, say, clockwise, from a designated starting entry), of length n, in which every maximal block of 1's is of length at least

3 — equivalently, these strings contain no blocks 010, nor 0110. Let B_n be the set of binary circular strings of length n containing no "isolated" 0's, and no isolated 1's. That is, the strings in B_n contain no blocks of the form 101, nor 010.

We will see that A_n and B_n are in one-to-one correspondence, and this will finish the proof, because it is shown in [1] and [3] that

$$|B_n| = \sum_{i=0}^{\lfloor \frac{n}{4} \rfloor} \frac{2n}{n-2i} \binom{n-2i}{2i},$$

for $n \geq 3$.

Suppose that $w \in A_n$ and imagine that the bits of w are attached as labels to the vertices $v_1, \ldots, v_n$ of C_n, in the natural order—say, clockwise—around C_n. Form a cyclic binary word $y = f(w)$ by attaching bits to the *edges* $v_1v_2, \ldots, v_nv_1$ of C_n, by the following rule: if both ends of $e \in E(C_n)$ are labeled 1 by w, then label e with 1; otherwise, label e with 0. If $f(w)$ contains 010 then w contains 0110; it is not possible that $f(w)$ contains 101, by the rule of formation of $f(w)$. So f maps A_n into B_n and it is easy to see that f is one-to-one.

If $y \in B_n$, regard y as a labeling of the edges of C_n. Obtain a cyclic binary word w by attaching labels to the vertices of C_n as follows: first, label with 1 both vertices at the ends of any edge labeled 1 by y; now label the rest of the vertices with 0. We leave it to the reader to verify that $w \in A_n$ and $f(w) = y$. $\square$

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(Received 25 Sep 2016; revised 23 Feb 2019, 26 Apr 2020)