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Some constraints on the missing Moore graph

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Abstract

We derive some constraints on the structure of the missing Moore graph.

1 Moore graphs

A **Moore** graph Γ_k is a regular graph of degree k with the property that every pair of adjacent vertices has no common neighbor, and every pair of non-adjacent vertices has precisely one common neighbor. Equivalently, the graph has diameter two and girth five. By simple counting, it follows that the adjacency matrix X_k of a Moore graph is a $v \times v$ matrix with $v = k^2 + 1$ satisfying

$$X_k^2 + X_k = (k-1)I_v + J_v, \tag{1}$$

where I_v (respectively, J_v) is the $v \times v$ identity matrix (respectively, all-ones matrix). Using spectral techniques, Hoffman and Singleton showed [9] that the only possibilities are $k = 2$ (the 5-cycle), $k = 3$ (the Petersen graph), $k = 7$ (the Hoffman-Singleton graph), and $k = 57$ (called the 'missing Moore graph', because no one knows whether it exists or not).[1]

Despite much effort on the problem the existence question remains undecided. Higman (unpublished, see [4], Proposition 5.4 or [3], Proposition 11.5.2), extending earlier work of Aschbacher [1], showed that, if Γ_{57} exists, it cannot be vertex-transitive. Mačaj and Şiráņ [10] and Makhnev and Paduchikh [11] have put limits on the possible size of the automorphism group of Γ_{57}. In this note we exhibit some constraints on the possible structure of the adjacency matrix of Γ_{57}.[2]

[1] For more about Moore graphs, see, e.g., Godsil and Royle [7], Miller and Şiráņ [12], and Dalfó [5].
[2] The Smith form of the Laplacian matrix of Γ_{57} was almost completely determined by Ducey [6].

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2 The Moore graphs for $k \in \{2, 3, 7\}$

Before investigating what can be said about X_{57}, we recall the "canonical constructions" for the adjacency matrices of the other Moore graphs. It turns out that the adjacency matrices of all the known Moore graphs can be built from the cyclic permutation matrix of size five:

$$P := \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

2.1 The 5-cycle

The adjacency matrix of the 5-cycle is just $X_2 := P + P^{-1}$. To see that this satisfies (1), we first note that $P^0 + P^1 + P^2 + P^3 + P^4 = J_5$, where $P^0 = I_5$. Thus,

$$X_2^2 + X_2 = P^2 + P^{-2} + 2I_5 + P + P^{-1} = I_5 + J_5.$$

2.2 The Petersen graph

The adjacency matrix of the Petersen graph is

$$X_3 := \begin{pmatrix} P + P^{-1} & I_5 \\ I_5 & P^2 + P^{-2} \end{pmatrix}.$$

To see this, observe that

$$\begin{pmatrix} P + P^{-1} & I_5 \\ I_5 & P^2 + P^{-2} \end{pmatrix} \begin{pmatrix} P + P^{-1} & I_5 \\ I_5 & P^2 + P^{-2} \end{pmatrix} = \begin{pmatrix} P^2 + P^{-2} + 3I_5 & J_5 - I_5 \\ J_5 - I_5 & P + P^{-1} + 3I_5 \end{pmatrix}.$$

Hence,

$$X_3^2 + X_3 = \begin{pmatrix} J_5 + 2I_5 & J_5 \\ J_5 & J_5 + 2I_5 \end{pmatrix} = 2I_{10} + J_{10},$$

and once again, (1) is satisfied.

2.3 The Hoffman-Singleton graph

Based on unpublished work of Robertson, Berlekamp, van Lint, and Seidel [2] constructed an adjacency matrix for the Hoffman-Singleton graph, as follows. Define

$$B := \begin{pmatrix} P^0 & P^0 & P^0 & P^0 & P^0 \\ P^0 & P^1 & P^2 & P^3 & P^4 \\ P^0 & P^2 & P^4 & P^6 & P^8 \\ P^0 & P^3 & P^6 & P^9 & P^{12} \\ P^0 & P^4 & P^8 & P^{12} & P^{16} \end{pmatrix} = \begin{pmatrix} P^0 & P^0 & P^0 & P^0 & P^0 \\ P^0 & P^1 & P^2 & P^3 & P^4 \\ P^0 & P^2 & P^4 & P^1 & P^3 \\ P^0 & P^3 & P^1 & P^4 & P^2 \\ P^0 & P^4 & P^3 & P^2 & P^1 \end{pmatrix}.$$

Note that

$$BB^T = \begin{pmatrix} 5I_5 & J_5 & J_5 & J_5 & J_5 \\ J_5 & 5I_5 & J_5 & J_5 & J_5 \\ J_5 & J_5 & 5I_5 & J_5 & J_5 \\ J_5 & J_5 & J_5 & 5I_5 & J_5 \\ J_5 & J_5 & J_5 & J_5 & 5I_5 \end{pmatrix} = 5I_{25} + J_{25} - (I_5 \otimes J_5),$$

where $M \otimes N$ is the Kronecker product of M and N. Also, observe that, although B is not symmetric, it is normal (i.e., $BB^T = B^T B$) and it has the property that

$$(I_5 \otimes (P + P^{-1}))B + B(I_5 \otimes (P^2 + P^{-2})) = J_{25} - B.$$

Now define

$$X_7 := \begin{pmatrix} I_5 \otimes (P + P^{-1}) & B \\ B^T & I_5 \otimes (P^2 + P^{-2}) \end{pmatrix}.$$

Then

$$X_7^2 = \begin{pmatrix} I_5 \otimes (P^2 + P^{-2} + 2I_5) + BB^T & J_{25} - B \\ J_{25} - B^T & B^T B + I_5 \otimes (P^1 + P^{-1} + 2I_5) \end{pmatrix}.$$

Hence,

$$X_7^2 + X_7 = \begin{pmatrix} 6I_{25} + J_{25} & J_{25} \\ J_{25} & 6I_{25} + J_{25} \end{pmatrix} = 6I_{50} + J_{50},$$

and once again, (1) holds.

3 The pattern ends

The obvious question is whether or not the adjacency matrix of the missing Moore graph can be written in a similar form (assuming it exists). More precisely, we may ask if X_{57} has the following form:

$$\begin{pmatrix} I_{325} \otimes (P + P^{-1}) & B \\ B^T & I_{325} \otimes (P^2 + P^{-2}) \end{pmatrix}, \tag{2}$$

where B is some 1625×1625 0-1 matrix. In this section we show that the answer is 'no'. Already we can see where things might go wrong. The row (and column) sums of X_{57} must be 57. The block matrices in the upper left and bottom right corners have row sums equal to 2, which means that B must have row sum 55. Each matrix of the form P^k has row sum equal to unity, so we can only have 55 copies of various powers of P as the blocks of the rows of B. These would have to be augmented by 270 5×5 zero matrices. But then one suspects intuitively that there are not enough 1's to satisfy (1). The following result confirms our intuitions.

Before stating our main result, we need some preliminaries. Suppose the adjacency matrix of Γ_k is written in the form

$$X_k = \begin{pmatrix} A & B \\ B^T & D \end{pmatrix}, \tag{3}$$

where A and D are square symmetric matrices of sizes v_1 and v_2, respectively. The matrices A and D may be viewed as the adjacency matrices of induced subgraphs Γ_A and Γ_D, respectively, of Γ_k, and these subgraphs partition the vertices of Γ_k. Hence,

$$v_1 + v_2 = v = k^2 + 1. \tag{4}$$

Denote the row sums of A by $(a_1, \dots, a_{v_1})$, the row sums of B by $(b_1, \dots, b_{v_1})$, the column sums of B by $(c_1, \dots, c_{v_2})$, and the row sums of D by $(d_1, \dots, d_{v_2})$. By the degree constraint,

$$a_i + b_i = k = c_j + d_j, \quad (1 \le i \le v_1, 1 \le j \le v_2). \tag{5}$$

Note that the row sums of A and D are just the degrees of the vertices of the corresponding induced subgraphs Γ_A and Γ_D. If Γ_A is regular of degree α and Γ_D is regular of degree δ, then we say that $\{A, D\}$ is a **biregular bipartition** of Γ_k of **bidegree** (α, δ).

Theorem 3.1. *If $k = 57$, then the only possible biregular bipartition of Γ_k has equal size parts and bidegree $(32, 32)$.*

Corollary 3.1. *The adjacency matrix of Γ_{57} cannot be of the form (2) (which corresponds to a biregular bipartition of bidegree $(2, 2)$).*

Proof. Suppose that the adjacency matrix of a Moore graph Γ_k is written in the block form (3). By the Moore graph condition (1), we have

$$A^2 + A + BB^T = (k-1)I_{v_1} + J_{v_1} \tag{6}$$

$$AB + BD + B = J_{v_1,v_2}, \tag{7}$$

$$B^T B + D^2 + D = (k-1)I_{v_2} + J_{v_2}, \tag{8}$$

where $J_{m,n}$ is the $m \times n$ all-ones matrix.

For any matrix M, set

$$|M| := \sum_{i,j} M_{ij}.$$

Then (6), (7), and (8) imply

$$|A^2| + |A| + |BB^T| = (k-1)v_1 + v_1^2 \tag{9}$$

$$|AB| + |BD| + |B| = v_1 v_2 \tag{10}$$

$$|B^T B| + |D^2| + |D| = (k-1)v_2 + v_2^2. \tag{11}$$

Suppose $\{A, D\}$ is a biregular bipartition of Γ_k of bidegree (α, δ). Then, for $0 \leq i \leq v_1$ and $0 \leq j \leq v_2$, we have $a_i = \alpha$, $b_i = k - \alpha$, $c_j = k - \delta$, and $d_j = \delta$. Hence, (9) implies

$$\alpha^2 + \alpha + (k - \delta)^2 = (k - 1) + v_1.$$

But $|B| = |B^T|$, so

$$(k - \alpha)v_1 = (k - \delta)v_2, \tag{12}$$

whence we obtain

$$\alpha^2 + \alpha + (k - \alpha)^2 \left(\frac{v_1}{v_2}\right)^2 = (k - 1) + v_1.$$

Substituting $k = 57$ into this equation and expanding (using (4)) gives

$$\begin{aligned} {v_1}^3 - \left(2\,\alpha^2 - 113\,\alpha + 9693\right) {v_1}^2 \\ + \left(6500\,\alpha^2 + 6500\,\alpha + 10198500\right) v_1 \\ - 10562500\,(\alpha + 8)\,(\alpha - 7) = 0. \end{aligned}$$

A computer check for $1 \leq \alpha \leq 56$ reveals that there is only one nontrivial integral solution for v_1, namely $v_1 = 1625$, corresponding to $\alpha = \delta = 32$. □

Observe that, if B were normal, (6) and (8) would imply $A^2 + A = D^2 + D$. Even without the assumption of normality, we may still deduce a relationship between $A^2 + A$ and $D^2 + D$, as long as Γ_{57} admits a biregular bipartition. In what follows, $[M, N] := MN - NM$ denotes the commutator of M and N.

Theorem 3.2. *Suppose $\{A, D\}$ is a biregular bipartition of Γ_{57}. Then $A^2 + A$ and $D^2 + D$ are cospectral.*

Proof. First, we observe that A, D, BB^T, and B^TB are all real symmetric matrices, hence diagonalizable by orthogonal transformations. Moreover, as B is square, BB^T and B^TB are cospectral (see, e.g., [13], Section 2.5).

Let j be the all-ones vector of size 1625, and let

$$U := \{u \in \mathbb{R}^{1625} : (u, j) = 0\}$$

be the orthogonal complement of the subspace spanned by j. (Here $(\cdot, \cdot)$ denotes the ordinary Euclidean inner product.) By Theorem 3.1, we have $Aj = Dj = \alpha j$ (where $\alpha = 32$). Hence, $A^2 + A$ and $D^2 + D$ both share $\alpha^2 + \alpha$ as an eigenvalue. As A and D are symmetric, their remaining eigenvectors can be taken to lie in U.

Define $M := (k - 1)I_{1625} + J_{1625}$. As A and D have constant row sums, $[A, J] = [D, J] = 0$, and therefore $[A^2 + A, J] = [D^2 + D, J] = 0$. Hence, $[A^2 + A, M] = [D^2 + D, M] = 0$. From (6) and (8) we may conclude that $[A^2 + A, BB^T] = [D^2 + D, B^TB] = 0$. It follows that $A^2 + A$ and BB^T are simultaneously diagonalizable, as are, respectively, $D^2 + D$ and B^TB.

We have

$$(\alpha^2 + \alpha)j = (A^2 + A)j = Mj - BB^T j = (k + 1624)j - BB^T j.$$

If $Au = \mu u$ with $u \in U$, then

$$(\mu^2 + \mu)u = (A^2 + A)u = Mu - BB^T u = (k - 1)u - BB^T u,$$

whence we conclude (plugging in $k = 57$ and $\alpha = 32$) that

$$\operatorname{spec}(BB^T) = \{625, 56 - \mu^2 - \mu\},$$

where μ runs over the eigenvalues of A associated to the eigenspaces orthogonal to j. A similar argument shows that

$$\operatorname{spec}(B^T B) = \{625, 56 - \nu^2 - \nu\},$$

where ν runs over the eigenvalues of D associated to the eigenspaces orthogonal to j. The theorem now follows. □

Remark. The equation $\mu(\mu + 1) = \nu(\nu + 1)$ has two solutions, namely, $\mu = \nu$ and $\mu = -(\nu + 1)$, so, although it could be true, we cannot conclude from Theorem 3.2 that A and D are cospectral.

Theorem 3.2 relates the eigenvalues of $A^2 + A$ and $D^2 + D$. The next theorem allows us to relate some of their eigenvectors.

Theorem 3.3. *Suppose $\{A, D\}$ is a biregular bipartition of Γ_{57}. Then*

$$(A^2 + A)B = B(D^2 + D). \tag{13}$$

Proof. By Theorem 3.1, $JD = AJ = \alpha J$. Now multiply (7) on the left by A and on the right by D and subtract the two resulting equations. □

Suppose $Du = \nu u$. Then (13) implies

$$(A^2 + A)Bu = (\nu^2 + \nu)Bu.$$

Hence, if $u \notin \ker B$, then Bu is an eigenvector of $A^2 + A$ with eigenvalue $\nu^2 + \nu$. Similarly, if $Au = \mu u$ and $u \notin \ker B^T$, then the transpose of (13) implies that $B^T u$ is an eigenvector of $D^2 + D$ with eigenvalue $\mu^2 + \mu$. (The kernel of B is nontrivial; see Theorem 4.1 below.)

4 More constraints

In this section we see what can be said about general bipartitions $\{A, D\}$ of Γ_{57} into equal size parts (i.e., $v_1 = v_2 = 1625$), with or without the assumption of biregularity. The primary tool used in this section is interlacing (see, e.g., [3], Section 2.5 or [7], Chapter 9). For brevity, in what follows we write X for X_{57} and J for J_{1625}.

4.1 Rank constraints

Let $\mathrm{rk}(M)$ denote the rank of M.

Theorem 4.1. *Suppose that X is written in the form* (3) *with $v_1 = v_2 = 1625$. Then*

$$\mathrm{rk}(B) \leq 1522,$$

and $\mathrm{rk}(A) = \mathrm{rk}(D) = 1625$*. In particular, B is singular, while A and D are both invertible.*

Proof. By ([3], Theorem 2.5.1 or [7], Theorem 9.1.1), the eigenvalues of A and D interlace those of X. The spectrum of X is well-known (e.g., ([7], Section 10.2) or [9]) to be $(57, 7^{1729}, (-8)^{1520})$ (where the exponents denote the multiplicities). Let the spectrum of A be denoted $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{1625}$. In this case, the interlacing inequalities read

$$\lambda_i \geq \mu_i \geq \lambda_{1625+i}, \qquad 1 \leq i \leq 1625.$$

Hence,

$$57 \geq \mu_1 \geq 7, \quad \mu_2 = \cdots = \mu_{105} = 7, \quad \text{and} \quad 7 \geq \mu_j \geq -8 \quad (j \geq 106), \tag{14}$$

and similarly for the eigenvalues of D.

Let E denote the eigenspace of A corresponding to $\mu = 7$. Then

$$\dim E \geq 104.$$

Let U be the orthogonal complement of the all-ones vector j in $\mathbb{R}^{1625}$. By the modular law of subspaces,

$$\dim(E \cap U) = \dim E + \dim U - \dim(E + U) \geq 103.$$

Let $u \in E \cap U$ be normalized to unity. Then

$$Au = 7u, \quad (u, u) = 1, \quad \text{and} \quad (u, j) = 0.$$

From (6), we get

$$(u, A^2 u) + (u, Au) + (u, BB^T u) = k - 1,$$

which gives

$$49 + 7 + (u, BB^T u) = 56 \quad \Rightarrow \quad |B^T u|^2 = 0 \quad \Rightarrow \quad B^T u = 0.$$

It follows that $\dim \ker B^T \geq 103$, so $\mathrm{rk}(B) = \mathrm{rk}(B^T) \leq 1522$.

Now, X has no zero eigenvalues and so is full rank. In particular, for every nonzero vector $w \in \mathbb{R}^{3250}$, $Xw \neq 0$. Write $w = \begin{pmatrix} x \\ y \end{pmatrix}$, where $x, y \in \mathbb{R}^{1625}$. Then, if $x \neq 0$ or $y \neq 0$, we must have

$$Ax + By \neq 0.$$

Choose $y \in \ker B$. Then $Ax \neq 0$ for every $x \in \mathbb{R}^{1625}$, which shows that A is full rank. (A similar argument shows that D is full rank.) ∎

Corollary 4.1. *Suppose that* $\{A, D\}$ *is a biregular bipartition of* Γ_{57}. *Then* Γ_A *and* Γ_D *are connected subgraphs.*

Proof. By Theorem 3.1, Γ_A and Γ_D are regular of degree 32. By (14), the largest eigenvalue of A (and D) is 32 and the second largest eigenvalue of A (and D) is 7. Now apply Proposition 1.3.8 in [3]. $\square$

4.2 Inertia and spectral constraints

As A and D are invertible, their Schur complements exist:

$$X/A := D - B^T A^{-1} B \quad \text{and} \quad X/D := A - BD^{-1}B^T.$$

We have

$$\det X = \det A \cdot \det(X/A) = \det D \cdot \det(X/D).$$

As A and D are integer matrices, their determinants are nonzero integers. This implies that their Schur complements have nonzero rational determinants. In particular, X/A and X/D are both full rank.

For any matrix M, define the ordered triple

$$\text{Inert}(M) = (n_+(M), n_-(M), n_0(M)),$$

where $n_+(M)$, $n_-(M)$, and $n_0(M)$ are the numbers of positive, negative, and zero eigenvalues of M, respectively. By the Haynsworth inertia additivity formula [8],

$$\text{Inert}(X) = \text{Inert}(A) + \text{Inert}(X/A) = \text{Inert}(D) + \text{Inert}(X/D).$$

Theorem 4.2. *Suppose that* $\{A, D\}$ *is a biregular bipartition of* Γ_{57}. *Then*

$$[D, B^T A^{-1} B] = [A, BD^{-1}B^T] = 0. \tag{15}$$

Proof. If $\{A, D\}$ is biregular then $v_1 = v_2$, so by (12), $AJ = JA = JD = DJ = \alpha J$. Also, if $\beta := k - \alpha$ then $BJ = JB = B^T J = JB^T = \beta J$. (By Theorem 3.1, $\beta = 25$.) Multiplying (7) on the left by $B^T A^{-1}$ gives

$$B^T B + B^T A^{-1} BD + B^T A^{-1} B = \alpha^{-1} \beta J.$$

Similarly, multiplying the transpose of (7) on the right by $A^{-1}B$ gives

$$B^T B + DB^T A^{-1} B + B^T A^{-1} B = \alpha^{-1} \beta J.$$

Subtracting these two equations yields the first equation in (15). The other equation in (15) follows similarly. $\square$

Corollary 4.2. *Suppose that* $\{A, D\}$ *is a biregular bipartition of* Γ_{57}. *Then*

$$\text{spec}\,(X/A) = \text{spec}\,(D) - \text{spec}\,(B^T A^{-1} B)$$
$$\text{spec}\,(X/D) = \text{spec}\,(A) - \text{spec}\,(BD^{-1}B^T),$$

where the notation means that the eigenvalues of the matrices on the left are differences of the eigenvalues of the matrices on the right.

4.3 Average degree constraints

For any vector $(x_1, \ldots, x_{v/2})$, define

$$\langle x \rangle = \frac{1}{v/2} \sum_{i=1}^{v/2} x_i.$$

Then, as $|B| = |B^T|$, we have

$$\langle b \rangle = \langle c \rangle, \tag{16}$$

which implies

$$\langle a \rangle = \langle d \rangle. \tag{17}$$

That is, the average degree of Γ_A must equal the average degree of Γ_D.

Theorem 4.3. *Suppose X is written in the form* (3) *with $v_1 = v_2 = 1625$. Then*

$$24.5 \le \langle a \rangle \le 32. \tag{18}$$

Similarly, $25 \le \langle b \rangle \le 32.5$, $25 \le \langle c \rangle \le 32.5$, *and* $24.5 \le \langle d \rangle \le 32$.

Proof. Define

$$Y := \begin{pmatrix} \langle a \rangle & \langle b \rangle \\ \langle c \rangle & \langle d \rangle \end{pmatrix} = \begin{pmatrix} \langle a \rangle & \langle b \rangle \\ \langle b \rangle & \langle a \rangle \end{pmatrix}.$$

By ([3], Corollary 2.5.4), the eigenvalues of Y must interlace the eigenvalues of X. The eigenvalues of Y are easily seen to be $\langle a \rangle + \langle b \rangle = k$ and $\langle a \rangle - \langle b \rangle = 2\langle a \rangle - k$. Define

$$y_1 := \max\{k, 2\langle a \rangle - k\} \quad \text{and} \quad y_2 := \min\{k, 2\langle a \rangle - k\}.$$

Then the interlacing inequalities ($\lambda_i \ge y_i \ge \lambda_{v-2+i}$) give

$$57 \ge y_1 \ge -8 \quad \text{and} \quad 7 \ge y_2 \ge -8.$$

This forces $y_1 = 57$ and

$$7 \ge 2\langle a \rangle - 57 \ge -8,$$

from which (18) follows. The other claims follow from Equations (16) and (17). □

We conclude that the average degrees of Γ_A and Γ_D are both approximately half of the degree of Γ. Observe that this is consistent with Theorem 3.1.

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References

[1] M. Aschbacher, The non-existence of rank three permutation groups of degree 3250 and subdegree 57, *J. Algebra* **19** (1971), 538–540.

[2] E. R. Berlekamp, J. H. van Lint and J. J. Seidel, A strongly regular graph derived from the perfect ternary Golay code, in *A Survey of Combinatorial Theory* (J. N. Srivastava et al., Eds.), North Holland, Amsterdam, 1973, pp. 25–30.

[3] A. E. Brouwer and W. H. Haemers, *Spectra of Graphs*, Springer, New York, 2012.

[4] P. J. Cameron, Automorphism groups of graphs, in *Selected Topics in Graph Theory 2* (L. W. Beineke and iR. J. Wilson, Eds.), Academic Press, 1983, 89–128.

[5] C. Dalfó, A survey on the missing Moore graph, *Lin. Alg. Appl.* **569** (2019), 1–14.

[6] J. E. Ducey, On the critical group of the missing Moore graph, *Discrete Math.* **340(5)** (2017), 1104–1109.

[7] C. D. Godsil and G. Royle, *Algebraic Graph Theory*, Springer, New York, 2001.

[8] E. V. Haynsworth, Determination of the inertia of a partitioned Hermitian matrix, *Lin. Alg. Appl.* **1** (1968), 73–81.

[9] A. J. Hoffman and R. R. Singleton, On Moore graphs with diameters 2 and 3, *IBM J. Res. Develop.* **4** (1960), 497–504.

[10] M. Mačaj and J. Şiráņ, Search for properties of the missing Moore graph, *Lin. Alg. Appl.* **432** (2010), 2381–2398.

[11] A. A. Makhnev and D. V. Paduchikh, On the automorphism group of the Aschbacher graph, *Dokl. Math.* **79(3)** (2009), 365–368.

[12] M. Miller and J. Şiráņ, Moore graphs and beyond: a survey of the degree/diameter problem, *Electron. J. Combin.* **20(2)** (2013), #DS14v2.

[13] F. Zhang, *Matrix Theory*, 2nd ed., Springer, New York, 2011.

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