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Sequences in dihedral groups with distinct partial products

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Abstract

Given a subset S of the non-identity elements of the dihedral group of order $2m$, is it possible to order the elements of S so that the partial products are distinct? This is equivalent to the sequenceability of the group when $|S| = 2m - 1$ and so it is known that the answer is yes in this case if and only if $m > 4$. We show that the answer is yes when $|S| \leq 9$ and m is an odd prime other than 3, when $|S| = 2m - 2$ and m is even or prime, and when $|S| = 2m - 2$ for many instances of the problem when m is odd and composite. We also consider the problem in the more general setting of arbitrary non-abelian groups and discuss connections between this work and the concept of strong sequenceability.

1 Introduction

Let G be a multiplicatively-written group with identity element e and let $\mathbf{g} = (g_1, g_2, \ldots g_k)$ be a sequence of elements of $G \setminus \{e\}$. Define the *partial product sequence* of $\mathbf{g}$ to be $\mathbf{h} = (h_0, h_1, h_2, \ldots, h_k)$ where $h_0 = e$ and $h_i = g_1 g_2 \cdots g_i$ for $1 \leq i \leq k$.

The following conjecture, which generalises an earlier one of Alspach for cyclic groups, is investigated in [11]:

Conjecture 1.1. *Let G be an abelian group and let $S \subseteq G \setminus \{e\}$ such that the product of all of the elements in S is not the identity. Then there exists an ordering of the elements of S such that the elements in its partial product sequence are distinct.*

Conjecture 1.1 is known to be true in the following cases:

- when $|S| \leq 9$ [6, 7, 9],
- when $|S| \leq 10$ and G is cyclic of prime order [18],

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- when $|S| = |G| - 1$ [16],
- when $|S| = |G| - 2$ and G is cyclic [9],
- when $|S| = |G| - 3$ and G is cyclic of prime order [18],
- when $|G| \leq 21$ [9, 11],
- when $|G| \leq 25$ and G is cyclic [7].

In [11] the question of dropping the requirement that G be abelian is raised and quickly rejected upon consideration of a counterexample. The question of which subsets of which non-abelian groups do satisfy the conditions remains, and that is the question we study here.

Question 1. *Let G be a finite group and let S be a subset of $G \setminus \{e\}$ of size k such that there is some ordering of the elements of S whose product is not the identity. Is there an ordering of the elements of S such that the elements in its partial product sequence are distinct?*

In order to begin to address this, we introduce some notation and terminology.

Let G be a group of order n. As in the first paragraph, let $\mathbf{g} = (g_1, g_2, \ldots, g_k)$ be an arrangement of elements of $G \setminus \{e\}$ with partial product sequence $\mathbf{h} = (h_0, h_1, h_2, \ldots, h_k)$. Suppose the elements of $\mathbf{g}$ are distinct and let $S = \{g_1, g_2, \ldots, g_k\}$.

If the elements of $\mathbf{h}$ are all distinct then $\mathbf{h}$ is a *basic directed S-terrace* for G and $\mathbf{g}$ is the associated *S-sequencing* of G. In the case $k = n - 1$ (and so $S = G \setminus \{e\}$), $\mathbf{h}$ is a *basic directed terrace* for G and $\mathbf{g}$ is the associated *sequencing* of G. A group with a sequencing is called *sequenceable*. When G has an S-sequencing, we also say that S is *sequenceable*.

The study of sequencings in non-abelian groups originated in [16] and is surveyed in [24]. Note that it is always possible to order the non-identity elements of a non-abelian group to give a non-identity product.

The three non-abelian groups of orders 6 and 8 are not sequenceable [16], hence the answer to Question 1 in the cases when S contains all of the non-identity elements of such a group is no [11]. *Keedwell's Conjecture* is that all other non-abelian groups are sequenceable; that is, that the answer to Question 1 is yes when S contains all of the non-identity elements of a non-abelian group of order at least 10.

In the next section, when cataloguing possible structures of sets for small k, we see more instances of sets S for which the answer to Question 1 is no. In Section 3 we use the Non-Vanishing Corollary to Alon's Combinatorial Nullenstatz to show that the answer is always yes in dihedral groups of order $2m$ when $m > 3$ is prime and $|S| \leq 9$. In Sections 4 and 5 we show that the answer is yes in dihedral groups of order $2m$ for $k \geq 2m - 2$ when $m > 4$ is even or prime, and for many instances of the question for composite m.

Alspach and Kalinowski, see [6], have asked a closely related question regarding "strong sequenceability" of groups. The main difference to Question 1 is that the product of all the elements of a successful ordering is permitted to be the identity (in abelian groups one has no control over this value). We give the necessary definitions and consider the implications of our work for the strong sequenceability question in Section 6.

2 Small k, general groups

For related conjectures that are more limited in their claims or restricted to abelian groups a case-based approach has been used prove them for small values of k, including Conjecture 1.1 for $k \leq 9$ [6, 7, 11]. In this section we start this process for Question 1.

Theorem 2.1. *The answer to Question 1 is yes for $k \leq 4$, with the following two exceptions:*

- $|S| = 4$, *with* $S = \{x, x^{-1}, y, z\}$ *and* $xyz = x^{-1}zy = xzx^{-1}y = e$,
- $|S| = 4$, *with* $S = \{w, x, y, z\}$ *and* $wxy = wyz = wzx = xzy = e$.

Proof. We consider potential S-sequencings and break into cases depending on what might cause a given sequence not to be an S-sequencing. Those causes will be subsequences of elements whose product is the identity. An added wrinkle compared to the abelian case is that it is not immediately obvious when a set of words which are the identity imply a contradiction. We use the group theory software package GAP [15] to determine when a set of such words implies that either the subgroup generated by the elements in question is trivial (which will imply a contradiction) or abelian (in which case the problem is reduced to one already solved in [7]). We frequently use that if $sx = e$ for some string of group elements s then also $xs = e$.

When $k \in \{1, 2\}$ the result is immediate. For $k = 3$ let $S = \{g_1, g_2, g_3\} \subseteq G \setminus \{e\}$ and suppose that $g_1g_2g_3 \neq e$. If (g_1, g_2, g_3) is not an S-sequencing then it must be that either $g_2 = g_1^{-1}$ or $g_2 = g_3^{-1}$. In the former case (g_1, g_3, g_2) is an S-sequencing, in the latter case (g_2, g_1, g_3) is.

For the $k = 4$ case, we consider S of three forms: $\{x, x^{-1}, y, y^{-1}\}$, $\{x, x^{-1}, y, z\}$ and $\{w, x, y, z\}$ where any elements that are inverses are so indicated.

If S is of the first form then without loss of generality a sequence whose product is not the identity is (x, y, x^{-1}, y^{-1}). This is an S-sequencing as the product of none of the two- or three-element subsequences is the identity.

Next consider S of the form $\{x, x^{-1}, y, z\}$. Conjugates of y or z by x cannot be the identity. There must be an ordering of the form (x, y, z, x^{-1}) whose product is not the identity. If this is not an S-sequencing, then it must be the case that either $xyz = e$ or $yzx^{-1} = e$; without loss of generality assume $xyz = e$. Now consider the ordering (x, y, x^{-1}, z). The product of all four cannot be the identity as this implies $x^{-1} = e$. The only way that this cannot be an S-sequencing is if $yx^{-1}z = e$. Now consider (x, z, x^{-1}, y). If $zx^{-1}y = e$ then y and z commute, implying $x = x^{-1}$. So for this not to be an S-sequencing it must be that $xzx^{-1}y = e$, which gives the first exception in the statement of the theorem.

Finally, consider $S = \{w, x, y, z\}$. There must be at least one sequence of all 4 elements whose product is not e, say (w, x, y, z). If this sequence is not an S-sequencing, a subsequence of three elements must have product e (as there are no mutually inverse elements). Assume without loss of generality that $wxy = e$. If (y, z, x, w) is not an S-sequencing then $yzx = e$, $zxw = e$ or $yzxw = e$. The first case implies $zxy = e$ and hence $z = w$ and so can be discounted. In the second case, consider (x, w, y, z). If $xwy = e$ then $y = z$ and if $xwyz = e$ then $y = e$, so to fail to be an S-sequencing we must have $wyz = e$. In this instance either (w, x, z, y) is a successful S-sequencing or $xzy = e$, as $wxz = e$ implies $y = z$ and $wxzy = e$ implies $z = e$. Considering $xzy = e$ leads to the second exception in the statement of the theorem. In the third case, consider (z, w, y, x). If $zwy = e$ then $x = e$ and if $wyx = e$ then $z = e$, so (z, w, y, x) fails to be an S-sequencing only if $zwyx = e$. In this instance (w, z, x, y) is an S-sequencing as $wzx = e$ implies $y = e$, $zxy = e$ implies $w = e$, and $wzxy = e$ implies $z = e$. □

Both exceptions in Theorem 2.1 are necessary. To see this for the first, let

$$D_{2m} = \langle u, v : u^m = e = v^2, vu = u^{m-1}v\rangle$$

be the dihedral group of order $2m$. Consider $S = \{u, u^2, v, u^2v\} \subseteq D_6$. It is straightforward to check there is no S-sequencing.

For the second, let $\mathrm{SL}(2, 3)$ be the special linear group of 2×2 matrices with determinant 1 over the field with three elements. Then there is no S-sequencing for

$$S = \left\{ \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \right\}$$

in $\mathrm{SL}(2, 3)$.

These two exceptions, and the three with sizes 5 and 7 that come from the non-sequenceable groups of orders 6 and 8, are not the only instances of sets S that are not sequenceable. For example, in D_8 the set

$$S = \{u^2, v, uv, u^2v, u^3v\}$$

of size 5 does not have an S-sequencing.

As a further example, let Q_8 be the quaternion group of order 8 and let z be its unique involution. Then $S = Q_8 \setminus \{e, z\}$ is a set of size 6 with no sequencing.

3 Small k, dihedral groups

Before moving to the main method of the section, we give a general construction that works in all dihedral groups and obviates the need for the most computationally intensive case in Theorem 3.4, the main result of this section.

Let $C_m = \langle u \rangle$, a normal cyclic subgroup of D_{2m} of order m, and let $C_m v$ be its coset.

Lemma 3.1. *If $S \subseteq C_m v \subseteq D_{2m}$, then D_{2m} has an S-sequencing.*

Proof. Let $S = \{u^{a_i} v : 1 \leq i \leq k\}$ with $a_1 > \cdots > a_k$. Then the sequence $(u^{a_1}v, \ldots, u^{a_k}v)$ is an S-sequencing with partial products

$$(e, u^{a_1}v, u^{a_1-a_2}, u^{a_1-a_2+a_3}v, u^{a_1-a_2+a_3-a_4}, \ldots).$$

The ordering of the a_i guarantees that these elements are distinct. □

We now pursue a new approach that applies specifically to dihedral groups of order twice an odd prime. It uses Alon's Non-Vanishing Corollary and we are able answer Question 1 for these groups up to $k = 9$. Perhaps more importantly, it embodies an approach that could plausibly be extended to all values of k in these groups. This generalises a method developed for cyclic groups in [18].

The Non-Vanishing Corollary was introduced in [3]; for a short direct proof see [23].

Theorem 3.2. (Non-Vanishing Corollary) *Let F be an arbitrary field, and let $f = f(x_1, x_2, \ldots, x_k)$ be a polynomial in $F[x_1, x_2, \ldots, x_k]$. Suppose the degree deg(f) of f is $\sum_{i=1}^k \gamma_i$, where each γ_i is a nonnegative integer, and suppose the coefficient of $\prod_{i=1}^k x_i^{\gamma_i}$ in f is nonzero. Then if $A_1, A_2, \ldots, A_k$ are subsets of F with $|A_i| > \gamma_i$, there are $a_1 \in A_1, \ldots, a_k \in A_k$ so that $f(a_1, a_2, \ldots, a_k) \neq 0$.*

For m prime, we will take F to be $\mathbb{Z}_m$, the integers modulo m considered as a field.

For a given S, we need a polynomial that is nonzero exactly when we have a sequencing of S. Suppose our set $S \subseteq D_{2m} \setminus \{e\}$ has r elements in C_m and s elements in $C_m v$. Then the variables of the polynomial will be x_i, for $1 \leq i \leq r$, and y_j, for $1 \leq j \leq s$. The x_i correspond to elements of C_m in the sense that, in the notation of the Non-Vanishing Corollary, we take $A_i = \{x : u^x \in S\}$. Similarly, the y_j correspond to the elements of $C_m v$, using $A_j = \{y : u^y v \in S\}$. If we can find a monomial such that the exponent on each x_i is less than $|A_i|$ and the exponent on each y_j is less than $|A_j|$, then the Non-Vanishing Corollary will give a positive answer to that instance of Question 1.

We look for a solution of a particular form, which varies slightly with the parity of s. Before considering the general case, we look at a small example.

Example 3.3. Consider the case when $|S| = 5$ with three elements in $C_m \setminus \{e\}$ and two in $C_m v$. There are various ways in which we might arrange such elements to look for a sequence with distinct partial products.

One potential form for a successful sequence is

$$(u^{x_1}, u^{y_1}v, u^{x_2}, u^{x_3}, u^{y_2}v),$$

which has partial products

$$(e, u^{x_1}, u^{x_1+y_1}v, u^{x_1+y_1-x_2}v, u^{x_1+y_1-x_2-x_3}v, u^{x_1+y_1-x_2-x_3-y_2}).$$

We want to build a polynomial that is 0 exactly when the sequence fails to be an S-sequencing. The elements in the sequence itself must be distinct, which is true exactly when

$$(x_2 - x_1)(x_3 - x_1)(x_3 - x_2)(y_2 - y_1) \neq 0.$$

The elements in the partial products must be distinct, which is true exactly when

$$(x_1 + y_1 - x_2 - x_3 - y_2)(y_1 - x_2 - x_3 - y_2)(-x_2 - x_3) \neq 0.$$

Note that here we have made use of the fact that $e \notin S$. This implies that adjacent elements of the partial products cannot be equal in any ordering of the elements of S, and so we do not need to include factors in the polynomial to check for this.

Combining these we get the polynomial

$$(x_2-x_1)(x_3-x_1)(x_3-x_2)(y_2-y_1)(x_1+y_1-x_2-x_3-y_2)(y_1-x_2-x_3-y_2)(-x_2-x_3)$$

(which will be called $\pi_{3,2}$ in the general method described in the remainder of this section). This polynomial is nonzero if and only if our sequence has the elements it must have and has distinct partial products.

To apply the Non-Vanishing Corollary we need a monomial that divides $x_1^2x_2^2x_3^2y_1y_2$ with a non-zero coefficient. The polynomial has degree 7, so this is plausible to ask for. Indeed, there is one: $x_1^2x_2x_3^2y_1y_2$, which has coefficient 6.

Hence for prime $m > 3$ (to which 6 is coprime) every subset S of size 5 of D_{2m} that has three elements in $C_m \setminus \{e\}$ and two in C_mv has an S-sequencing.

For the general case, we need to arrange the r elements of C_m and s elements of C_mv. There are many potentially-successful ways to do this. For a given form, we can get some crude information about how likely it is that it is possible to assign the elements so that the partial products are distinct based on our knowledge of which coset each of the partial products is in.

Looking at that list of partial products, a hard rule is that we cannot have more than m of the products in C_m or in C_mv. Two softer rules that guided the choice are to a) try to make the number of elements in the partial product in each coset roughly equal to each other (which tends to lower the degree of the polynomial under consideration compared to other options), and to b) mimic patterns that have been successful in finding sequencings for dihedral groups (that is, the case $r = m - 1$ and $s = m$).

We first consider the case when s is odd. Let $r = 2p + \delta$, for $\delta \in \{0, 1\}$, and $s = 2q + 1$. In this case use the form

$$(u^{x_1}, u^{x_2}, \ldots, u^{x_p}, u^{y_1}v, u^{y_2}v, \ldots, u^{y_s}v, u^{x_{p+1}}, u^{x_{p+2}}, \ldots, u^{x_r}).$$

The sequence of partial products is

$$(u^{z_0}, u^{z_1}, \dots, u^{z_p}, u^{t_1}v, u^{z_{p+1}}, u^{t_2}v, \dots, u^{z_{p+q}}, u^{t_{q+1}}v, u^{t_{q+2}}v, \dots, u^{t_{q+p+\delta}}v).$$

Each z_i and t_i is a linear combination of the x_i and y_i with all coefficients ± 1. In particular,

$$z_i = \begin{cases} 0 & \text{if } i = 0 \\ x_1 + x_2 + \cdots + x_i & \text{if } 1 \le i \le p \\ z_p + y_1 - y_2 + y_3 - y_4 + \cdots + y_{2(i-p)-1} - y_{2(i-p)} & \text{if } p+1 \le i \le p+q \end{cases}$$

and

$$t_i = \begin{cases} z_p + y_1 - y_2 + y_3 - y_4 + y_5 + \cdots - y_{2(i-1)} + y_{2i-1} & \text{if } i \le q+1 \\ t_{q+1} - x_{p+1} - x_{p+2} - \cdots - x_{i-q+p-1} & \text{if } q+2 \le i \le p+q+1+\delta. \end{cases}$$

The polynomial

$$\prod_{1 \le i < j \le 2p+\delta} (x_j - x_i) \prod_{1 \le i < j \le 2q+1} (y_j - y_i) \prod_{0 \le i < j \le p+q} (z_j - z_i) \prod_{1 \le i < j \le p+q+1+\delta} (t_j - t_i).$$

is not 0 if and only if the assignment to the variables $x_1, \dots, x_r, y_1, \dots, y_s$ solves our problem (we know that $e \notin S$ from the formulation of the problem and so $x_i \neq 0$ for $1 \le i \le r$).

The Non-Vanishing Corollary is generally easier to apply when the polynomial in question has a lower degree. To this end, we look to remove some redundant factors.

We know that $e \notin S$, and so when u^{z_i} and $u^{z_{i+1}}$ appear as adjacent elements we know that $z_{i+1} = z_i + x_{i+1} \neq z_i$. When u^{z_i} and $u^{z_{i+1}}$ are not adjacent, we have $z_{i+1} = z_i + y_j - y_{j+1}$ for some j. As we do not allow $y_j = y_{j+1}$ we do not need to check that $z_{i+1} \neq z_i$ in this case either. Hence we may omit factors of the form $(z_{i+1} - z_i)$ from the polynomial. Similar reasoning shows that we may also omit factors of the form $(t_{i+1} - t_i)$.

Hence we define $\pi_{r,s}$ for $r = 2p + \delta$, for $\delta \in \{0, 1\}$, and $s = 2q + 1$ by:

$$\pi_{r,s} = \prod_{1 \le i < j \le 2p+\delta} (x_j - x_i) \prod_{1 \le i < j \le 2q+1} (y_j - y_i) \prod_{\substack{0 \le i < j \le p+q \\ j \neq i+1}} (z_j - z_i) \prod_{\substack{1 \le i < j \le p+q+1+\delta \\ j \neq i+1}} (t_j - t_i).$$

This is the polynomial to which we shall apply the Non-Vanishing Corollary.

If $s = 0$ then the problem reduces to one in the cyclic group C_m, and is addressed in [18]. If $s \ge 2$ is even, let $r = 2p + \delta$, with $\delta \in \{0, 1\}$ as before, and set $s = 2q + 2$. We look for a solution of the form

$$(u^{x_1}, u^{x_2}, \dots, u^{x_p}, u^{y_1}v, u^{y_2}v, \dots, u^{y_{s-1}}v, u^{x_{p+1}}, u^{x_{p+2}}, \dots, u^{x_r}, u^{y_s}v).$$

That is, we adjoin the additional element of $C_m v$ to the end of the form used for odd s. The sequence of partial products is

$$(u^{z_0}, u^{z_1}, \dots, u^{z_p}, u^{t_1}v, u^{z_{p+1}}, u^{t_2}v, \dots, u^{z_{p+q}}, u^{t_{q+1}}v, u^{t_{q+2}}v, \dots, u^{t_{q+p+1+\delta}}v, u^{z_{p+q+1}}).$$

Table 1: Some details of $\pi_{r,s}$ for $|S| = 5$

r	$\deg(\pi)$	monomial	coefficient	prime factors
1	9	$y_1^3 y_3^3 y_4^3$	4	2
2	6	$x_2 y_1 y_2^2 y_3^2$	-3	3
3	7	$x_1^2 x_2 x_3^2 y_1 y_2$	6	$2, 3$
4	8	$x_2^3 x_3^2 x_4^3$	1	–

Now, z_i and t_i have the same values as before with the addition that $z_{p+q+1} = t_{p+q+1+\delta} - y_{2q+2}$. If $r \neq 0$, let $\pi_{r,s}$ be

$$(z_{p+q+1} - z_{p+q}) \prod_{1 \le i < j \le 2p+\delta} (x_j - x_i) \prod_{1 \le i < j \le 2q+2} (y_j - y_i) \prod_{\substack{0 \le i < j \le p+q+1 \\ j \neq i+1}} (z_j - z_i) \prod_{\substack{1 \le i < j \le p+q+1+\delta \\ j \neq i+1}} (t_j - t_i)$$

Again $\pi_{r,s}(x_1, \ldots, x_r, y_1, \ldots, y_s) \neq 0$ if and only if $x_1, \ldots, x_r, y_1, \ldots, y_s$ gives a solution to the problem.

For completeness, note that when $r = 0$ and s is even we have $z_{p+q+1} - z_{p+q} = y_{s-1} - y_s$ and hence we can omit the factor $(z_{p+q+1} - z_{p+q})$ in this case as the negative of this factor is included in the second product. However, this case is covered by Lemma 3.1 anyway.

As $\pi_{r,s}$ is homogeneous, any monomial where the exponent on each x_i is less than r and the exponent on each y_j is less than s is suitable for use in the Non-Vanishing Corollary.

Theorem 3.4. *Let $m > 3$ be prime. If $S \subseteq D_{2m} \setminus \{e\}$ with $|S| \le 9$ then D_{2m} has an S-sequencing.*

Proof. If $|S| \le 4$ the only possible way that S might not be sequenceable is via the first exception of Theorem 2.1. In this case S has the form $\{u^a, u^{-a}, u^b v, u^{a+b} v\}$ for some a, b with $3a \equiv 0 \pmod{m}$. As m is prime and $m > 3$, this cannot occur.

For larger values of $|S|$, let r be the number of elements of S that are in C_m and $s = |S| - r$ the number that are in $C_m v$. If $r = 0$ then the result follows from Lemma 3.1. If $r = |S|$ then, as noted earlier, the result follows from the cyclic group version of the conjecture which is proved for $|S| \le 10$ for prime m in [18].

For each r with $1 \le r \le |S| - 1$ we apply the method of Example 3.3. Tables 1–5 collect the pertinent information concerning the polynomial $\pi_{r,s}$ and one of its monomials for $|S|$ from 5 through to 9. In each case the coefficient is not congruent to 0 $\pmod{m}$; the prime factors of the coefficients are included to make this immediately evident. $\square$

Note that in the proof of Theorem 3.4 when $|S| = 5$ and $r = 2$ the coefficient is not coprime to 3 (and neither is the coefficient on any other viable monomial). This is necessarily the case as D_6 is not sequenceable.

Table 2: Some details of $\pi_{r,s}$ for $|S| = 6$

r	deg(π)	monomial	coefficient	prime factors
1	14	$y_2^2y_3^4y_4^4y_5^4$	-4	2
2	12	$x_2y_1^3y_2^2y_3^3y_4^3$	16	2
3	10	$x_1^2x_3^2y_1^2y_2^2y_3^2$	-3	3
4	12	$x_1^3x_2^3x_3^2x_4^3y_2$	-4	2
5	14	$x_1^2x_2^4x_4^4x_5^4$	-2	2

Table 3: Some details of $\pi_{r,s}$ for $|S| = 7$

r	deg(π)	monomial	coefficient	prime factors
1	22	$y_2^5y_3^5y_4^5y_5^5y_6^2$	-16	2
2	17	$x_2y_2^4y_3^4y_4^4y_5^4$	-4	2
3	16	$x_1^2x_3^2y_1^3y_2^3y_3^3y_4^3$	32	2
4	16	$x_1^3x_2^3x_3^3x_4^3y_2y_3^2$	2	2
5	18	$x_1^4x_2^4x_4^4x_5^4y_1y_2$	12	$2, 3$
6	21	$x_2^5x_3^5x_4x_5^5x_6^5$	-2	2

Table 4: Some details of $\pi_{r,s}$ for $|S| = 8$

r	deg(π)	monomial	coefficient	prime factors
1	30	$y_2y_3^6y_4^5y_5^6y_6^6y_7^6$	-64	2
2	26	$x_2y_2^5y_3^5y_4^5y_5^5y_6^5$	-72	$2, 3$
3	22	$x_1^2x_2^2x_3^2y_1^4y_3^4y_4^4y_5^4$	-1	–
4	23	$x_1^2x_2^2x_3^3x_4^3y_1^3y_2^3y_3^3y_4^3$	-48	$2, 3$
5	24	$x_1^4x_2^4x_3^3x_4^4x_5^4y_2y_3^2$	-3	3
6	26	$x_1^4x_2^5x_3^5x_4x_5^5x_6^5y_2$	48	$2, 3$
7	30	$x_2^6x_3x_4^5x_5^6x_6^6x_7^6$	1	–

Table 5: Some details of $\pi_{r,s}$ for $|S| = 9$

r	deg(π)	monomial	coefficient	prime factors
1	41	$y_2y_3^7y_4^7y_5^7y_6^7y_7^6y_8^6$	720	$2, 3, 5$
2	34	$x_2y_2^6y_3^6y_4^6y_5^6y_6^5y_7^4$	-512	2
3	31	$x_1x_2^2x_3^2y_1^5y_2^5y_3^5y_4y_5^5y_6^5$	-384	$2, 3$
4	28	$x_1^3x_2^3x_3^3x_4^3y_1^4y_2y_3^4y_4^3y_5^4$	12	$2, 3$
5	29	$x_1^3x_2^4x_3^3x_4^4x_5^4y_1^3y_2^3y_3^3y_4^2$	8	2
6	30	$x_1x_2^5x_3^5x_4^5x_5^5x_6^5y_1^2y_3^2$	-16	2
7	35	$x_1^5x_2^6x_3^6x_5^5x_6^6x_7^6y_2$	64	2
8	40	$x_2^7x_3x_4^7x_5^4x_6^7x_7^7x_8^7$	-3	3

In all cases in the proof of Theorem 3.4 there were many monomials with non-zero coefficients. The ones in the tables were chosen to have only small prime factors. Other monomials could have been used in combination, provided that their greatest common divisor has only small prime factors. A more general theoretical approach to solving the problem by finding monomial coefficients might take advantage of this.

Thus the answer to Question 1 is yes for $|S| \leq 9$ in dihedral groups of order twice a prime, with the exceptions noted for D_6 and $|S| \in \{4, 5\}$ in the previous section. For D_{10} this answers the question completely. A deeper understanding of $\pi_{r,s}$ is a conceivable route to removing (or weakening) this condition on S.

4 Large k

In this section we consider subsets of $D_{2m} \setminus \{e\}$ of size at least $2m-2$. An affirmative answer to Question 1 for $k = 2m-1$ and $m \geq 5$ follows immediately from known constructions:

Theorem 4.1. [19, 22] *The dihedral group of order n is sequenceable if and only if $n \geq 10$.*

Existing results also get us some of the way for the $2m-2$ case, via the general result given in Lemma 4.2. In order to state that we need a notion closely related to sequenceability.

Let $\mathbf{a} = (a_1, a_2, \ldots, a_{n-1})$ be a cyclic arrangement of the non-identity elements of G (i.e. a_{n-1} is considered to be adjacent to a_1) and define $\mathbf{b} = (b_1, b_2, \ldots, b_{n-1})$ by $b_i = a_i^{-1}a_{i+1}$ where the indices are considered modulo $(n-1)$ (so $b_{n-1} = a_{n-1}^{-1}a_1$). If the elements of $\mathbf{b}$ are distinct then $\mathbf{a}$ is a *directed rotational terrace* for G and $\mathbf{b}$ is its associated *rotational sequencing.* Clearly, the directed rotational terrace determines the rotational sequencing; the reverse is also true.

(Note: there are several different but equivalent definitions in the literature; see, for example, [2, 13, 20, 26]. We avoid the more common, but less descriptive, names "R-sequencing" and "directed R-terrace" for these concepts to bypass confusion with our S-sequencings and directed S-terraces.)

Lemma 4.2. *Let G be a group of order n and let $S \subseteq G \setminus \{e\}$ with $|S| = n-2$. If G has a rotational sequencing then it also has an S-sequencing.*

Proof. Let $S = G \setminus \{e, x\}$ for some $x \in G$. Let $\mathbf{b} = (b_1, b_2, \ldots, b_{n-1})$ be a rotational sequencing of G indexed so that $b_{n-1} = x$. The partial products of $(b_1, b_2, \ldots, b_{n-2})$ are all distinct because otherwise there would be a repeat in the directed rotational terrace associated with $\mathbf{b}$. Hence $(b_1, b_2, \ldots, b_{n-2})$ is the required S-sequencing. □

Bode and Harborth use this method (but not this terminology) to prove Conjecture 1.1 for the $k = n-2$ case for cyclic groups of odd order. More generally, the recent proof [5] of a conjecture of Friedlander, Gordon and Miller [13] similarly implies that Conjecture 1.1 holds for the $k = n - 2$ case for all abelian groups that do not have a single involution.

For dihedral groups, this approach covers the cases where the group has order a multiple of 4:

Theorem 4.3. *Let m be even and $S \subseteq D_{2m} \setminus \{e\}$ with $|S| = 2m-2$. Then D_{2m} has an S-sequencing.*

Proof. The dihedral group D_{2m} has a rotational sequencing if and only if m is even [20]. Apply Lemma 4.2. □

We also take our cue from Bode and Harborth's methodology when m is odd. Their proof that Conjecture 1.1 holds when $k = n-2$ for cyclic groups of even order implicitly uses the following result:

Lemma 4.4. *If G has a sequencing with first element x then G has an S-sequencing for $S = G \setminus \{e, x\}$.*

Proof. If the sequencing is $(x, b_2, \ldots, b_{n-1})$ then $(b_2, \ldots, b_{n-1})$ must be an S-sequencing else we would have a repeat somewhere in the directed terrace associated with the sequencing. □

Therefore, for odd m, our task becomes to construct a sequencing for D_{2m} with first element x, for each possible choice of x. Noting that there is an automorphism that maps one element to another in D_{2m} (m still odd) if and only if the two elements have the same order reduces the problem to finding sequencings that have first elements of all possible orders. In order to follow this path, we introduce and generalise some of the constructions of Isbell [19] for sequencings of dihedral groups.

To begin, we say a bit more about how sequencings function and introduce graceful permutations.

Let G be a group of order n and let $\mathbf{g} = (g_1, g_2, \ldots g_{n-1})$ be a sequencing with basic directed terrace $\mathbf{h} = (h_0, h_1, h_2, \ldots, h_{n-1})$. Then any sequence $\mathbf{h'} = (h'_0, h'_1, h'_2, \ldots, h'_{n-1})$ such that $h'^{-1}_{i-1} h'_i = g_i$ for each i is called a *directed terrace* for G. The basic directed terrace $\mathbf{h}$ is a directed terrace and a sequence is a directed terrace if and only if it is of the form $x\mathbf{h} = (xh_0, xh_1, xh_2, \ldots, xh_{n-1})$. Not requiring that directed terraces be basic removes an unnecessary restriction when attempting to build a sequencing via a directed terrace.

Given a directed terrace $\mathbf{h} = (h_0, h_1, h_2, \ldots, h_{n-1})$ for G there are two simple ways to obtain further directed terraces [8]. First, we may reverse $\mathbf{h}$ to give the directed terrace $(h_{n-1}, h_{n-2}, \ldots, h_0)$. Second we may find the unique value of i such that $h^{-1}_{n-1} h_0 = g_i$ and produce the directed terrace $(h_{i+1}, h_{i+2}, \ldots, h_{n-1}, h_0, \ldots, h_i)$. Call this the *translation* of $\mathbf{h}$. Thus from each directed terrace we can produce three new directed terraces: its reverse, its translation and the reverse of its translation.

Let $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ be an arrangement of the integers $\{0, 1, \ldots, n-1\}$. If the *sequence of absolute differences* $\mathbf{b} = (b_1, b_2, \ldots, b_{n-1})$ defined by $b_i = |a_{i+1} - a_i|$ consists of the integers $\{1, 2, \ldots, n-1\}$ then $\mathbf{a}$ is a *graceful permutation.* A graceful permutation is equivalent to a graceful labelling of a path with n vertices; see [14] for more details about graceful labellings.

We shall need graceful permutations with various properties in our constructions and will investigate them further in the next section. For now, the following example has the constructions we use to prove the conjecture when m is prime.

Example 4.5. The sequence $(0, 2\ell-1, 1, 2\ell-2, 2, 2\ell-3, \ldots, \ell-1, \ell)$ is a graceful permutation of length 2ℓ and $(0, 2\ell, 1, 2\ell-1, 2, 2\ell-2, \ldots, \ell, \ell+2, \ell+1)$ is a graceful permutation of length $2\ell+1$. These are known as the *Walecki Constructions* [4].

When ℓ is odd, the sequence

$$(\ell, 0, 2\ell, 1, 2\ell-1, \ldots, (\ell-1)/2, (3\ell+1)/2, \\ (3\ell-1)/2, (\ell+1)/2, (3\ell-3)/2, (\ell+3)/2, \ldots, \ell+1, \ell-1)$$

is a graceful permutation of length $2\ell+1$ [19]. The similar construction

$$(\ell; 2\ell-1, 0, 2\ell-2, 1, \ldots, (3\ell-1)/2, (\ell-1)/2; (\ell+1)/2, (3\ell-3)/2, (\ell+3)/2, \ldots, \ell+1, \ell-1)$$

is also a graceful permutation, of length 2ℓ. Between the semicolons it has the absolute differences $(2\ell-1, 2\ell-2, \ldots, \ell)$, after the second semicolon it has the absolute differences $(\ell-2, \ell-3, \ldots, 2)$ and the differences $\ell-1$ and 1 appear at the semicolons.

Isbell [19] gives three constructions for sequencings of dihedral groups D_{2m} where m is odd. In that paper the concern is to get one sequencing for each order. We require more, so the following descriptions work with arbitrary sequences of integers that have the properties on which Isbell relied (graceful permutations in the first two, a slight generalisation thereof in the third) in place of the specific sequences used by Isbell. Further, [19] only covers the $m \equiv 3 \pmod 4$ cases for the second and third constructions; in addition to these we give slight variations that include the $m \equiv 1 \pmod 4$ cases (although we do not have the integer sequences required to make use of the third construction).

Isbell's first construction. Let $m = 4\ell+1$. Let $(a_1, a_2, \ldots, a_{2\ell})$ be a graceful permutation of length 2ℓ with differences $(b_1, b_2, \ldots, b_{2\ell-1})$ (not absolute differences; here $b_i = a_{i+1} - a_i$) and such that $a_{2\ell} = \ell$. Consider the a_i and b_i modulo $4\ell+1$ rather than as integers, then the sequence

$$(u^{b_1}, u^{b_2}, \ldots, u^{b_{2\ell-1}}; u^{2\ell}; v, uv, u^2v, \ldots, u^{2\ell-1}v; u^{4\ell}v; \\ u^{2\ell}v, u^{2\ell+1}v, \ldots, u^{4\ell-1}v; u^{2\ell+1}; u^{-b_{2\ell-1}}, u^{-b_{2\ell-2}}, \ldots, u^{-b_1})$$

is a sequencing for $D_{8\ell+2}$ (where semi-colons are used to help indicate the pattern). The associated directed terrace is

$$(u^{a_1}, u^{a_2}, \ldots, u^{a_{2\ell}}; u^{3\ell}; u^{3\ell}v, u^{3\ell-1}, u^{3\ell+1}v, u^{3\ell-2}, \ldots, u^{4\ell-1}v, u^{2\ell};$$

$$u^{2\ell-1}v, u^{4\ell}, u^{2\ell}v, u^{4\ell-1}, \ldots, u^{3\ell+1}, u^{3\ell-1}v; u^{2\ell-2-a_{2\ell}}v, u^{2\ell-2-a_{2\ell-1}}v, \ldots, u^{2\ell-2-a_1}v).$$

Isbell's second construction. Let $m = 4\ell + 3$ with ℓ odd. Let $(a_1, a_2, \ldots, a_{2\ell+1})$ be a graceful permutation of length $2\ell+1$ with differences $(b_1, b_2, \ldots, b_{2\ell})$ (again, not absolute differences) and such that $a_1 = \ell$ and $a_{2\ell+1} = \ell - 1$. Consider the symbols modulo $4\ell + 3$ rather than as integers, then the sequence

$$(u^{b_1}, u^{b_2}, \ldots, u^{b_{2\ell}}; u^{2\ell+2}; v, uv, u^2v, \ldots, u^{2\ell}v; u^{4\ell+2}v;$$

$$u^{2\ell+1}v, u^{2\ell+2}v, \ldots, u^{4\ell+1}v; u^{2\ell+1}; u^{-b_1}, u^{-b_2}, \ldots, u^{-b_{2\ell}})$$

is a sequencing for $D_{8\ell+6}$. The associated directed terrace is

$$(u^{a_1}, u^{a_2}, \ldots, u^{a_{2\ell+1}}; u^{3\ell+1}; u^{3\ell+1}v, u^{3\ell}, u^{3\ell+2}v, u^{3\ell-1}, \ldots, u^{2\ell+1}, u^{4\ell+1}v;$$

$$u^{4\ell+2}, u^{2\ell}v, u^{4\ell+1}, u^{2\ell+1}v, \ldots, u^{3\ell+2}, u^{3\ell}v; u^{a_1-1}v, u^{a_2-1}v, \ldots, u^{a_{2\ell+1}-1}v).$$

Although Isbell did not consider this case, essentially the same construction works for $m = 4\ell + 1$ with ℓ odd. Let $(a_1, a_2, \ldots, a_{2\ell})$ be a graceful permutation of length 2ℓ with differences $(b_1, b_2, \ldots, b_{2\ell-1})$ (not absolute differences) and such that $a_1 = \ell$ and $a_{2\ell} = \ell - 1$. Consider the symbols modulo $4\ell + 1$ rather than as integers, then the sequence

$$(u^{b_1}, u^{b_2}, \ldots, u^{b_{2\ell-1}}; u^{2\ell+1}; v, uv, u^2v, \ldots, u^{2\ell-1}v; u^{4\ell}v;$$

$$u^{2\ell}v, u^{2\ell+1}v, \ldots, u^{4\ell-1}v; u^{2\ell}; u^{-b_1}, u^{-b_2}, \ldots, u^{-b_{2\ell-1}})$$

is a sequencing for $D_{8\ell+2}$. The associated directed terrace is

$$(u^{a_1}, u^{a_2}, \ldots, u^{a_{2\ell}}; u^{3\ell}; u^{3\ell}v, u^{3\ell-1}, u^{3\ell+1}v, u^{3\ell-2}, \ldots, u^{4\ell-1}v, u^{2\ell};$$

$$u^{2\ell-1}v, u^{4\ell}, u^{2\ell}v, u^{4\ell-1}, \ldots, u^{3\ell+1}, u^{3\ell-1}v; u^{a_1-1}v, u^{a_2-1}v, \ldots, u^{a_{2\ell}-1}v).$$

The only reason that this construction does not work for even ℓ is that the required graceful permutation cannot exist [17]; we shall see more about this in the next section.

For the third construction we need a new concept, closely related to those of $\hat{\rho}$-labellings (also known as nearly graceful labellings) and holey α-labellings described in [14, Section 3.3]. Let $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ be an arrangement of n of the integers $\{0, 1, \ldots, n\}$. If the *sequence of absolute differences* $\mathbf{b} = (b_1, b_2, \ldots, b_{n-1})$ defined by $b_i = |a_{i+1} - a_i|$ consists of the integers $\{1, 2, \ldots, n-1\}$ then we call $\mathbf{a}$ a *cracked graceful permutation*. The missing element of $\{0, 1, \ldots, n\}$ is called the *crack*.

Example 4.6. For even values of ℓ we give the cracked permutation of length $2\ell - 1$ with crack $\ell - 2$ given by Isbell [19]. For $\ell = 2, 4, 8$ the sequences $(3, 1, 2)$, $(5, 0, 6, 7, 3, 1, 4)$ and

$$(9, 5, 7, 12, 11, 2, 10, 4, 14, 3, 0, 13, 1, 15, 8)$$

respectively have the required properties.

For other ℓ the first part of the construction varies as ℓ varies modulo 6, always having the form of an ad hoc sequence of elements followed by "zigzag" sequences of length 6 in a regular pattern.

For $\ell \equiv 0 \pmod 6$ it starts

$$\ell+1, \ell-3, \ell-1; \ell+4, \ell-6, \ell+3, \ell-5, \ell+2, \ell-4; \ell+7, \ell-9, \ell+6, \ell-8, \ell+5, \ell-7; \ldots$$

For $\ell \equiv 2 \pmod 6$ with $\ell \geq 14$ it starts

$$\ell + 1, \ell - 6, \ell + 3, \ell - 1, \ell - 3, \ell + 2, \ell - 4, \ell + 4, \ell - 7, \ell + 5, \ell - 5;$$

$$\ell + 8, \ell - 10, \ell + 7, \ell - 9, \ell + 6, \ell - 8; \ell + 11, \ell - 13, \ell + 10, \ell - 12, \ell + 9, \ell - 11; \ldots$$

For $\ell \equiv 4 \pmod 6$ with $\ell \geq 10$ it starts

$$\ell + 1, \ell - 1, \ell - 5, \ell + 3, \ell - 4, \ell + 2, \ell - 3;$$

$$\ell + 6, \ell - 8, \ell + 5, \ell - 7, \ell + 4, \ell - 6; \ell + 9, \ell - 11, \ell + 8, \ell - 10, \ell + 7, \ell - 9; \ldots$$

In each case the last zigzag sequence of length 6 is

$$(3\ell - 4)/2, \ell/2, (3\ell - 6)/2, (\ell + 2)/2, (3\ell - 8)/2, (\ell + 4)/2.$$

At this point we have used the elements $[\ell/2, (3\ell-4)/2] \setminus \{\ell, \ell-2\}$ and generated the differences $[2, \ell-2] \setminus \{3\}$. The cracked graceful permutation concludes with $(l-2)/2$, which gives the difference 3, followed by a long zigzag and two final ad hoc elements,

$$(3\ell - 2)/2, (\ell - 4)/2, 3\ell/2, (\ell - 6)/2, \ldots, 0, 2\ell - 2; 2\ell - 1, \ell,$$

which gives the remaining elements and differences.

Isbell's third construction. Let $m = 4\ell + 1$ with ℓ even. Let $(a_1, a_2, \ldots, a_{2\ell-2})$ be a cracked graceful permutation of length $2\ell - 2$ with crack $\ell - 4$ and differences $(b_1, b_2, \ldots, b_{2\ell-3})$ (not absolute differences) and such that $a_1 = \ell-1$ and $a_{2\ell-2} = \ell - 2$. Consider the symbols modulo $4\ell+1$ rather than as integers, then the sequence

$$(u^{b_1}, u^{b_2}, \ldots, u^{b_{2\ell-3}}; u^{2\ell+1}, u^{2\ell-2}, u^{2\ell+2}; v, uv, u^2v, \ldots, u^{2\ell-2}v; u^{4\ell-2}v, u^{4\ell-1}, u^{4\ell}v;$$

$$u^{2\ell-1}v, u^{2\ell}v, \ldots, u^{4\ell-3}v; u^{2\ell-1}; u^{-b_1}, u^{-b_2}, \ldots, u^{-b_{2\ell-3}}; u^{2\ell}, u^{2\ell+3})$$

is a sequencing for $D_{8\ell+2}$. The associated directed terrace is

$$(u^{a_1}, u^{a_2}, \dots, u^{a_{2\ell-2}}; u^{3\ell-1}, u^{\ell-4}, u^{3\ell-2}; u^{3\ell-2}v, u^{3\ell-3}, u^{3\ell-1}v, u^{3\ell-4}, \dots, u^{2\ell-1}, u^{4\ell-3}v;$$
$$u^{4\ell}, u^{4\ell-2}v, u^{4\ell-1}; u^{2\ell-3}v, u^{4\ell-2}, u^{2\ell-2}v, u^{4\ell-3}, \dots, u^{3\ell}, u^{3\ell-4}v;$$
$$u^{a_1-2}v, u^{a_2-2}v, \dots, u^{a_{2\ell-2}-2}v; u^{3\ell-3}v, u^{\ell-6}v).$$

Let $m = 4\ell+3$ with ℓ even. Let $(a_1, a_2, \dots, a_{2\ell-1})$ be a cracked graceful permutation of length $2\ell - 1$ with crack $\ell - 2$ and differences $(b_1, b_2, \dots, b_{2\ell-2})$ (not absolute differences) and such that $a_1 = \ell+1$ and $a_{2\ell-1} = \ell$. Consider the symbols modulo $4\ell+3$ rather than as integers, then the sequence

$$(u^{b_1}, u^{b_2}, \dots, u^{b_{2\ell-2}}; u^{2\ell+2}, u^{2\ell-1}, u^{2\ell+3}; v, uv, u^2v, \dots, u^{2\ell-1}v; u^{4\ell}v, u^{4\ell+1}, u^{4\ell+2}v;$$
$$u^{2\ell}v, u^{2\ell+1}v, \dots, u^{4\ell-1}v; u^{2\ell}; u^{-b_1}, u^{-b_2}, \dots, u^{-b_{2\ell-2}}; u^{2\ell+1}, u^{2\ell+4})$$

is a sequencing for $D_{8\ell+6}$. The associated directed terrace is

$$(u^{a_1}, u^{a_2}, \dots, u^{a_{2\ell-1}}; u^{3\ell+2}, u^{\ell-2}, u^{3\ell+1}; u^{3\ell+1}v, u^{3\ell}, u^{3\ell+2}v, u^{3\ell-1}, \dots, u^{4\ell}v, u^{2\ell+1};$$
$$u^{2\ell-2}v, u^{2\ell}, u^{2\ell-1}v; u^{4\ell+2}, u^{2\ell}v, u^{4\ell+1}, u^{2\ell+1}v, \dots, u^{3\ell+3}, u^{3\ell-1}v;$$
$$u^{a_1-2}v, u^{a_2-2}v, \dots, u^{a_{2\ell-1}-2}v; u^{3\ell}v, u^{\ell-4}v).$$

Similarly to the second construction, the parity restrictions on ℓ are because otherwise the required cracked graceful permutations do not exist. We prove this in the next section.

Theorem 4.7. *Let m be an odd prime. If $S \subseteq D_{2m} \setminus \{e\}$ with $|S| = 2m - 2$, then D_{2m} has an S-sequencing.*

Proof. As m is prime each pair of elements of $C_m \setminus \{e\}$ are equivalent by automorphisms, as are each pair of elements of C_mv. By Lemma 4.4 it is therefore sufficient to find a sequencing with first element in $C_m \setminus \{e\}$ and a sequencing with first element in C_mv.

Let $\mathbf{g} = (g_1, g_2, \dots g_{2m-1})$ be the sequencing for D_{2m} constructed from Isbell's first construction when $m \equiv 1 \pmod 4$, Isbell's second construction when $m \equiv 7 \pmod 8$ and Isbell's third construction when $m \equiv 3 \pmod 8$, in each case using the (cracked) graceful permutations given in the examples. Let $\mathbf{h} = (h_1, h_2, \dots h_{2m})$ be the associated directed terrace.

In all cases $g_1 = u^{b_1}$, where b_1 is the first element of the differences of the (cracked) graceful permutation. We have $g_1 \in C_m \setminus \{e\}$.

Now consider the translation of $\mathbf{h}$. When $m = 4\ell + 1$, the first element of the associated sequencing of the reverse of the translation is $u^{2\ell-3}v$. When $m = 4\ell+3$ for even ℓ, the first element of its associated sequencing is $u^{4\ell-1}v$. When $m = 4\ell + 3$ for odd ℓ, the first element of the associated sequencing of the reverse of the translation is $u^{4\ell}v$. Each is in C_mv. □

To move to composite values of m we need to vary the first element in the sequencings we construct. We do this by varying the (cracked) graceful permutations used in Isbell's constructions. Note that when m is odd that any two elements of $C_m v$ are conjugate and hence there is an automorphism of D_{2m} mapping one to the other. This means that the proof of Theorem 4.7 implies the existence of sequencings that start with any such elements for arbitrary odd m. We can therefore focus on sequencings starting with elements of $C_m \setminus \{e\}$.

Lemma 4.8. *Let $(a_1, a_2, \dots, a_{2\ell})$ be a graceful permutation of length 2ℓ with sequence of absolute differences $(b_1, b_2, \dots, b_{2\ell-1})$. Let $(c_1, c_2, \dots, c_{2\ell+1})$ be a graceful permutation of length $2\ell+1$ with sequence of absolute differences $(d_1, d_2, \dots, d_{2\ell})$. Then*

1. *if $a_{2\ell} = \ell$, then $D_{8\ell+2}$ has a sequencing with first element u^{b_1};*
2. *if $a_1 = \ell$ and $a_{2\ell} = \ell - 1$, then $D_{8\ell+2}$ has a sequencing with first element u^{b_1} and a sequencing with first element $u^{b_{2\ell-1}}$;*
3. *if $c_1 = \ell$ and $c_{2\ell+1} = \ell - 1$, then $D_{8\ell+6}$ has a sequencing with first element u^{d_1} and a sequencing with first element $u^{d_{2\ell}}$.*

Proof. Part 1 follows from Isbell's first construction. Parts 2 and 3 follow from Isbell's second, the first of each clause directly and the second after taking the reverse. The distinction between absolute differences (here) and differences (in the Isbell constructions) is rendered moot as they give elements of D_{2m} that have the same order and hence are equivalent under automorphisms (as m is odd). $\square$

Lemma 4.9. *Let $(a_1, a_2, \dots, a_{2\ell-2})$ be a cracked graceful permutation of length $2\ell-2$ with sequence of absolute differences $(b_1, b_2, \dots, b_{2\ell-2})$ and crack $\ell-4$. Let $(c_1, c_2, \dots, c_{2\ell-1})$ be a cracked graceful permutation of length $2\ell-1$ with sequence of absolute differences $(d_1, d_2, \dots, d_{2\ell-2})$ with crack $\ell-2$. Then*

1. *if $a_1 = \ell-1$ and $a_{2\ell-2} = \ell-2$, then $D_{8\ell+2}$ has a sequencing with first element u^{b_1} and a sequencing with first element $u^{2\ell+3}$;*
2. *if $c_1 = \ell+1$ and $c_{2\ell-1} = \ell$, then $D_{8\ell+6}$ has a sequencing with first element u^{d_1} and a sequencing with first element $u^{2\ell+4}$.*

Proof. These follow from Isbell's third construction and its reverse. $\square$

In the next section we investigate the existence of (cracked) graceful permutations for use with these lemmas.

5 Graceful Permutations

Our task in this section is to construct (cracked) graceful permutations that meet the criteria for use in one of Isbell's constructions and have endpoints and first/last differences that give a variety of orders for the first element of the dihedral group sequencing.

Given a graceful permutation $\mathbf{a} = (a_1, a_2, \dots, a_n)$ on the symbols $\{0, \dots, n-1\}$ define the *complement* to be $\bar{\mathbf{a}} = (n-1-a_1, n-1-a_2, \dots, n-1-a_n)$. This is also a graceful permutation.

The most flexible result from the previous section is the first clause of Lemma 4.8, which requires a graceful permutation with specified first difference and last element and gives results for $D_{8\ell+2}$. We work with this first. It is known that for a graceful permutation of length n, a first difference of d, for $1 \leq d \leq n-1$, is possible except when $d = 2$ and $n \in \{4, 5, 8\}$ [18].

To make the most of these constructions, we need to be able to control the elements at each end of a graceful permutation. Necessary conditions are known:

Theorem 5.1. [17] *Let $0 \leq x, y \leq n-1$ and $x \neq y$. If there is a graceful permutation of length n with first element x and last element y then*

- *$|x - y|$ has the same parity as $\lfloor n/2 \rfloor$,*
- *$|x - y| \leq n/2$,*
- *$(n-1)/2 \leq x + y \leq (3n-3)/2$.*

Gvözdjak conjectures that these conditions are also sufficient [17]. Call this *Gvözdjak's Conjecture*. It is also known that for any $x < n$ there is a graceful sequence of length n that starts with x, see any of [10, 12, 17].

Suppose we write $n = pq + r$, where $p, q \geq 1$ and $r \geq p/2$. An "imperfect p-twizzler terrace" $\mathbf{g}$, which is a graceful permutation and here we shall call a *p-twizzler permutation*, is one constructed as follows. Start with the Walecki Construction of length n. Divide the first pq elements into q subsequences of length p. Reverse each subsequence while keeping the order of the subsequences intact. Rearrange the final r elements into a translate of a graceful permutation of length r such that the absolute difference between the pqth and $(pq+1)$th elements of the sequence is r (the condition that $r \geq p/2$ and the fact that there is a graceful permutation of length r that starts with any element are the crucial pieces that guarantee this is possible; a full proof of the construction's correctness can be found in [25]).

Example 5.2. Consider $n = 2\ell = 22$ with $p = 4$, $q = 3$ and $r = 10$. The following is a p-twizzler permutation with these parameters where the last 10 elements are a translate of $(8, 2, 6, 1, 9, 0, 7, 4, 3, 5)$:

$$(20, 1, 21, 0; 18, 3, 19, 2; 16, 5, 17, 4; 14, 8, 12, 7, 15, 6, 13, 10, 9, 11)$$

(semicolons separate the subsequences).

Theorem 5.3. *Assume Gvözdjak's Conjecture holds. Then there is a S-sequencing for any $S \subseteq D_{8\ell+2} \setminus \{e\}$ with $|S| = 8\ell$.*

Proof. By Theorem 4.7, and the comment following it that notes that the proof goes through for missing elements in $C_m v$ regardless of the primality of $4\ell+1$, it is sufficient to consider the cases where $4\ell+1$ is composite and the non-identity element x missing from S has order o with $3 \leq o \leq 4\ell+1$. We construct a sequencing for $D_{8\ell+2}$ that has first element x, from which the result follows by Lemma 4.4.

Our general method does not work for $l = 2$ and $o = 9$, so we give a sequencing for D_{18} that covers this case first:

$$u, u^8v, u^7, u^3v, u^6v, u^4, u^4v, uv, v, u^6, u^2v, u^2, u^5, u^5v, u^8, u^3, u^7v.$$

The associated directed terrace is:

$$e, u, v, u^2v, u^8, u^5v, uv, u^6, u^7v, u^7, u^4, u^6v, u^4v, u^8v, u^3, u^2, u^5, u^3v.$$

For the general case, we want d such that $d \in [\ell+1, 2\ell-1]$ and d has order o in $\mathbb{Z}_{4\ell+1}$. If $o < 4\ell+1$ then set $d = (4\ell+1)(o-1)/2o$. If $o = 4\ell+1$ and ℓ is a not power of 2, then set d to be the unique power of 2 in $[\ell+1, 2\ell-1]$. If $o = 4\ell+1$ and ℓ is a power of 2, then let π be the smallest odd prime that does not divide $4\ell+1$. As $\pi < 2\ell$ (since $\ell \neq 2$), there is a unique element of the form $2^s\pi$ in $[\ell+1, 2\ell-1]$; take this to be d.

We construct a p-twizzler permutation of length $2\ell = pq + r$ with $p = 2\ell - d + 1$ and $q = 1$. It shall have first difference d and final element in $\{\ell-1, \ell\}$.

Suppose p is even. If Gvözdjak's Conjecture holds, then there is a graceful permutation $\mathbf{g}$ of length $r = 2\ell - p - 1$ with first element $2\ell - 3p/2 - 1$ and final element in $\{\ell - 1 - p/2, \ell - p/2\}$. Add $p/2$ to each element to get a translate with first element $2\ell - p - 1$ and last element in $\{\ell-1, \ell\}$. As we are reversing only one section of the Walecki Construction, we require $|(2\ell - p - 1) - 0| = r$ for the p-twizzler construction to be valid, and this is the case. The result now follows by the first part of Lemma 4.8, possibly after taking the complement.

The odd case is similar. □

While the ability to vary q in the twizzler construction was not used in this proof, we can use this flexibility to exploit that it is known Gvözdjak's Conjecture holds for $n \leq 20$ [17]:

Theorem 5.4. *Suppose that 2ℓ can be written $2\ell = pq + r$ with $q \geq 1$ and $p/2 \leq r \leq 20$. Then there is an S-sequencing for any $S \subseteq D_{8\ell+2} \setminus \{e\}$, where $|S| = 8\ell$ and the missing non-identity element of S has the same order as $u^{2\ell-p+1}$.*

Proof. The method is exactly as in the proof of Theorem 5.3, except we do not insist that $q = 1$. In this more general setting, we need the graceful permutation of length $r = 2\ell - pq$ to have final element in $\{\ell - 1 - pq/2, \ell - pq/2\}$ and an appropriate first element. As Gvözdjak's Conjecture holds for r, we have complete choice of possible first element. □

Theorem 5.4 implies $D_{8\ell+2}$ has an S-sequencing for all $S \subseteq D_{8\ell+2} \setminus \{e\}$ with $|S| = 8\ell$ for many values of ℓ, including all $\ell < 35$ for which $4\ell + 1$ is composite and

$$\ell \in \{36, 40, 42, 46, 51, 52, 54, 55, 63, 72, 75, 82, 85, 90, 94\}.$$

It gives partial results for many more. The smallest value of ℓ for which $4\ell + 1$ is composite and Theorem 5.4 adds no new S-sequencings is 420.

When considering Isbell's second and third constructions, we need (cracked) graceful permutations with both first and last element specified. Further, the first and last elements are both near the center of the possible values. Twizzler permutations, and other similar extension constructions in the literature, tend to push at least one of the values of the endpoints closer to an extreme and so are not very helpful for the current situation. We are able to make some partial progress by introducing a new extension construction for graceful permutations.

Let $\mathbf{c} = (c_1, \ldots, c_{2p})$ be a graceful permutation of even length $2p$ on $[0, 2p-1]$. Say that $\mathbf{c}$ is *bipartite* if the odd-index elements $\{c_1, c_3, \ldots, c_{2p-1}\}$ are either all at most $p-1$ or all at least p. That is, a bipartite graceful permutation has alternating "small" and "large" elements.

In the more general theory of graceful labelings of graphs, bipartite graceful labelings are often known as *α-labelings*, see [14].

Lemma 5.5. [21] *For any $x \in [0, p-1]$ there is a bipartite graceful permutation $(c_1, \ldots, c_{2p})$ with $c_1 = x$. If $(c_1, \ldots, c_{2p})$ is a bipartite graceful permutation of length $2p$ with $c_1 < p$, then $c_{2p} = c_1 + p$.*

We can now give the main construction.

Theorem 5.6. *Let $\mathbf{a} = (a_1, \ldots, a_q)$ be a graceful permutation of length q. Suppose $\mathbf{a}$ has adjacent elements x and y with $x < y < 2(y - x) = p$. Then there is a graceful permutation of length $2p + q$ with first element $a_1 + p$ and last element $a_q + p$.*

Proof. Let $\mathbf{c} = (c_1, \ldots, c_{2p})$ be a bipartite graceful permutation with $c_1 = y$. This exists by Lemma 5.5. Also by that lemma, we know that $c_{2p} = y + p$.

Without loss of generality, assume that $a_i = x$ and $a_{i+1} = y$ (if x and y appear in the other order, replace $\mathbf{a}$ with its reverse and then reverse the resulting graceful permutation once the construction is complete). Consider the sequence

$$(a_1 + p, \ldots, a_i + p, c_1, c_2 + q, \ldots, c_{2p-1}, c_{2p} + q, a_{i+1} + p, \ldots, a_q + p).$$

That is, the first i elements of $\mathbf{a}$ with p added to each, followed by $\mathbf{c}$ with q added to each element in an even position within $\mathbf{c}$, followed by the last $q - i$ elements of $\mathbf{a}$ with p added to each. We have

$$\{c_1, c_3, \ldots, c_{2p-1}\} = [0, p-1], \quad \{a_1 + p, \ldots, a_q + p\} = [p, p+q-1]$$

and

$$\{c_2 + q, c_4 + q, \ldots, c_{2p} + q\} = [p+q, 2p+q-1],$$

so the sequence has the required elements.

As **a** is a graceful permutation, we have the absolute differences $[1, q-1] \setminus \{y-x\}$ and as **c** is a bipartite graceful permutation, we have the absolute differences $[q+1, 2p+q-1]$. Two additional absolute differences are given at the joins:

$$|c_1 - (a_i + p)| = (x + 2(y-x)) - y = y - x$$

and

$$|(a_{i+1} + p) - (c_{2p} + q)| = (y + p + q) - (y + p) = q.$$

Hence our sequence is a graceful permutation. Its first element is $a_1 + p$ and its last element is $a_q + p$. $\square$

Call the graceful permutation obtained from a graceful permutation **a** and a bipartite graceful permutation **c** via the method of Theorem 5.6's proof the *insertion* of **c** into **a** at i.

Example 5.7. The insertion of

$$\mathbf{c} = (4, 11, 5, 10, 6, 9, 7, 8, 0, 15, 1, 14, 2, 13, 3, 12)$$

into $\mathbf{a} = (1, 6, 0, 4, 3, 5, 2)$ at $i = 3$ (where $x = 0$, $y = 4$, $2(y-x) = p = 8$ and $q = 7$) is

$$(9, 14, 8, 4, 18, 5, 17, 6, 16, 7, 15, 0, 22, 1, 21, 2, 20, 3, 19, 12, 11, 13, 10),$$

a graceful permutation of length 23.

We can often guarantee options to choose as x and y in Theorem 5.6:

Lemma 5.8. *Let* **a** *be a graceful permutation of length q. There is a valid insertion point in at least one of* **a** *and its complement for a bipartite graceful permutation of length $2p$ whenever p is even and $2(q-1)/3 < p < 2q$.*

Proof. The conditions that p be even and $p < 2q$ guarantee that $p/2$ is an absolute difference of **a**. Let $p/2 = y - x$ for some adjacent elements x, y of **a**.

If $q \leq p$ then $2(y-x) \geq q > y$ and the conditions of Theorem 5.6 are met. Similarly, if $q > p > y$ then the conditions are met, so assume that $q > p$ and $p \leq y$. In the complement of **a** there are adjacent elements $y' = q-1-x$ and $x' = q-1-y$ with $y' - x' = p/2$. Now, $y' = q-1-y+p/2$ as $y - p/2 = x$. Using $2(q-1)/3 < p$, we see that $y' < 2p - y$ and so $y' < p$. We may therefore apply Theorem 5.6 to the complement of **a**. $\square$

We often care about the end-points of graceful permutations. Taking the complement of a graceful permutation, inserting a bipartite graceful permutation of length $2p$, and taking the complement of the result gives the same end-points as if we had directly inserted a bipartite graceful permutation of length $2p$ into the original graceful permutation.

Lemma 5.9. *Let $n \equiv 3 \pmod 4$, with $n > 3$. There is a graceful permutation of length n with first element $(n-1)/2$, last element $(n-3)/2$, and first absolute difference 3.*

Proof. We use induction. For base cases we require $n \in \{7, 11, 15, 19\}$. The construction in Example 4.5 covers the case $n = 7$. Here are graceful permutations for the remaining base cases:

$$(5, 8, 0, 10, 1, 6, 7, 3, 9, 2, 4),$$
$$(7, 10, 0, 14, 1, 13, 2, 8, 3, 12, 4, 11, 9, 5, 6),$$
$$(9, 12, 0, 18, 1, 17, 2, 16, 3, 7, 13, 5, 14, 4, 15, 8, 6, 11, 10).$$

Now consider $n \geq 23$ and write $n = 3q - 2$, $n = 3q + 2$ or $3q + 6$ according to its congruence class modulo 3 and let $p = (n - q)/2$. As $n \equiv 3 \pmod 4$, we have $3q \equiv 1 \pmod 4$ and so $q \equiv 3 \pmod 4$. As $n \geq 23$ we have that $q \geq 7$, that $2(q - 1)/3 < p < 2q$, and that $p/2 \neq 3$.

By the inductive hypothesis, we have a bipartite graceful permutation into a graceful permutation of length q with first element $(q - 1)/2$, last element $(q - 3)/2$, and first absolute difference 3 into which we can insert a bipartite graceful permutation of length $2p$ to get a graceful permutation of length n with first element

$$(q - 1)/2 + p = (q - 1)/2 + (n - q)/2 = (n - 1)/2,$$

last element

$$(q - 3)/2 + p = (q - 3)/2 + (n - q)/2 = (n - 3)/2,$$

and first absolute difference 3. □

Combining Lemma 5.9 with Lemma 4.8 we can prove the following result about the existence of S-sequencings.

Theorem 5.10. *Let ℓ be an odd multiple of 3. Let $S \subseteq D_{8\ell+6} \setminus \{e\}$ with $|S| = 8\ell + 4$ and x the non-identity element not in S. If x has order $(4\ell + 3)/3$ then $D_{8\ell+6}$ has an S-sequencing.*

Proof. We use the third clause of Lemma 4.8 and so are looking to construct graceful permutations of length $2\ell + 1$ with first element ℓ and final element $\ell - 1$. Lemma 5.9 gives us such a graceful permutation with first absolute difference 3. Hence the resulting sequencing will have first element with order $(4\ell + 3)/3$. □

More generally, the method of Lemma 5.9 and Theorem 5.10 can be used to provide sequencings for $D_{8\ell+6} \setminus \{e, x\}$ where x has order $(4\ell + 3)/o$ rather than $(4\ell + 3)/3$. The combination of the construction from Example 4.5 with the insertion method gives sequencings for $D_{8\ell+6}$ that start with x for many values of ℓ. However, to get a complete result along the lines of Theorem 5.10 requires increasingly many base cases as o grows.

For cracked graceful permutations the same proof as Gvözdjak's for part 1 of Theorem 5.1 applies:

Lemma 5.11. *Let $0 \leq x, y \leq n$ and $x \neq y$. If there is a cracked graceful permutation of length n with first element x and last element y then $|x-y|$ has the same parity as $\lfloor n/2 \rfloor$.*

Proof. Let $(a_1, a_2, \ldots, a_n)$, where $a_1 = x$ and $a_n = y$, be a cracked graceful permutation. Then

$$\begin{aligned} |x-y| &\equiv |a_1 - a_2| + |a_2 - a_3| + \cdots + |a_{n-1} - a_n| \\ &\equiv n(n-1)/2 \equiv \lfloor n/2 \rfloor \pmod 2 \end{aligned}$$

as required. □

This result, along with Theorem 5.1, implies that neither Isbell's second nor third constructions can successfully cover all dihedral groups of the form $D_{8\ell+6}$ alone.

It is possible to extend the insertion method to cracked graceful permutations and then employ a similar approach to that of Theorem 5.10. However, we are lacking a graceful permutation that would get us started in the case $m \equiv 3 \pmod 8$. Rather than pursue this route here, we use the specific construction of the cracked graceful permutation in Example 4.6, which allows a more efficient way to cover some small initial elements of sequencings of $D_{8\ell+6}$.

Theorem 5.12. *For $d \in \{5, 6, 7\}$ and even $\ell > 2$, there is a cracked graceful permutation of length $2\ell - 1$ with crack $\ell - 2$ that starts with $\ell + 1$, ends with ℓ and has first absolute difference d, except when $\ell = 4$ and $d \in \{6, 7\}$.*

Proof. The main method of proof is to take the Isbell construction given in Example 4.6 and replace the start of the sequence with an alternative subsequence that uses the same elements and gives the same differences.

When $\ell \equiv 0 \pmod 6$, for any $t < \ell/6$ the first $6t+3$ elements of Isbell's cracked graceful permutation are a translate of an arrangement of the elements $[-3t-2, 3t+2] \setminus \{-1, 1\}$, starting with 2 and ending with $-3t$, and having absolute differences $[2, 6t+4] \setminus \{3\}$.

When $\ell \equiv 2 \pmod 6$ with $\ell \geq 14$, for any $t < (\ell - 8)/6$ the first $6t + 11$ elements of Isbell's cracked graceful permutation are a translate of an arrangement of the elements $[-3t-6, 3t+6] \setminus \{-1, 1\}$, starting with 2 and ending with $-3t-4$, and having absolute differences $[2, 6t+12] \setminus \{3\}$.

When $\ell \equiv 4 \pmod 6$ with $\ell \geq 10$, for any $t < (\ell - 4)/6$ the first $6t + 7$ elements of Isbell's cracked graceful permutation are a translate of an arrangement of the elements $[-3t-4, 3t+4] \setminus \{-1, 1\}$, starting with 2 and ending with $-3t-2$, and having absolute differences $[2, 6t+8] \setminus \{3\}$.

In each case we may substitute an alternative sequence with these properties to obtain an alternative cracked graceful permutation. Table 6 gives the sequences required to prove the result, except when (ℓ, d) is one of

$$(4,5), (6,5), (6,6), (6,7), (8,5), (8,6), (8,7), (10,6), (10,7), (12,5), (14,6), (16,7).$$

Table 6: Sequences for use in the proof of Lemma 5.12

$\ell \pmod 6$	d	t	sequence
0	5	2	$(2, -3, 3, -4, 7, -7, 8, -8, 5, -5, 4, 0, -2, 6, -6)$
	6	1	$(2, -4, 5, -5, 3, -2, 0, 4, -3)$
	7	1	$(2, -5, 5, -4, 4, 0, -2, 3, -3)$
2	5	0	$(2, -3, 3, -6, 6, -5, 5, -2, 0, 4, -4)$
	6	1	$(2, -4, 3, -2, 0, 4, -6, 7, -8, 8, -9, 9, -5, 6, -3, 5, -7)$
	7	0	$(2, -5, 5, -6, 6, -3, 3, -2, 0, 4, -4)$
4	5	0	$(2, -3, 3, -4, 4, 0, -2)$
	6	1	$(2, -4, 3, -2, 0, 4, -7, 7, -6, 6, -3, 5, -5)$
	7	2	$(2, -5, 4, 0, -2, 3, -3, 5, -7, 6, -4, 7, -10, 10, -9, 9, -6, 8, -8)$

Cracked graceful permutations with the stated properties for these parameters are given in Table 7.

□

This immediately implies:

Corollary 5.13. *Let ℓ be even. Let $S \subseteq D_{8\ell+6} \setminus \{e\}$ with $|S| = 8\ell + 4$ and x the non-identity element not in S. If x has order $(4\ell + 3)/d$ for $d \in \{3, 5, 7\}$ then $D_{8\ell+6}$ has an S-sequencing.*

6 Strong Sequenceability

As usual, let $\mathbf{g} = (g_1, g_2, \ldots, g_k)$ be an arrangement of elements of $G \setminus \{e\}$ with partial product sequence $\mathbf{h} = (h_0, h_1, h_2, \ldots, h_k)$. Suppose the elements of $\mathbf{g}$ are distinct and let $S = \{g_1, g_2, \ldots, g_k\}$.

If the elements $(h_1, \ldots, h_k)$ are all distinct with $h_k = h_0 = e$ then $\mathbf{h}$ is a *rotational directed S-terrace* for G and $\mathbf{g}$ is the associated *rotational S-sequencing* of G. If $|G| = n$ and $k = n - 1$ we get a rotational directed terrace and associated rotational sequencing of G as in Section 4.

In [6] the following notion is introduced: A group is *strongly sequenceable* if for all $S \subseteq G \setminus \{e\}$ there is either an S-sequencing or a rotational S-sequencing (or both). In an abelian group the element h_k is independent of the ordering and so we cannot have both an S-sequencing and a rotational S-sequencing. Thus for abelian groups, the property of being strongly sequenceable indicates that all subsets have the sequenceability properties we might wish to use in combinatorial constructions.

Alspach and Kalinowski, see [6], have posed the problem of determining which groups are strongly sequenceable. As observed in [6], D_6 has neither a sequencing

Table 7: Some cracked graceful permutations

ℓ	d	cracked graceful permutation
4	5	$(5, 0, 1, 7, 3, 6, 4)$
6	5	$(7, 2, 5, 11, 3, 1, 8, 9, 0, 10, 6)$
	6	$(7, 1, 5, 10, 0, 3, 11, 2, 9, 8, 6)$
	7	$(7, 0, 2, 10, 1, 11, 5, 9, 8, 3, 6)$
8	5	$(9, 4, 12, 3, 13, 14, 2, 15, 1, 5, 11, 0, 7, 10, 8)$
	6	$(9, 3, 10, 14, 2, 1, 15, 5, 7, 12, 4, 13, 0, 11, 8)$
	7	$(9, 2, 13, 11, 7, 1, 14, 0, 10, 15, 3, 12, 4, 5, 8)$
10	6	$(11, 5, 13, 15, 0, 1, 19, 2, 18, 4, 16, 3, 12, 9, 14, 7, 17, 6, 10)$
	7	$(11, 4, 13, 15, 0, 5, 6, 12, 9, 17, 7, 18, 1, 19, 3, 16, 2, 14, 10)$
12	5	$(13, 8, 15, 17, 0, 23, 1, 2, 22, 3, 21, 5, 19, 4, 16, 6, 14, 11, 7, 20, 9, 18, 12)$
14	6	$(15, 9, 17, 19, 1, 27, 2, 26, 3, 25, 4, 0, 20,$ $10, 21, 7, 16, 23, 8, 11, 24, 5, 22, 6, 18, 13, 14)$
16	7	$(17, 10, 19, 21, 1, 31, 2, 30, 3, 29, 4, 28, 5, 27, 6,$ $0, 18, 15, 11, 25, 20, 8, 23, 7, 26, 9, 22, 12, 13, 24, 16)$

nor a rotational sequencing. The same is true for the quaternion group Q_8 [13, 16]. Hence we have:

Theorem 6.1. *If G has a subgroup isomorphic to either D_6 or Q_8, then G is not strongly sequenceable.*

The other sets S that do not have S-sequencings discussed in Section 2 either have rotational S-sequencings or do not imply that any further groups are not strongly sequenceable. It is known that all abelian groups of order at most 21 and all cyclic groups of order at most 25 are strongly sequenceable [7, 11]. By the remark at the end of Section 3 we can add D_{10} to this list.

In non-abelian groups it is possible for a subset to have both a sequencing and a rotational sequencing. For example, the dihedral groups of order a multiple of 4 have both sequencings and rotational sequencings [19, 20, 22]. For a combinatorial construction we might require, for example, that a subset that has an ordering with product e has a rotational sequencing; strong sequenceability of the group does not guarantee this for us in the non-abelian setting.

We therefore strengthen the concept of strong sequenceability: a group G is *very strongly sequenceable* if every subset $S \subseteq G \setminus \{e\}$ with an ordering of the elements whose product is not the identity has a sequencing and every subset $T \subseteq G \setminus \{e\}$ with an ordering whose product is the identity has a rotational sequencing.

An abelian group is very strongly sequenceable if and only if it is strongly sequenceable. For non-abelian groups, very strong sequenceability is more restrictive. For example, D_8 is not sequenceable (and there is an ordering of its non-identity elements whose product is not the identity) and so is not very strongly sequenceable. On the other hand, a computer search shows that D_8 is strongly sequenceable.

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