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Hamiltonicity in directed Toeplitz graphs $T_n\langle 1, 2; t_1, t_2\rangle$

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Abstract

A square matrix of order n is called a Toeplitz matrix if it has constant values along all diagonals parallel to the main diagonal. A directed Toeplitz graph $T_n\langle s_1, \ldots, s_k; t_1, \ldots, t_l\rangle$ of order n, where the edge (i, j) occurs if and only if $j - i = s_p$ or $i - j = t_q$ for some $1 \leq p \leq k$ and $1 \leq q \leq l$, is a digraph whose adjacency matrix is a Toeplitz matrix. In this paper, we study hamiltonicity in directed Toeplitz graphs $T_n\langle 1, 2; t\rangle$ and $T_n\langle 1, 2; t_1 \leq 5, t_2\rangle$. We obtain new results and improve existing results on $T_n\langle 1, 2; t\rangle$ and $T_n\langle 1, 2; t_1 \leq 5, t_2\rangle$. We then extend these results to $T_n\langle 1, 2; t_1 \geq 6, t_2\rangle$, and discuss the hamiltonicity for all $t_1 \geq 6$ and all n.

1 Introduction

We use [18] for basic terminology and notation not defined here. Since multiple edges and loops do not play any role in the study of hamiltonicity, the graphs we consider are finite, directed and simple.

A *directed Toeplitz graph* $T_n\langle s_1, \ldots, s_k; t_1, \ldots, t_l\rangle$ of order n, in which the edge (i, j) occurs if and only if $j - i = s_p$ or $i - j = t_q$ for some $1 \leq p \leq k$ and $1 \leq q \leq l$, is a digraph whose adjacency matrix is a *Toeplitz matrix* (a square matrix having constant values along all diagonals parallel to the main diagonal). The edges of $T_n\langle s_1, s_2, \ldots, s_k; t_1, t_2, \ldots, t_l\rangle$ are of two types: *increasing edges* (u, v), for which $u < v$, and *decreasing edges* (u, v), where $u > v$. Note that any increasing edge has length s_p for some p, and any decreasing edge has length t_q for some q, and that $T_n\langle s_1, \ldots, s_k; t_1, \ldots, t_l\rangle$ and $T_n\langle t_1, \ldots, t_l; s_1, \ldots, s_k\rangle$ are obtained from each other by reversing the orientation of all edges. We define the *length* of an edge (u, v) to be $|u - v|$.

Suppose that H is a hamiltonian cycle in $T_n\langle s_1, s_2, \ldots, s_k; t_1, t_2, \ldots, t_l\rangle$. The

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hamiltonian cycle H is determined by two paths $H_{1\to n}$ (from 1 to n) and $H_{n\to 1}$ (from n to 1), i.e., $H = H_{1\to n} \cup H_{n\to 1}$.

Properties of Toeplitz graphs, such as colourability, planarity, bipartiteness, connectivity, cycle discrepancy, edge irregularity strength, decomposition, labeling, and metric dimension have been studied in [1–5, 7–15, 17]. Hamiltonian properties of Toeplitz graphs were first investigated by van Dal et al. in [6] and then studied in [12, 16, 23], while the hamiltonicity in directed Toeplitz graphs was first studied by Malik and Qureshi in [18], and then by Malik in [19, 20, 22] and by Malik and Zamfirescu in [21].

In [18], the hamiltonicity of the Toeplitz graphs $T_n\langle 1, 2; t\rangle$ and $T_n\langle 1, 2, 3; t\rangle$ was investigated. We improve upon [18] by adding negative and positive results about $T_n\langle 1, 2; t\rangle$. In [18], it was shown that $T_n\langle 1, 2; 5\rangle$ is hamiltonian for all $n \geq 29$, and that for odd $t \geq 7$, $T_n\langle 1, 2; t\rangle$ is hamiltonian for all $n > 3t + 5$. Here we discuss the hamiltonicity of $T_n\langle 1, 2; 5\rangle$ for $n < 29$, and we see that for $n < 29$, $T_n\langle 1, 2; 5\rangle$ is hamiltonian if and only if $n \in \{6, 7, 9, 11, 14, 16, 17, 19, 21, 22, 24, 26, 27\}$. For odd $t \geq 7$, we also discuss the hamiltonicity of $T_n\langle 1, 2; t\rangle$ for $n \leq 3t + 5$, and we see that for $n \leq 3t + 5$ and odd $t \geq 7$, $T_n\langle 1, 2; t\rangle$ is hamiltonian if and only if $n \in \{t + 1, t + 2, t + 4, \ldots, 2t + 1, 2t + 4, 2t + 6, 2t + 7, \ldots, 3t + 2, 3t + 4\}$.

Paper [22] extended the investigation of [18] to $T_n\langle 1, 2; t_1 \leq 5, t_2\rangle$, where we saw that $T_n\langle 1, 2; 1, t_2\rangle$ is hamiltonian for all n and t_2; for $t_1 \in \{2, 4\}$, $T_n\langle 1, 2; t_1, t_2\rangle$ is hamiltonian for all n and odd t_2; for even t_2, $T_n\langle 1, 2; 3, t_2\rangle$ is hamiltonian for all n; for odd t_2, if $t_2 \equiv 1 \pmod 3$, then $T_n\langle 1, 2; 3, t_2\rangle$ is hamiltonian for all n, and if $t_2 \equiv 0, 2 \pmod 3$, then $T_n\langle 1, 2; 3, t_2\rangle$ is hamiltonian for all $n \neq t_2 + 3$. We refine the results for $t_1 = 3$ in [22] as, $T_n\langle 1, 2; 3, t\rangle$ is hamiltonian for all n and t, by proving that $T_n\langle 1, 2; 3, t\rangle$ is hamiltonian for $n = t + 3$ for all t. And we also prove that $T_{12}\langle 1, 2; 5, 7\rangle$ and $T_{12}\langle 1, 2; 5, 9\rangle$ are non-hamiltonian.

Then we extend the investigation of $T_n\langle 1, 2; t_1 \leq 5, t_2\rangle$ to $t_1 \geq 6$, so we discuss the hamiltonicity in Toeplitz graphs $T_n\langle 1, 2; t_1 \geq 6, t_2\rangle$, which is our main result. The paper concludes with a conjecture which completes the hamiltonicity investigation in Toeplitz graphs $T_n\langle 1, 2; t_1, t_2\rangle$.

2 Toeplitz Graphs $T_n\langle 1, 2; t\rangle$

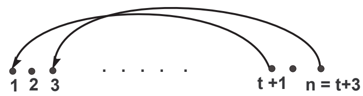
Remark 1: Let H be a hamiltonian cycle in the Toeplitz graph $T_n\langle 1, 2; t\rangle$. Then H is determined by the paths $H_{1\to n}$ and $H_{n\to 1}$. The path $H_{n\to 1}$ contains no pair of successive vertices (except $1, 2$ or $n - 1, n$) and hence its increasing edges are of length 2 only, because the path $H_{1\to n}$ is hamiltonian in the subgraph of $T_n\langle 1, 2; t\rangle$ induced by $(V(G) \setminus V(H_{n\to 1})) \cup \{1, n\}$ in which increasing edges of lengths 1 and 2 only are used.

Theorem 2.1 *[18] For even t, $T_n\langle 1, 2; t\rangle$ is hamiltonian if and only if n is odd.*

We will prove the following lemmas, which will be used in proving Theorems 2.7 and 2.10.

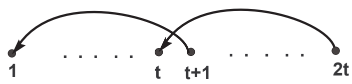
Lemma 2.2 *$T_{t+3}\langle 1,2;t\rangle$ is hamiltonian if and only if t is even.*

Proof. Let t be even. Then by Theorem 2.1, $T_{t+3}\langle 1,2;t\rangle$ is hamiltonian. Conversely, let $T_{t+3}\langle 1,2;t\rangle$ be hamiltonian and H be a hamiltonian cycle in $T_{t+3}\langle 1,2;t\rangle$. Then $H = H_{1\to t+3} \cup H_{t+3\to 1}$. Since decreasing edges are of length t, we have $(t+3,3),(t+1,1) \in E(H_{t+3\to 1})$. Suppose then that t is odd, and there is no path $P_{3\to t+1}$ as vertices 3 and $t+1$ are of opposite parity and by Remark 1, the only increasing edges of $H_{t+3\to 1}$ are of length 2. This is a contradiction. Hence t is even.



Lemma 2.3 *$T_{2t}\langle 1,2;t\rangle$ is non-hamiltonian for all t.*

Proof. Assume, to the contrary, that $T_{2t}\langle 1,2;t\rangle$ is hamiltonian. Let H be a hamiltonian cycle of $T_{2t}\langle 1,2;t\rangle$. Then $(2t,t),(t+1,1) \in E(H_{2t\to 1})$. But now $t, t+1 \in V(H_{2t\to 1})$. By Remark 1, $H_{2t\to 1}$ does contains a pair of successive vertices. This is a contradiction.



Lemma 2.4 *$T_{2t+2}\langle 1,2;t\rangle$ is non-hamiltonian for all t.*

Proof. Assume, to the contrary, that $T_{2t+2}\langle 1,2;t\rangle$ is hamiltonian. Let H be a hamiltonian cycle of $T_{2t+2}\langle 1,2;t\rangle$. Since by Remark 1, $(2t+2,t+2),(t+1,1) \in E(H_{2t+2\to 1})$ is not true, we have $t+1, t+2 \in V(H_{2t+2\to 1})$. Hence $T_{2t+2}\langle 1,2;t\rangle$ is not hamiltonian for each t.

Theorem 2.5 *[18] $T_n\langle 1,2;5\rangle$ is hamiltonian for all $n \geqslant 29$.*

Theorem 2.6 *[19] $T_n\langle s;t\rangle$ is a cycle if and only if $gcd(s,t) = 1$ and $s+t = n$.*

By Theorem 2.5, $T_n\langle 1,2;5\rangle$ is hamiltonian for every $n \geqslant 29$. In the proof of Theorem 2.5, it was also shown that $T_n\langle 1,2;5\rangle$ is hamiltonian for $n \in \{6,9,11,14,16,17,19,21,22,24,25,26,27\}$. By Theorem 2.6, $T_7\langle 1,2;5\rangle$ is hamiltonian. We show that $T_n\langle 1,2;5\rangle$ is not hamiltonian for $n \in \{8,10,12,13,15,18,20,23,28\}$. This refines Theorem 2.5.

Theorem 2.7 *$T_n\langle 1,2;5\rangle$ is hamiltonian if and only if $n \geq 29$ or $n \in \{6,7,9,11,14,16,17,19,21,22,24,25,26,27\}$.*

Proof. By Theorem 2.5 and Theorem 2.6, $T_n\langle 1,2;5\rangle$ is hamiltonian for all $n \geq 29$ and $n \in \{6,7,9,11,14,16,17,19,21,22,24,25,26,27\}$. For the converse, we show that $T_n\langle 1,2;5\rangle$ is not hamiltonian for $n \in \{8,10,12,13,15,18,20,23,28\}$. By Lemma 2.2, Lemma 2.3, and Lemma 2.4, $T_n\langle 1,2;5\rangle$ is not hamiltonian for $n = 8, n = 10$ and $n = 12$, respectively. For $n \in \{13,15,18,20,23,28\}$, suppose, to the contrary, that $T_n\langle 1,2;5\rangle$ is hamiltonian. Let H_n be a hamiltonian cycle in $T_n\langle 1,2;5\rangle$. The decreasing edges in $T_n\langle 1,2;5\rangle$ are of length 5 only. Since $n \not\equiv 1 \bmod 5$ for $n \in \{13,15,18,20,23,28\}$, $H_{n\to 1}$ cannot use only the decreasing edges. By Remark 1, $H_{n\to 1}$ cannot contain the path $(n, n-5, n-3, n-1, n-6)$, for otherwise $H_{n\to 1}$ contains a pair of successive vertices, namely $n-5$ and $n-6$, see Fig. 1a. Hence $H_{n\to 1} = (n, n-5, n-3, n-8) \cup H_{n-8\to 1}$, for each $n \in \{13,15,18,20,23, 28\}$. See Fig. 1b.

Figure 1: Possible paths of $H_{n\to 1}$ in $T_n\langle 1,2;5\rangle$; $n \in \{13,15,18,20,23,28\}$

Consider $n = 13$. $H_{13\to 1} = (13,8,10,5) \cup H_{5\to 1}$. Since the decreasing edges in $T_{13}\langle 1,2;5\rangle$ are of length 5 only, $(6,1) \in E(H_{13\to 1})$. But then $H_{13\to 1}$ contains pair of successive vertices, namely 5 and 6, see Fig. 2a. By Remark 1, this is not possible. Hence there is no path $H_{13\to 1}$ such that $H_{13} = H_{1\to 13} \cup H_{13\to 1}$.

Figure 2: Possible paths of $H_{n\to 1}$ in $T_n\langle 1,2;5\rangle$; $n \in \{13,15\}$

Consider $n = 15$. $H_{15\to 1} = (15,10,12,7) \cup H_{7\to 1}$. Since the decreasing edges in $T_{15}\langle 1,2;5\rangle$ are of length 5 only, $(6,1) \in E(H_{15\to 1})$. But then $H_{15\to 1}$ contains a pair of successive vertices, namely 6 and 7, see Fig. 2b. This is a contradiction.

Consider $n = 18$. $H_{18\to 1} = (18,13,15,10) \cup H_{10\to 1}$. This is shown in Fig. 3a. By Lemma 2.3, there is no path $H_{10\to 1}$. This is a contradiction.

Consider $n = 20$. $H_{20\to 1} = (20,15,17,12) \cup H_{12\to 1}$. This is shown in Fig. 3b. By Lemma 2.4, there is no path $H_{12\to 1}$. This a contradiction.

Consider $n = 23$. $H_{23\to 1} = (23,18,20,15) \cup H_{15\to 1}$. This is shown in Fig. 3c. As in the case of $n = 15$, there is no path $H_{15\to 1}$. This is a contradiction.

Consider $n = 28$. $H_{28\to 1} = (28,23,25,20) \cup H_{20\to 1}$. This is shown in Fig. 3d. As in the case of $n = 20$, there is no path $H_{20\to 1}$. This is a contradiction.

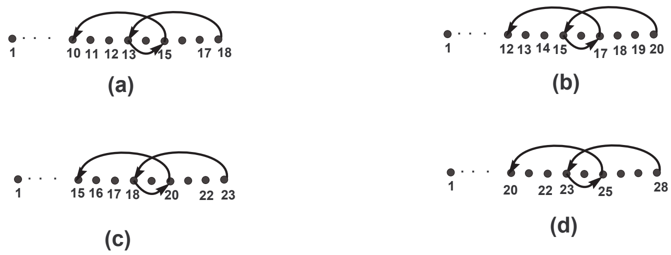


Figure 3: Possible paths of $H_{n\to 1}$ in $T_n\langle 1,2;5\rangle$; $n \in \{18, 20, 23, 28\}$

This completes the proof.

The following lemma will be applied in the proof of Theorem 2.10.

Lemma 2.8 *For every odd $t \geq 7$, $T_n\langle 1,2;t\rangle$ is hamiltonian for $n \in \{3t, 3t{+}2, 3t{+}4\}$.*

Proof. In Fig. 4, we display a hamiltonian cycle $(1,2,3,5,7,\ldots,t+2,t+3,t+5,t+6,\ldots,2t-1,2t+1,2t+3,2t+5,2t+6,\ldots,3t,2t,2t+2,2t+4,t+4,4,6,\ldots,t+1,1)$ in $T_{3t}\langle 1,2;t\rangle$.

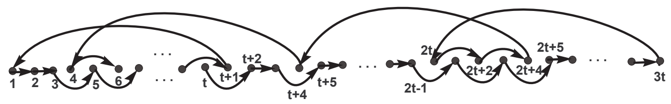


Figure 4: A hamiltonian cycle in $T_{3t}\langle 1,2;t\rangle$

In Fig. 5a, we show a hamiltonian cycle $(1,2,3,5,7,\ldots,t+2,t+3,t+5,t+6,\ldots,2t+1,2t+3,2t+5,2t+6,\ldots,3t+2,2t+2,2t+4,t+4,4,6,\ldots,t+1,1)$ in $T_{3t+2}\langle 1,2;t\rangle$.

In Fig. 5b, we show a hamiltonian cycle $(1,2,3,5,7,\ldots,t+2,t+3,t+5,t+6,\ldots,2t+3,2t+5,2t+6,\ldots,3t+4,2t+4,t+4,4,6,\ldots,t+1,1)$ in $T_{3t+4}\langle 1,2;t\rangle$.

Theorem 2.9 *[18] Let $t \geq 7$ be an odd integer. Then $T_n\langle 1,2;t\rangle$ is hamiltonian for all $n > 3t+5$.*

In Theorem 2.9, it was shown that for odd $t \geq 7$, $T_n\langle 1,2;t\rangle$ is hamiltonian for all $n > 3t+5$. We now address the cases for $n \leq 3t+5$.

Theorem 2.10 *For odd $t \geq 7$ and $n \leq 3t+5$, $T_n\langle 1,2;t\rangle$ is hamiltonian if and only if $n \in \{t+1,t+2,t+4,\ldots,2t+1,2t+4,2t+6,2t+7\ldots,3t+2,3t+4\}$.*

Proof. Let $t \geq 7$ be odd and $n \leq 3t{+}5$. It will be shown that $T_n\langle 1,2;t\rangle$ is hamiltonian for $n \in \{t+1,t+2,t+4,\ldots,2t+1,2t+4,2t+6,2t+7\ldots,3t+2,3t+4\}$.

Consider $n = t+2$. By Theorem 2.6, $T_{t+2}\langle 2;t\rangle$ is hamiltonian.

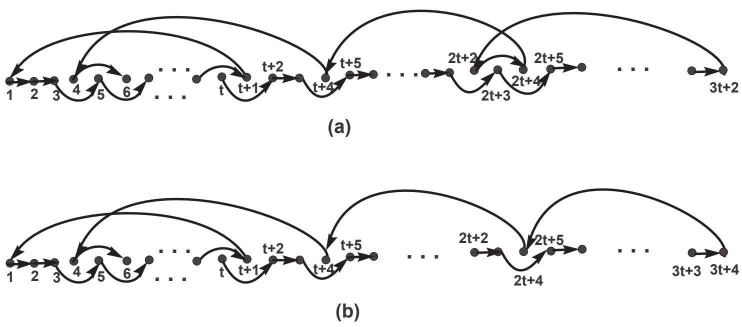


Figure 5: A hamiltonian cycle in (a) $T_{3t+2}\langle 1,2;t\rangle$, and (b) $T_{3t+4}\langle 1,2;t\rangle$

Consider $n \in \{t+1, 2t+1, 3t+1\}$. Since $n \cong 1 \bmod t$, $H_{n\to 1} = (n, n-t, n-2t, \ldots, t+1, 1)$. Then $H_{1\to n}$ is induced by $(V(G) \setminus V(H_{n\to 1})) \cup \{1, n\}$. See Fig. 6 for an illustration.

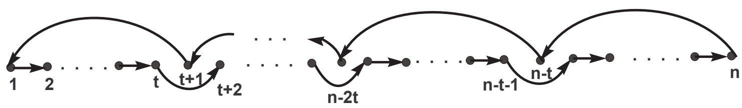


Figure 6: A hamiltonian cycle in $T_n\langle 1,2;t\rangle$; $n \in \{t+1, 2t+1, 3t+1\}$

Consider $n \in \{t+a, 2t+a\}$, $a \in \{4, 6, 8, \ldots, t-1\}$; that is, $n \in \{t+4, t+6\ldots, 2t-1, 2t+4, 2t+6, \ldots, 3t-1\}$. Here $n \cong a \bmod t$, so $n-(2t+a)+1 \cong 1 \bmod t$, which shows that there is a path $H_{n\to 2t+a} = (n, n-t, n-2t, \ldots, 2t+a)$ in $H_{n\to 1}$. Then $(2t+a, t+a), (t+a, a) \in E(H_{n\to 1})$. Since a and $t+1$ are both even, there is a path $H_{a\to t+1} = (a, a+2, \ldots, t+1)$ in $H_{n\to 1}$. Hence $H_{n\to 1} = H_{n\to 2t+a} \cup (2t+a, t+a, a) \cup H_{a\to t+1} \cup (t+1, 1)$. Then $H_{1\to n}$ is induced by $(V(G) \setminus V(H_{n\to 1})) \cup \{1, n\}$. See Fig. 7 for an illustration.

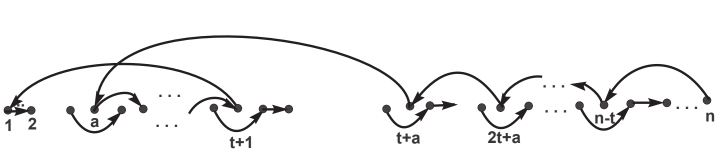


Figure 7: A hamiltonian cycle in $T_{n\in\{t+a,2t+a\}}\langle 1,2;t\rangle$; $a \in \{4, 6, 8, \ldots, t-1\}$

Consider $n \in \{3t, 3t+2, 3t+4\}$. By Lemma 2.8, $T_n\langle 1,2;t\rangle$ is hamiltonian for $n \in \{3t, 3t+2, 3t+4\}$.

Consider $n \in \{2t+7, 2t+9, \ldots, 3t-2\}$. Here $n-t$ and $n-3$ are both even and $n-t < n-3$. Hence there is a path $H_{n-t\to n-3} = (n-t, n-t+2, \ldots, n-3)$ in $H_{n\to 1}$. Similarly $n-3-2t$ and $t+1$ are both even. Hence there is a path

$H_{n-3-2t\to t+1} = (n-3-2t, n-1-2t, \ldots, t+1)$ in $H_{n\to 1}$. Hence $H_{n\to 1} = (n, n-t) \cup H_{n-t\to n-3} \cup (n-3, n-3-t, n-3-2t) \cup H_{n-3-2t\to t+1} \cup (t+1, 1)$. Fig. 8 shows a hamiltonian cycle $(1, 2, \ldots, n-2t-4, n-2t-2, \ldots, t+2, t+3, \ldots, n-4-t, n-2-t, n-1-t, n+1-t, n+3-t, \ldots, n-2, n-1, n, n-t, n-t+2, n-t+4, \ldots, n-3, n-3-t, n-3-2t, n-1-2t, n+1-2t, \ldots, t+1, 1)$ in $T_n\langle 1, 2; t\rangle$ for $n \in \{2t+7, 2t+9, \ldots, 3t-2\}$.

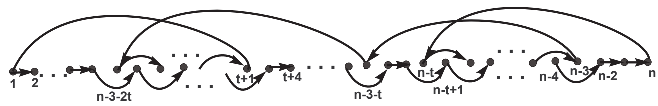


Figure 8: A hamiltonian cycle in $T_n\langle 1, 2; t\rangle$; $n \in \{2t+7, 2t+9, \ldots, 3t-2\}$

In summary, all the above cases show that, for odd $t \geq 7$ and $n \leq 3t+5$, $T_n\langle 1, 2; t\rangle$ is hamiltonian for $n \in \{t+1, t+2, t+4, \ldots, 2t+1, 2t+4, 2t+6, 2t+7 \ldots, 3t+2, 3t+4\}$.

Conversely, in order to prove that for odd $t \geq 7$ and $n \leq 3t+5$, $T_n\langle 1, 2; t\rangle$ is not hamiltonian for every $n \notin \{t+1, t+2, t+4, \ldots, 2t+1, 2t+4, 2t+6, 2t+7 \ldots, 3t+2, 3t+4\}$, we shall prove that $T_n\langle 1, 2; t\rangle$ is not hamiltonian for $n \in \{t+3, t+5, \ldots, 2t+2, 2t+3, 2t+5, 3t+3, 3t+5\}$.

Consider $n = 2t$. By Lemma 2.3, $T_{2t}\langle 1, 2; t\rangle$ is non-hamiltonian.

Consider $n = 2t+2$. By Lemma 2.4, $T_{2t+2}\langle 1, 2; t\rangle$ is non-hamiltonian.

Consider $n = t+b$, where $b = 3, 5, \ldots, t-2$, i.e, $n \in \{t+3, t+5, t+7, \ldots, 2t-2\}$. Assume, to the contrary, that $T_n\langle 1, 2; t\rangle$ is hamiltonian. Let H be a hamiltonian cycle in $T_n\langle 1, 2; t\rangle$. Then $H = H_{1\to n} \cup H_{n\to 1}$. Since the decreasing edges in $T_n\langle 1, 2; t\rangle$ are of length t only, $(n, n-t = b), (t+1, 1) \in E(H_{n\to 1})$. We have $b \leq t-2$. That is, there is no decreasing edge, say $(b, b-t)$ in $T_n\langle 1, 2; t\rangle$. Hence $H_{n\to 1} = (n, b) \cup H_{b\to t+1} \cup (t+1, 1)$. By Remark 1, $H_{n\to 1}$ does not contain a pair of successive vertices. Hence $H_{b\to t+1}$ uses increasing edges of length 2 only. Since b and $t+1$ are of opposite parity, there is no path $H_{b\to t+1}$ in $H_{n\to 1}$, for it terminates at vertex $t-2$. See Fig. 9 for an illustration. This is a contradiction.

Figure 9: Possible edges in $H_{t+b\to 1}$; $b = 3, 5, \ldots, t-2$

Consider $n = 2t+3$. Assume, to the contrary, that $T_{2t+3}\langle 1, 2; t\rangle$ is hamiltonian. Let H be a hamiltonian cycle in $T_{2t+3}\langle 1, 2; t\rangle$. Then $H = H_{1\to 2t+3} \cup H_{2t+3\to 1}$. Since the decreasing edges in $T_{2t+3}\langle 1, 2; t\rangle$ are of length t only, we have $(2t+3, t+3), (t+1, 1) \in E(H_{2t+3\to 1})$. For $n = t+3$, we saw in the previous case that there is no path $H_{t+3\to 1}$ in $T_{t+3}\langle 1, 2; 5\rangle$. Hence $(t+3, t+5) \in E(H_{2t+3\to 1})$, for $H_{2t+3\to 1}$ uses increasing edges of length 2 only. As in the previous case, there is no path $H_{t+5\to 1}$ in $T_{t+5}\langle 1,2;5\rangle$. Continuing in this way, there is not path $H_{t+b\to 1}$ in $T_{t+b}\langle 1,2;5\rangle$ for $b=3,5,\ldots,(t-2)$. See Fig. 10 for an illustration. Hence there is no path $H_{2t+3\to 1}$ in $T_{2t+3}\langle 1,2;5\rangle$. This is a a contradiction.

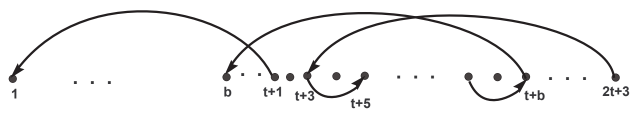


Figure 10: Possible edges in $H_{2t+3\to 1}$

Consider $n=2t+5$. This is similar to the case for $n=2t+3$.

Consider $n=3t+3$. Assume, to the contrary, that $T_{3t+3}\langle 1,2;t\rangle$ is hamiltonian. Let H be a hamiltonian cycle in $T_{3t+3}\langle 1,2;t\rangle$. Then $H=H_{1\to 3t+3}\cup H_{3t+3\to 1}$. Since the decreasing edges in $T_{3t+3}\langle 1,2;t\rangle$ are of length t only, we have $(3t{+}3,2t{+}3),(t{+}1,1)\in E(H_{3t+3\to 1})$. As in the case for $n=2t+3$, there is no path $H_{2t+3\to 1}$ in $T_{2t+3}\langle 1,2;5\rangle$. Since $H_{3t+3\to 1}$ uses increasing edges of length 2 only, we have $(2t+3,2t+5)\in E(H_{3t+3\to 1})$. As in the preceding case, there is no path $H_{2t+5\to 1}$ in $T_{2t+5}\langle 1,2;5\rangle$. Hence $(2t{+}5,2t{+}7)\in E(H_{3t+3\to 1})$. If $(2t{+}7,t{+}7)\in E(H_{3t+3\to 1})$, then as in the case for $n\in\{t{+}a,2t{+}a\}$, there is no path $H_{t+7\to 1}$ in $T_{t+7}\langle 1,2;5\rangle$. Hence $(2t{+}7,2t{+}9)\in E(H_{3t+3\to 1})$. Similarly, $(2t{+}9,2t{+}11),(2t{+}11,2t{+}13),\ldots(3t{-}2,3t)\in E(H_{3t+3\to 1})$. Then $(3t,2t)\in E(H_{3t+3\to 1})$. See Fig. 11 for an illustration. By Lemma 2.3, there is no path $H_{2t\to 1}$. Hence the path $E(H_{3t+3\to 1})$ terminates at vertex $2t$. This is a contradiction.

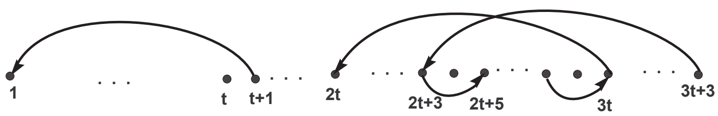


Figure 11: Possible edges in $H_{3t+3\to 1}$

Consider $n=3t+5$. This is similar to the case for $n=3t+3$.

This completes the proof.

3 Toeplitz Graphs $T_n\langle 1,2;t_1,t_2\rangle$

In this section, we make an improvement upon the work of [22]. For $t_1=3$, the following results were proved in [22].

Theorem 3.1 *[22] For even t, $T_n\langle 1,2;3,t\rangle$ is hamiltonian for all n.*

Theorem 3.2 *[22] Let t be an odd integer.*

(i) If $t \equiv 1 (mod\ 3)$ then $T_n\langle 1, 2; 3, t\rangle$ is hamiltonian for all n.

(ii) If $t \equiv 0, 2 (mod\ 3)$ then $T_n\langle 1, 2; 3, t\rangle$ is hamiltonian for all n different from $t + 3$.

We will prove that $T_{t+3}\langle 1, 2; 3, t\rangle$ is hamiltonian for all t, which will allow us to combine Theorems 3.1 and 3.2 as follows:

Theorem 3.3 *$T_n\langle 1, 2; 3, t\rangle$ is hamiltonian for all n and t.*

Proof. Let t be either even or odd and $t \equiv 1 \pmod 3$. By Theorems 3.1 and 3.2, $T_n\langle 1, 2; 3, t\rangle$ is hamiltonian for all n. If t is odd and $t \equiv 0, 2 \pmod 3$, then by Theorem 3.2, $T_n\langle 1, 2; 3, t\rangle$ is hamiltonian for all $n \neq t+3$. We now show that $T_{t+3}\langle 1, 2; 3, t\rangle$ is hamiltonian for all t. Fig. 12 shows a hamiltonian cycle $(1, 3, 5, 6, \ldots, t-1, t+1, t+3, t, t+2, 2, 4, 1)$ in $T_{t+3}\langle 1, 2; 3, t\rangle$. Thus $T_n\langle 1, 2; 3, t\rangle$ is hamiltonian for all n and t.

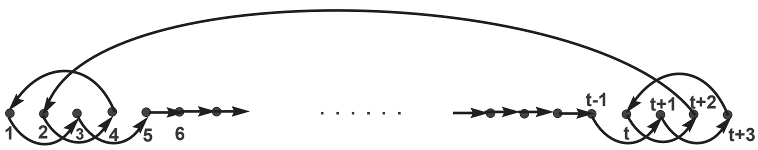


Figure 12: A hamiltonian cycle in $T_n\langle 1, 2; 3, t\rangle$

For $t_1 = 5$, the following results were proved in [22].

Theorem 3.4 *[22] $T_n\langle 1, 2; 5, t\rangle$ is hamiltonian for all n, and for all t different from 7 and 9.*

Theorem 3.5 *[22] $T_n\langle 1, 2; 5, 7\rangle$ is hamiltonian for all $n \notin \{10, 12\}$.*

Theorem 3.6 *[22] $T_n\langle 1, 2; 5, 9\rangle$ is hamiltonian for all n different from 12.*

Theorem 3.7 *[22] $T_{10}\langle 1, 2; 5, 7\rangle$ is non-hamiltonian.*

In Theorem 3.5, it was proved that $T_n\langle 1, 2; 5, 7\rangle$ is hamiltonian for all n different from 10 and 12. Theorem 3.7 asserts that $T_{10}\langle 1, 2; 5, 7\rangle$ is non-hamiltonian. We now prove that $T_{12}\langle 1, 2; 5, 7\rangle$ is non-hamiltonian. Theorem 3.6 asserts that $T_n\langle 1, 2; 5, 9\rangle$ is hamiltonian for all n different from 12. We now prove that $T_{12}\langle 1, 2; 5, 9\rangle$ is non-hamiltonian. Thus we can combine and refine Theorem 3.4-3.7 as follows:

Theorem 3.8 *$T_n\langle 1, 2; 5, t\rangle$ is hamiltonian if and only if (i) $t \neq 7$ and $n \notin \{10, 12\}$ or (ii) $t \neq 9$ and $n \neq 12$.*

Proof. By Theorem 3.4 and Theorem 3.5, $T_n\langle 1,2;5,t\rangle$ is hamiltonian for all $t \neq 7$ and all $n \notin \{10,12\}$. By Theorem 3.4 and Theorem 3.6, $T_n\langle 1,2;5,t\rangle$ is hamiltonian for all $t \neq 9$ and $n \neq 12$.

Conversely, (i) suppose that $t = 7$ and $n \in \{10,12\}$. Then we show that $T_n\langle 1,2;5,t\rangle$ is non-hamiltonian. By Theorem 3.7, $T_n\langle 1,2;5,t\rangle$ is non-hamiltonian for $t = 7$ and $n = 10$. We show that $T_n\langle 1,2;5,t\rangle$ is non-hamiltonian for $t = 7$ and $n = 12$. Assume, to the contrary, that $T_{12}\langle 1,2;5,7\rangle$ is hamiltonian. Let H be a hamiltonian cycle in $T_{12}\langle 1,2;5,7\rangle$. Thus $H = H_{1\to 12} \cup H_{12\to 1}$. Since the decreasing edges in $T_{12}\langle 1,2;5,7\rangle$ are of lengths 7 and 5 only, either $(12,7) \in E(H_{12\to 1})$ or $(12,5) \in E(H_{12\to 1})$. Since $H_{12\to 1}$ contains no pair of successive vertices, it has increasing edges of length 2 only. Thus the only possible paths that $H_{12\to 1}$ can follow are $(12,5,7,9,2)$, $(12,7,2,4)$ or $(12,7,9,2,4)$. Then the path $H_{12\to 1}$ terminates at vertices 2, 4 and 4, respectively. See Fig. 13. This is a contradiction. (ii) suppose

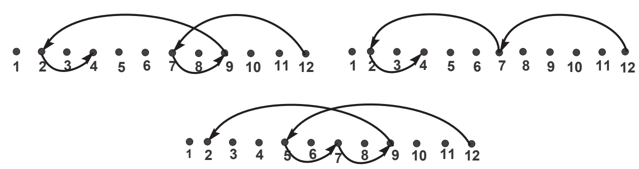


Figure 13: Possible paths of $H_{12\to 1}$ in $T_{12}\langle 1,2;5,7\rangle$

that $t = 9$ and $n = 12$. Then we show that $T_n\langle 1,2;5,t\rangle$ is non-hamiltonian. Assume, to the contrary, that $T_{12}\langle 1,2;5,9\rangle$ is hamiltonian. Let H be a hamiltonian cycle in $T_{12}\langle 1,2;5,9\rangle$. Then $H = H_{1\to 12} \cup H_{12\to 1}$. The decreasing edges in $T_{12}\langle 1,2;5,9\rangle$ are of lengths 9 and 5 only. Hence, either $(12,3)$ or $(12,7) \in E(H_{12\to 1})$. Since $H_{12\to 1}$ contains no pair of successive vertices, it has increasing edges of length 2 only. The only paths that $H_{12\to 1}$ can follow are $(12,3,5,7,9,11)$ or $(12,7,2,4)$ only. These paths terminate at vertices 11 and 4, respectively, for otherwise $H_{12\to 1}$ would have two successive vertices. See Fig. 14. This is a contradiction.

The proof is complete.

Figure 14: Possible paths of $H_{12\to 1}$ in $T_{12}\langle 1,2;5,9\rangle$

The hamiltonicity of $T_n\langle 1,2;t_1 \leq 5,t_2\rangle$ was investigated in [22]. We now discuss the hamiltonicity of $T_n\langle 1,2;t_1 \geq 6,t_2\rangle$.

Theorem 3.9 *Let* $G = T_n\langle 1,2;t_1 \geq 6,t_2\rangle$.

1. *If* t_1 *and* t_2 *both are even, then* G *is hamiltonian if and only if* n *is odd.*

2. If t_1 *and* t_2 *are of opposite parity, then* G *is hamiltonian for all* n.

3. If t_1 *and* t_2 *both are odd, and*

(*a*) *if* $t_2 \geq 2t_1 + 1$, *then* G *is hamiltonian for all* n.

(*b*) *if* $t_2 < 2t_1 + 1$, *then* G *is hamiltonian for all* $n \notin \{t_2+3, t_2+5, \ldots, 2t_1+2\}$.

Proof.

Case 1: Let t_1 and t_2 be both even. By Theorem 2.1, G is hamiltonian if and only if n is odd.

Case 2: Let t_1 and t_2 be of opposite parity.

Let t_1 be odd, and hence $t_1 \geq 7$. By Theorem 2.9, G is hamiltonian for all $n > 3t_1+5$. For $n \leq 3t_1+5$, Theorem 2.10 asserts that $T_n\langle 1, 2; t_1\rangle$ is hamiltonian for all $n \notin \{t_1+3, t_1+5, \ldots, 2t_1+2, 2t_1+3, 2t_1+5, 3t_1+3, 3t_1+5\}$. Hence for $n \leq 3t_1+5$, G is hamiltonian for all $n \notin \{t_1+3, t_1+5, \ldots, 2t_1+2, 2t_1+3, 2t_1+5, 3t_1+3, 3t_1+5\}$. Furthermore, by Theorem 2.1, G is hamiltonian for $n \in \{2t_1+3, 2t_1+5\}$, because here n is odd and t_2 is even. We now verify the hamiltonicity of G for $n \in \{t_1+3, t_1+5, \ldots, 2t_1+2, 3t_1+3, 3t_1+5\}$. Since $t_2 \geq t_1+1 \Rightarrow t_2+2 \geq t_1+3$ and $n \geq t_2+1 \Rightarrow n \geq t_2+2$, and n and t_2 are both even, $t_2+2 \leq n \leq 3t_1+5$. Thus $\{t_1+3, t_1+5, \ldots, 2t_1+2, 3t_1+3, 3t_1+5\} = \{t_2+2, t_2+4, \ldots, 2t_1+2, 3t_1+3, 3t_1+5\}$. We show that G is hamiltonian for $n \in \{t_2+2, t_2+4, \ldots, 2t_1+2, 3t_1+3, 3t_1+5\}$. If $t_2 \leq 2t_1$, then $n \in \{t_2+2, t_2+4, \ldots, 2t_1+2, 3t_1+3, 3t_1+5\}$. If $2t_1 < t_2 \leq 3t_1+1$, then $n \in \{3t_1+3, 3t_1+5\}$. If $t_2 = 3t_1+3$, then $n = 3t_1+5$. If $n \in \{t_2+2, t_2+4, \ldots, 2t_1+2\}$, then $t_1+1 \leq t_2 \leq 2t_1$. Then $n - t_2 \leq t_1+1$ and $n-t_2$ and t_1+1 are of the same parity. For $n-t_2 = t_1+1$, Fig. 15a shows a hamiltonian cycle $(1, 2, \ldots, t_1, t_1+2, t_1+3, \ldots, n, n-t_2 = t_1+1, 1)$. For $n-t_2 < t_1+1$, Fig. 15b shows a hamiltonian cycle $(1, 2, \ldots, n-t_2-1, n-t_2+1, \ldots, t_1+2,\ t_1+3, \ldots, n, n-t_2, n-t_2+2, \ldots, t_1+1, 1)$ in G.

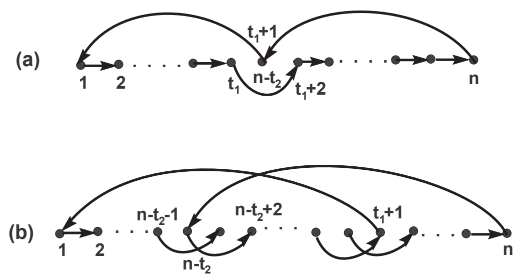


Figure 15: $n - t_2 \leq t_1 + 1$

If $n \in \{3t_1+3, 3t_1+5\}$, then $t_1+1 \leq t_2 \leq 3t_1+3$. For $t_2 \neq t_1+1$, $n-t_2 < t_1+t_2+1$. Since $n-t_2$ and t_1+t_2+1 are of the same parity, Fig. 16 shows a hamiltonian cycle $(1, 2, \ldots, t_1, t_1+2, t_1+3 \ldots, n-t_2-1, n-t_2+1, \ldots, t_1+t_2+2, t_1+t_2+3, \ldots, n, n-t_2, n-t_2+2, \ldots, t_1+t_2+1, t_1+1, 1)$ in G.

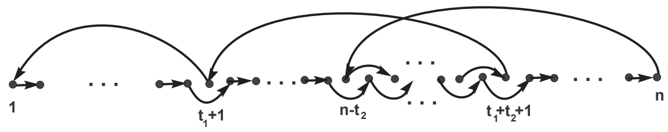


Figure 16: $n \in \{3t_1+3, 3t_1+5\}$ and $t_2 \neq t_1+1$

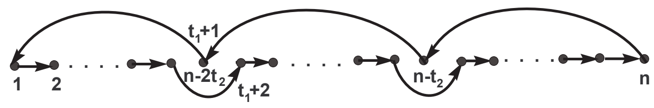


Figure 17: $n = 3t_1+3$ and $t_2 = t_1+1$

For $t_2 = t_1+1$ and $n = 3t_1+3$, Fig. 17 shows a hamiltonian cycle $(1, 2, \ldots, t_1, t_1+2, t_1+3, \ldots, n-t_2-1, n-t_2+1, n-t_2+2, \ldots, n, n-t_2, n-2t_2 = t_1+1, 1)$ in G.

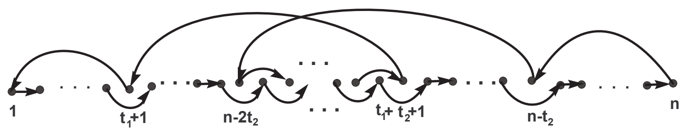


Figure 18: $n = 3t_1+5$ and $t_2 = t_1+1$

For $t_2 = t_1+1$ and $n = 3t_1+5$, $n-2t_2 < t_1+t_2+1$, Fig. 18 shows a hamiltonian cycle $(1, 2, \ldots, t_1, t_1+2, t_1+3, \ldots, n-2t_2-1, n-2t_2+1, \ldots, t_1+t_2+2, t_1+t_2+3, \ldots, n-t_2-1, n-t_2+1, n-t_2+2, \ldots, n, n-t_2, n-2t_2, n-2t_2+2, \ldots, t_1+t_2+1, t_1+1, 1)$ in G.

Let t_2 be odd. Then $t_2 \geq 7$. By Theorem 2.9, G is hamiltonian for all $n > 3t_2+5$. For $n \leq 3t_2+5$, by Theorem 2.10, G is hamiltonian for all $n \notin \{t_2+3, t_2+5, \ldots, 2t_2+2, 2t_2+3, 2t_2+5, 3t_2+3, 3t_2+5\}$. Since t_1 is even and n is odd, by Theorem 2.1, G is hamiltonian for $n \in \{2t_2+3, 2t_2+5\}$. We now show that G is hamiltonian for $n \in \{t_2+3, t_2+5, \ldots, 2t_2+2, 3t_2+3, 3t_2+5\}$. We have $n-t_2$ and t_1+1 are both odd. Hence we consider the cases $n-t_2 = t_1+1$, $n-t_2 < t_1+1$ and $n-t_2 > t_1+1$. If $n-t_2 = t_1+1$, then Fig. 15a shows a hamiltonian cycle $(1, 2, \ldots, t_1, t_1+2, t_1+3, \ldots, n, t_1+1, 1)$ in G. If $n-t_2 < t_1+1$, then Fig. 15b shows a hamiltonian cycle $(1, 2, \ldots, n-t_2-1, n-t_2+1, \ldots, t_1+2, t_1+3, \ldots, n, n-t_2, n-t_2+2, \ldots, t_1+1, 1)$ in G. If $n-t_2 > t_1+1$, then $n-t_1 > t_2+1$. Let $k = \lceil \frac{n-t_2-1}{t_1} \rceil$. Then $n-kt_1 \leq t_2+1$, since $k = \lceil \frac{n-t_2-1}{t_1} \rceil \geq \frac{n-t_2-1}{t_1}$. Then $n-kt_1$ and t_2+1 are both even. Fig. 19 shows a hamiltonian cycle $(1, 2, \ldots, n-kt_1-1, n-kt_1+1, \ldots, t_2+2, t_2+3, \ldots, n-(k-1)t_1-1, n-(k-1)t_1+1, n-(k-1)t_1+2, \ldots, n-(k-2)t_1-1, \ldots, n-2t_1-1, n-2t_1+1, n-2t_1+2, \ldots, n-t_1-1, n-t_1+1, n-t_1+2, \ldots, n, n-t_1, n-2t_1, \ldots, n-kt_1, n-kt_1+2, \ldots, t_2+1, 1)$ in G.

Case 3: If t_1 and t_2 are both odd, then $t_1 \geq 7$ and $t_2 \geq 9$. By Theorem 2.9,

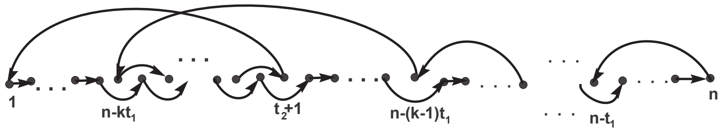


Figure 19: $n \leq 3t_1 + 5$, $n - t_2 > t_1 + 1$ and $n - kt_1 \leq t_2 + 1$

G is hamiltonian for all $n > 3t_1 + 5$. For $n \leq 3t_1 + 5$, we proved in Theorem 2.10 that $T_n\langle 1, 2; t_1\rangle$ is hamiltonian for all $n \notin \{t_1 + 3, t_1 + 5, \ldots, 2t_1 + 2, 2t_1 + 3, 2t_1 + 5, 3t_1 + 3, 3t_1 + 5\}$. Now for $n \leq 3t_1 + 5$, we verity the hamiltonicity of G for $n \in \{t_1 + 3, t_1 + 5, \ldots, 2t_1 + 2, 2t_1 + 3, 2t_1 + 5, 3t_1 + 3, 3t_1 + 5\}$. We will consider two cases: $t_2 \geq 2t_1 + 1$ and $t_2 < 2t_1 + 1$.

(a) $t_2 \geq 2t_1 + 1$.

Since $n \geq t_2 + 1$ and $2t_1 + 1 \leq t_2$, $2t_1 + 2 \leq t_2 + 1 \leq n$. We shall verify the hamiltonicity for $n \in \{2t_1 + 2, 2t_1 + 3, 2t_1 + 5, 3t_1 + 3, 3t_1 + 5\}$. If $n = 2t_1 + 2$, then $t_2 = 2t_1 + 1$. Then $(1, 2, \ldots, n, n - t_2 = 1)$ is a hamiltonian cycle in G. If $n = 2t_1 + 3$, then $t_2 = 2t_1 + 1$. Fig. 20 shows a hamiltonian cycle $(1, 3, \ldots, t_1 + 2, t_1 + 3, \ldots, 2t_1 + 3, n - t_2 = 2, 4, \ldots, t_1 + 1, 1)$ in G.

If $n = 2t_1 + 5$, then $t_2 \in \{2t_1 + 1, 2t_1 + 3\}$. For $t_2 = 2t_1 + 1$, Fig. 21 shows a hamiltonian cycle $(1, 2, 3, 5, \ldots, t_1+2, t_1+3, \ldots, n, n-t_2 = 4, 6, \ldots, t_1+1, 1)$ in G. For $t_2 = 2t_1+3$, Fig. 22 shows a hamiltonian cycle $(1, 3, 5, \ldots, t_1+2, t_1+3, \ldots, n, n-t_2 = 2, 4, \ldots, t_1 + 1, 1)$ in G. If $n = 3t_1 + 3$, then $2t_1 + 1 \leq t_2 \leq 3t_1 + 2$. For $t_2 = 3t_1 + 2$, G has a hamiltonian cycle $(1, 2, \ldots, n, n - t_2 = 1)$. For $2t_1 + 1 \leq t_2 \leq 3t_1$, Fig. 23 shows a hamiltonian cycle $(1, 2, \ldots, n - 2 - t_2, n - t_2, \ldots, t_1 + 2, t_1 + 3, \ldots, n - t_1 - 1, n - t_1 + 1, \ldots, n, n - t_1, n - t_1 + 2, \ldots, n - 1, n - 1 - t_2, n + 1 - t_2, \ldots, t_1 + 1, 1)$ in G.

If $n = 3t_1 + 5$, then $2t_1 + 1 \leq t_2 \leq 3t_1 + 4$. For $t_2 = 2t_1 + 1$, Fig. 24 shows a hamiltonian cycle $(1, 2, \ldots, t_1, t_1+2, t_1+3, \ldots, n-t_1-1, n-t_1+1, \ldots, n, n-t_1, n-t_1 + 2, \ldots, n - 3, n - 3 - t_2 = t_1 + 1, 1)$ in G. For $2t_1 + 3 \leq t_2 \leq 3t_1 + 2$, Fig. 25 shows a hamiltonian cycle $(1, 2, \ldots, n - 2 - t_2, n - t_2, \ldots, t_1 + 2, t_1 + 3, \ldots, n - t_1 - 1, n - t_1 + 1, \ldots, n, n - t_1, n - t_1 + 2, \ldots, n - 1, n - 1 - t_2, n + 1 - t_2, \ldots, t_1 + 1, 1)$ in G. For $t_2 = 3t_1 + 4$, G has a hamiltonian cycle $(1, 2, 3, \ldots, 3t_1 + 5, 1)$.

(b) $t_2 < 2t_1 + 1$.

Since $t_2 \geq t_1+1$ and t_1 and t_2 are of same parity, $t_2 \geq t_1+2$. Now $t_1+3 \leq t_2+1 \leq n$ and here $t_2 + 1 < 2t_1 + 2$. Hence $\{t_1 + 3, t_1 + 5, \ldots, 2t_1 + 2, 2t_1 + 3, 2t_1 + 5, 3t_1 + 3, 3t_1 + 5\} = \{t_2 + 1, t_2 + 3, t_2 + 5 \ldots, 2t_1 + 2, 2t_1 + 3, 2t_1 + 5, 3t_1 + 3, 3t_1 + 5\}$. By Theorem 2.10, G is hamiltonian for $n = t_2 + 1$. In order to show that G is hamiltonian for all $n \notin \{t_2 + 3, t_2 + 5, \ldots, 2t_1 + 2\}$, we show that G is hamiltonian for $n \in \{2t_1 + 3, 2t_1 + 5, 3t_1 + 3, 3t_1 + 5\}$. For $n \in \{2t_1 + 3, 2t_1 + 5\}$, then $n - t_2$ and $t_1 + 1$ are of same parity and $n - t_2 \leq t_1 + 1$. Thus G has a hamiltonian cycle $(1, 2, \ldots, n - t_2 - 1, n - t_2 + 1, \ldots, t_2 + 2, t_2 + 3, \ldots, n, n - t_2, n - t_2 + 2, \ldots, t_2 + 1, 1)$. See Fig. 15. For $n \in \{3t_1+3, 3t_1+5\}$, then $n-t_2$ and t_1+t_2+1 are of same parity and

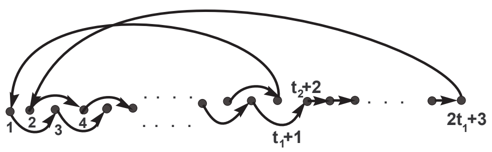


Figure 20: $n = 2t_1 + 3$ and $t_2 = 2t_1 + 1$

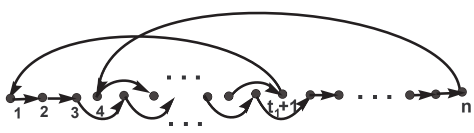


Figure 21: $n = 2t_1 + 5$ and $t_2 = 2t_1 + 1$

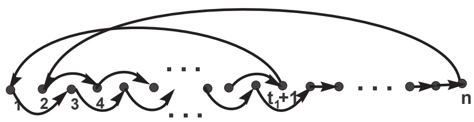


Figure 22: $n = 2t_1 + 5$ and $t_2 = 2t_1 + 3$

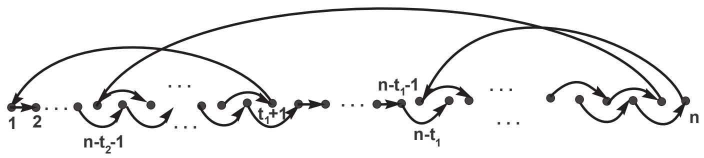


Figure 23: $n = 3t_1 + 3$ and $2t_1 + 1 \leq t_2 \leq 3t_1$

$n - t_2 \leq t_1 + t_2 + 1$. Then G has a hamiltonian cycle $(1, 2, \ldots, t_1, t_1+2, t_1+3, \ldots, n-t_2-1, n-t_2+1, \ldots, t_1+t_2+2, t_1+t_2+3, \ldots, n, n-t_2, n-t_2+2, \ldots, t_1+t_2+1, t_1+1, 1)$. See Fig. 16.

This completes the proof.

Conjecture: For odd $t_1 \geq 7$ and odd $t_2 < 2t_1 + 1$, $T_n\langle 1, 2; t_1, t_2\rangle$ is non-hamiltonian for $n \in \{t_2 + 3, t_2 + 5, \ldots, 2t_1 + 2\}$.

4 Concluding Remarks

The main result of this paper is an investigation of hamiltonicity of the Toeplitz graph $T_n\langle 1, 2; t_1 \geq 6, t_2\rangle$. An affirmative resolution of the conjecture above for $T_n\langle 1, 2; t_1 \geq 6, t_2\rangle$ would complete the study of hamiltonicity of $T_n\langle 1, 2; t_1, t_2\rangle$.

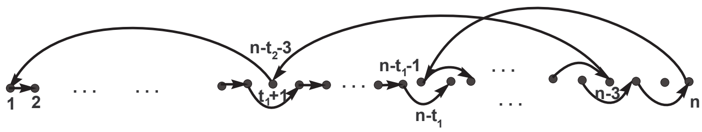


Figure 24: $n = 3t_1 + 5$ and $t_2 = 2t_1 + 1$

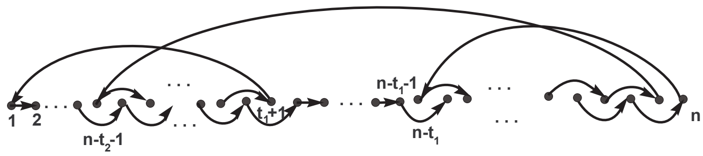


Figure 25: $n = 3t_1 + 5$ and $2t_1 + 3 \leq t_2 \leq 3t_1 + 2$

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