AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **79(1)** (2021), Pages 31–54

The Bose representation of PG$(2, q^3)$ in PG$(8, q)$

S.G. Barwick Wen-Ai Jackson

School of Mathematical Sciences
University of Adelaide, Adelaide, 5005
Australia
susan.barwick@adelaide.edu.au wen.jackson@adelaide.edu.au

Peter Wild

Royal Holloway
University of London, TW20 0EX
U.K.
peterrwild@gmail.com

Abstract

This article looks at the Bose representation of PG$(2, q^3)$ in PG$(8, q)$ which represents points of PG$(2, q^3)$ by planes of a regular 2-spread $\mathbb{S}$. This representation is also known as field reduction. We determine the representation of conics, $\mathbb{F}_q$-sublines, $\mathbb{F}_q$-subplanes and $\mathbb{F}_q$-conics of PG$(2, q^3)$ in PG$(8, q)$. The main focus is to study the interaction of the corresponding varieties with the transversal planes of $\mathbb{S}$ in PG$(8, q^3)$ and also in PG$(8, q^6)$, and to illustrate why the transversal planes play a pivotal role in characterising objects in this representation.

1 Introduction

Bose [8] gave a construction to represent PG$(2, q^2)$ using a regular 1-spread in PG$(5, q)$. Bose's construction generalises to the Bose representation of PG$(2, q^h)$ using a regular $(h-1)$-spread in PG$(3h-1, q)$. The Bose representation is an example of the technique of field reduction. This idea goes back to Segre [21] who introduced Desarguesian spreads arising from field reduction. The technique of field reduction has been an area of much recent interest, see [18] for a survey.

The Bruck-Bose representation [1, 9, 10] of PG$(2, q^2)$ in PG$(4, q)$ and the Bose representation [8] of PG$(2, q^2)$ in PG$(5, q)$ have been extensively studied, see [2] for a survey. The Bruck-Bose representation uses a regular 1-spread $\mathcal{S}$ in the hyperplane at infinity $\Sigma_\infty \cong$ PG$(3, q)$ of PG$(4, q)$; and the Bose representation uses a regular 1-spread $\mathbb{S}$ in PG$(5, q)$. The regular 1-spread $\mathcal{S}$ has two unique transversal lines g, g^q in the quadratic extension PG$(4, q^2)$, and the regular 1-spread $\mathbb{S}$ has two

ISSN: 2202-3518

unique transversal planes in the quadratic extension $\mathrm{PG}(5, q^2)$. The interaction of varieties of $\mathrm{PG}(4, q)$ with the transversals lines of $\mathcal{S}$ (and varieties of $\mathrm{PG}(5, q)$ with the transversal planes of $\mathbb{S}$) is intrinsic to characterisations of the corresponding sets in $\mathrm{PG}(2, q^2)$. We list four examples in the Bruck-Bose representation in $\mathrm{PG}(4, q)$.
(A) A non-degenerate conic $\mathcal{C}$ in $\mathrm{PG}(4, q)$ corresponds to a Baer subline of $\mathrm{PG}(2, q^2)$ if and only if the extension of $\mathcal{C}$ to $\mathrm{PG}(4, q^2)$ contains a point of g and a point of g^q; see [12].
(B) A ruled cubic surface $\mathcal{V}_2^3$ in $\mathrm{PG}(4, q)$ corresponds to a Baer subplane of $\mathrm{PG}(2, q^2)$ if and only if the extension of $\mathcal{V}_2^3$ to $\mathrm{PG}(4, q^2)$ contains g and g^q; see [12].
(C) An orthogonal cone $\mathcal{U}$ corresponds to a classical unital of $\mathrm{PG}(2, q^2)$ if and only if the extension of $\mathcal{U}$ to $\mathrm{PG}(4, q^2)$ contains g and g^q; see [20].
(D) A normal rational curve $\mathcal{N}$ in $\mathrm{PG}(4, q)$ corresponds to an $\mathbb{F}_q$-conic of $\mathrm{PG}(2, q^2)$ if and only if $\mathcal{N}$ is 2-special: a notion which describes how the extension of $\mathcal{N}$ to $\mathrm{PG}(4, q)$ meets the transversal lines g, g^q; see [6, Theorem 6.2].
This last characterisation is more complex to prove than the others. The proof uses the interplay between the Bruck-Bose representation of $\mathrm{PG}(2, q^2)$ in $\mathrm{PG}(4, q)$, and the Bose representation of $\mathrm{PG}(2, q^2)$ in $\mathrm{PG}(5, q)$. Moreover, the first three of these characterisations arise naturally when working in the Bose representation in $\mathrm{PG}(5, q)$.

This article aims to generalise these ideas to $\mathrm{PG}(2, q^3)$. The Bruck-Bose representation of $\mathrm{PG}(2, q^3)$ uses a regular 2-spread in the hyperplane at infinity $\Sigma_\infty \cong \mathrm{PG}(5, q)$. This regular 2-spread has three unique transversal lines g, g^q, g^{q^2} in the cubic extension. These transversal lines again play a pivotal role in characterising objects. For example: a twisted cubic $\mathcal{N}$ in $\mathrm{PG}(6, q)$ corresponds to a Baer subline of $\mathrm{PG}(2, q^3)$ if and only if the extension of $\mathcal{N}$ to $\mathrm{PG}(6, q^3)$ meets the transversal lines; see [3, Theorem 2.5]. A ruled quintic surface $\mathcal{V}_2^5$ in $\mathrm{PG}(6, q)$ corresponds to a Baer subplane of $\mathrm{PG}(2, q^3)$ if and only if the extension of $\mathcal{V}_2^5$ to $\mathrm{PG}(6, q^3)$ contains g, g^q, g^{q^2}; see [4]. Further, in [5], the carriers and sublines of an exterior splash are characterised using the transversals lines.

In this article we look at varieties in the Bose representation of $\mathrm{PG}(2, q^3)$ in $\mathrm{PG}(8, q)$, and investigate their relationship with the three transversal planes of the associated regular 2-spread. In particular, we determine the representation of conics, $\mathbb{F}_q$-sublines, $\mathbb{F}_q$-subplanes and $\mathbb{F}_q$-conics of $\mathrm{PG}(2, q^3)$ in $\mathrm{PG}(8, q)$. Further, by determining the extensions of the corresponding varieties to $\mathrm{PG}(8, q^3)$ and $\mathrm{PG}(8, q^6)$, we look at their interplay with the transversal planes of the regular 2-spread. This interplay illustrates why the characterisations in the previous paragraph hold. In further work [7], the authors use these varieties to characterise which normal rational curves in $\mathrm{PG}(6, q)$ correspond to $\mathbb{F}_q$-conics in the Bruck-Bose representation of $\mathrm{PG}(2, q^3)$ in $\mathrm{PG}(6, q)$.

The article is set out as follows. In Section 2, we describe the Bose representation of $\mathrm{PG}(2, q^3)$ in $\mathrm{PG}(8, q)$ using a regular 2-spread $\mathbb{S}$ of $\mathrm{PG}(8, q)$. We introduce coordinates for the Bose representation. In order to study the interaction of varieties with transversal planes of the regular 2-spread $\mathbb{S}$, we calculate coordinates for the transversal planes. Further, we need a suitable coordinate description of certain points in $\mathrm{PG}(8,q^3)$, and we determine these by looking at the conjugacy map with respect to an $\mathbb{F}_q$-subplane. Section 3 looks at a variety of $\mathrm{PG}(2,q^3)$, and uses coordinates to describe the corresponding variety in $\mathrm{PG}(8,q)$. We discuss the extension of the variety to $\mathrm{PG}(8,q^3)$, give a geometric description, and describe how the variety meets the transversal planes of $\mathbb{S}$. Section 3.3 describes a convention used in the literature when discussing extensions in the Bruck-Bose and Bose representations. In Section 4, we discuss some well known geometrical objects, namely quadrics, scrolls, and Segre varieties. These are essential objects in proving the remaining results of the article. The machinery that has been developed in the article is then used to look at substructures of $\mathrm{PG}(2,q^3)$ in the $\mathrm{PG}(8,q)$ representation; to determine the extension of the resulting varieties to $\mathrm{PG}(8,q^3)$ and to $\mathrm{PG}(8,q^6)$; and to describe their interplay with the transversal planes of the regular 2-spread. Section 5 looks at conics of $\mathrm{PG}(2,q^3)$, Section 6 looks at $\mathbb{F}_q$-sublines and $\mathbb{F}_q$-subplanes of $\mathrm{PG}(2,q^3)$, and Section 7 looks at $\mathbb{F}_q$-conics of $\mathrm{PG}(2,q^3)$.

2 The Bose representation

2.1 Preliminaries

We denote the unique finite field of prime power order q by $\mathbb{F}_q$, and let $\mathbb{F}_q^* = \mathbb{F}_q\backslash\{0\}$. We use uppercase letters to denote points of $\mathrm{PG}(n,q)$, and boldface to denote the (vector) homogeneous coordinates of a point. That is, if X is a point of $\mathrm{PG}(n,q)$, then X has homogenous coordinates $\boldsymbol{X} = (x_0,\ldots,x_n)$ for some $x_i \in \mathbb{F}_q$, not all 0. The Frobenius map $x \mapsto x^q$ for $x \in \mathbb{F}_{q^h}$ gives rise to an automorphic collineation $\boldsymbol{X} = (x_0,\ldots,x_n) \longmapsto \boldsymbol{X}^q = (x_0^q,\ldots,x_n^q)$ in $\mathrm{P\Gamma L}(n,q^h)$ of order h acting on points of $\mathrm{PG}(n,q^h)$ that fixes the points of $\mathrm{PG}(n,q)$ pointwise. We say the points $X, X^q \ldots, X^{q^{h-1}}$ are conjugate points with respect to the conjugacy map $X \mapsto X^q$.

A 2-*spread* of $\mathrm{PG}(8,q)$ is a set of planes that partition the points of $\mathrm{PG}(8,q)$. We use the following construction of a regular 2-spread of $\mathrm{PG}(8,q)$, see [11]. Embed $\mathrm{PG}(8,q)$ in $\mathrm{PG}(8,q^3)$ and consider the collineation $\boldsymbol{X} = (x_0,\ldots,x_8) \longmapsto \boldsymbol{X}^q = (x_0^q,\ldots,x_8^q)$ acting on $\mathrm{PG}(8,q^3)$. Let Γ be a plane in $\mathrm{PG}(8,q^3)$ which is disjoint from $\mathrm{PG}(8,q)$, such that $\Gamma,\Gamma^q,\Gamma^{q^2}$ span $\mathrm{PG}(8,q^3)$ (so any two span a 5-space which is disjoint from the third). For a point $X \in \Gamma$, the plane $\langle X, X^q, X^{q^2}\rangle$ of $\mathrm{PG}(8,q^3)$ meets $\mathrm{PG}(8,q)$ in a plane. The planes $\langle X, X^q, X^{q^2}\rangle \cap \mathrm{PG}(8,q)$ for $X \in \Gamma$ form a regular 2-spread of $\mathrm{PG}(8,q)$. The planes Γ, Γ^q and Γ^{q^2} are called the three *transversal planes* of the 2-spread. Conversely, any regular 2-spread of $\mathrm{PG}(8,q)$ has a unique set of three transversal planes in $\mathrm{PG}(8,q^3)$, and can be constructed in this way, see [11, Theorem 6.1].

We need the following general result on planes in 8-dimensional projective space.

Lemma 2.1 *Let α,β,γ be three planes which span $\mathrm{PG}(8,q)$. Let P be a point of $\mathrm{PG}(8,q)$ not on a line meeting two of α,β,γ. Then P lies on a unique plane that meets each of α,β,γ.*

Proof Let P be a point of $\mathrm{PG}(8,q)$ which is not on a line joining two of α,β,γ. So $\langle P,\alpha\rangle$ is a 3-space that does not meet β or γ. Hence $\Sigma_6=\langle P,\alpha,\beta\rangle$ is a 6-space. As α,β,γ span $\mathrm{PG}(8,q)$, Σ_6 meets γ in a point we denote by Q. So $\Sigma_4=\langle P,Q,\alpha\rangle$ is a 4-space contained in Σ_6. As α,β span a 5-space contained in Σ_6, Σ_4 meets β in a point denoted R. The two planes $\pi=\langle P,Q,R\rangle$ and α lie in the 4-space Σ_4, and so meet in a point S. That is, π meets α in the point S, meets β in the point R and meets γ in the point Q. That is, P lies on at least one plane that meets each of α,β,γ.

Suppose P lies in two distinct planes π_1,π_2 that meet each of α,β,γ, so $\langle\pi_1,\pi_1\rangle$ has dimension 3 or 4. Consider the set of six (possibly repeated) points $\mathcal{K}=\{\pi_i\cap\alpha,\ \pi_i\cap\beta,\ \pi_i\cap\gamma\,|\,i=1,2\}$. Suppose $\langle\pi_1,\pi_2\rangle$ is a 3-space, so $\pi_1\cap\pi_2=\ell$ is a line. As $P\in\ell$, by assumption at most one of the planes α,β,γ meets ℓ. Hence $|\mathcal{K}|\geq 5$, so the 3-space $\langle\pi_1,\pi_2\rangle$ meets two of the planes α,β,γ in a line, and meets the other in at least a point. This contradicts the three planes α,β,γ spanning $\mathrm{PG}(8,q)$. If $\langle\pi_1,\pi_2\rangle$ is a 4-space, then $|\mathcal{K}|=6$, and so $\langle\pi_1,\pi_2\rangle$ meets each of α,β,γ in a line, contradicting the three planes α,β,γ spanning $\mathrm{PG}(8,q)$. Hence P lies on at most one plane that meets each of α,β,γ. We conclude that P lies on exactly one plane that meets each of α,β,γ. □

2.2 The Bose representation of $\mathrm{PG}(2,q^3)$ in $\mathrm{PG}(8,q)$

Let $\mathbb{S}$ be a regular 2-spread in $\mathrm{PG}(8,q)$, we use the term $\mathbb{S}$-plane for a plane in $\mathbb{S}$. Let $\mathcal{I}_{\text{Bose}}$ be the incidence structure with *points* the q^6+q^3+1 $\mathbb{S}$-planes; *lines* the 5-spaces of $\mathrm{PG}(8,q)$ that contain two (and so q^3+1) $\mathbb{S}$-planes; and *incidence* is inclusion. The 5-spaces of $\mathrm{PG}(8,q)$ that meet $\mathbb{S}$ in q^3+1 planes form a dual spread $\mathbb{H}$ (that is, each 7-space of $\mathrm{PG}(8,q)$ contains a unique 5-space in $\mathbb{H}$). Then $\mathcal{I}_{\text{Bose}}\cong\mathrm{PG}(2,q^3)$, and is called the *Bose representation* of $\mathrm{PG}(2,q^3)$ in $\mathrm{PG}(8,q)$. The three transversal planes of $\mathbb{S}$ in $\mathrm{PG}(8,q^3)$ are denoted by Γ, Γ^q, Γ^{q^2} throughout this article.

We use the following notation. We use uppercase letters to denote points in $\mathrm{PG}(8,q)$ and its extensions; and uppercase letters with a bar to denote points in $\mathrm{PG}(2,q^3)$. If Π_r is an r-dimensional subspace of $\mathrm{PG}(8,q)$, then $\Pi_r^{\star}$ denotes the natural extension to an r-dimensional subspace of $\mathrm{PG}(8,q^3)$, and $\Pi_r^{\bigstar}$ denotes the extension to $\mathrm{PG}(8,q^6)$. Moreover, if Σ_r is an r-dimensional subspace of $\mathrm{PG}(8,q^3)$ (possibly disjoint from $\mathrm{PG}(8,q)$), then we use $\Sigma_r^{\bigstar}$ to denote the extension to $\mathrm{PG}(8,q^6)$. Let $\bar X$ be a point of $\mathrm{PG}(2,q^3)$, then the Bose representation of $\bar X$ is an $\mathbb{S}$-plane denoted by $[\![X]\!]$. In $\mathrm{PG}(8,q^3)$, we have $[\![X]\!]^{\star}\cap\Gamma=X$ and $[\![X]\!]^{\star}=\langle X,X^q,X^{q^2}\rangle$. Thus $\bar X$ corresponds to a unique point X of Γ, and the points of Γ and $\mathrm{PG}(2,q^3)$ are in one-to-one correspondence.

More generally, if $\bar{\mathcal{K}}$ is a set of points of $\mathrm{PG}(2,q^3)$, then $[\![\mathcal{K}]\!]=\{[\![X]\!]\,|\,\bar X\in\bar{\mathcal{K}}\}$ denotes the corresponding set of $\mathbb{S}$-planes, and $\mathcal{K}=\{[\![X]\!]^{\star}\cap\Gamma\,|\,\bar X\in\bar{\mathcal{K}}\}$ denotes the corresponding set of points of Γ. In particular, we have the following correspondences:

$$\begin{array}{ccccc}\mathrm{PG}(2,q^3) & \cong & \Gamma & \cong & \mathcal{I}_{\text{Bose}}\\ \bar X & \longleftrightarrow & X & \longleftrightarrow & [\![X]\!]=\langle X,X^q,X^{q^2}\rangle\cap\mathrm{PG}(8,q).\end{array}$$

We will need some properties of planes of $\mathrm{PG}(8,q^3)$ that meet all three transversal planes of the regular 2-spread $\mathbb{S}$. We call a plane of $\mathrm{PG}(8,q^3)$ that meets all three transversal planes a *T-plane*; call a line of $\mathrm{PG}(8,q^3)$ that meets two transversal planes a *T-line*; and call a point of $\mathrm{PG}(8,q^3)$ that lies in a transversal plane a *T-point*.

Corollary 2.2 *Two T-planes of* $\mathrm{PG}(8,q^3)$ *are either equal, disjoint, meet in a T-point, or meet in a T-line.*

Proof Let P be a point of $\mathrm{PG}(8,q^3)$ which is not a T-point, and does not lie on a T-line. As P is not on a T-line, by Lemma 2.1, P lies on a unique T-plane. The result follows. □

Next we choose a basis and give coordinates to describe the representation of $\mathrm{PG}(2,q^3)$ in $\mathrm{PG}(8,q)$. This idea is not new and has been used in field reduction and for coordinatising a regular 2-spread in $\mathrm{PG}(8,q)$. However, we need to formalise the technique and notation we use to prove the later results in this article.

2.3 Coordinates for the Bose representation

If $\bar{P}$ is a point in $\mathrm{PG}(2,q^3)$, then $\bar{P}$ has homogeneous coordinates $\bar{\boldsymbol{P}} = (x,y,z)$ for some $x,y,x \in \mathbb{F}_{q^3}$, not all zero. Further, the homogeneous coordinates $(x,y,z) \equiv \lambda(x,y,z)$ for any $\lambda \in \mathbb{F}_{q^3}^*$. If Q is a point in $\mathrm{PG}(8,q)$, then Q has homogeneous coordinates $\boldsymbol{Q} = (x_0,x_1,x_2,\ y_0,y_1,y_2,\ z_0,z_1,z_2)$ for some $x_i,y_i,z_i \in \mathbb{F}_q$, not all zero. Further,

$$(x_0,x_1,x_2,\ y_0,y_1,y_2,\ z_0,z_1,z_2) \equiv \rho(x_0,x_1,x_2,\ y_0,y_1,y_2,\ z_0,z_1,z_2)$$

for any $\rho \in \mathbb{F}_q^*$.

The Bose representation maps a point $\bar{P}$ in $\mathrm{PG}(2,q^3)$ to an $\mathbb{S}$-plane $[\![P]\!]$ in $\mathrm{PG}(8,q)$. We introduce notation to describe this algebraically. Let τ be a primitive element of $\mathbb{F}_q$ with primitive polynomial

$$x^3 - t_2x^2 - t_1x - t_0$$

for $t_0,t_1,t_2 \in \mathbb{F}_q$. If $x,y,z \in \mathbb{F}_{q^3}$, we can write $x = x_0+x_1\tau+x_2\tau^2$, $y = y_0+y_1\tau+y_2\tau^2$, $z = z_0+z_1\tau+z_2\tau^2$ for unique $x_i,y_i,z_i \in \mathbb{F}_q$. Define the following two maps

$$\begin{aligned} \theta\colon \mathbb{F}_{q^3} &\longrightarrow \mathbb{F}_q^3 \\ x &\longmapsto (x_0,x_1,x_2), \end{aligned}$$

$$\begin{aligned} \Theta\colon \mathbb{F}_{q^3}^3 &\longrightarrow \mathbb{F}_q^9 \\ (x,y,z) &\longmapsto (\theta(x),\theta(y),\theta(z)) = (x_0,x_1,x_2,\ y_0,y_1,y_2,\ z_0,z_1,z_2). \end{aligned}$$

Note that Θ maps the vector (x,y,z) which is one 'coordinate representation' of a point in $\mathrm{PG}(2,q^3)$ to a vector which is one 'coordinate representation' of a point in $\mathrm{PG}(8,q)$. The Bose representation maps the q^3-1 'coordinate representations' $\lambda(x,y,z)$, $\lambda\in\mathbb{F}_{q^3}^*$ to $(q^3-1)/(q-1)=q^2+q+1$ distinct points of $\mathrm{PG}(8,q)$. That is, if $\bar{\boldsymbol{P}}=(x,y,z)\equiv(\lambda x,\lambda y,\lambda z)$, then the corresponding $\mathbb{S}$-plane $[\![P]\!]$ is the set of points

$$[\![P]\!]=\{\Theta(\lambda x,\lambda y,\lambda z)=(\theta(\lambda x),\theta(\lambda y),\theta(\lambda z))\mid\lambda\in\mathbb{F}_{q^3}^*\}.$$

For reference, we compare our notation to the notation $\langle v\rangle_{q^3}$ and $\langle v\rangle_q$ used for field reduction in [17]. For a point $\bar{P}$ in $\mathrm{PG}(2,q^3)$ with homogeneous coordinates $\boldsymbol{P}=v=(x,y,z)$, we have

$$\begin{array}{rcccl}\boldsymbol{P}&=&\langle v\rangle_{q^3}&=&\{(\lambda x,\lambda y,\lambda z)\mid\lambda\in\mathbb{F}_{q^3}^*\}\\ [\![P]\!]&=&\langle v\rangle_q&=&\{\,\Theta(\lambda x,\lambda y,\lambda z)\mid\lambda\in\mathbb{F}_{q^3}^*\}\end{array}$$

2.4 Coordinates for $\mathbb{S}$-planes

Let P be a point of $\mathrm{PG}(8,q)$. We can write the $\mathbb{S}$-plane $[\![P]\!]$ in $\mathrm{PG}(8,q)$ as $[\![P]\!]=\langle P_0,P_1,P_2\rangle$ where P_0, P_1, P_2 are the three non-collinear points of $\mathrm{PG}(8,q)$ whose homogeneous coordinates are

$$\begin{array}{rcl}\boldsymbol{P}_0&=&\Theta(x,y,z),\\ \boldsymbol{P}_1&=&\Theta(\tau x,\tau y,\tau z),\\ \boldsymbol{P}_2&=&\Theta(\tau^2x,\tau^2y,\tau^2z).\end{array}$$

It is straightforward to expand and simplify these; the first coordinates $\theta(x)$, $\theta(\tau x)$ and $\theta(\tau^2x)$ are:

$$\begin{array}{rcl}\theta(x)&=&(x_0,x_1,x_2),\\ \theta(\tau x)&=&(x_2t_0,\ x_0+x_2t_1,\ x_1+x_2t_2),\\ \theta(\tau^2x)&=&(t_0(x_1+t_2x_2),\ t_0x_2+t_1(x_1+t_2x_2),\ x_0+t_1x_2+t_2(x_1+t_2x_2)\,).\end{array}$$

Further, if $\rho\in\mathbb{F}_{q^3}^*$, write $\rho=p_0+p_1\tau+p_2\tau^2$ for unique $p_0,p_1,p_2\in\mathbb{F}_q$. Straightforward expanding and simplifying yields

$$\Theta(\rho x,\rho y,\rho z)=p_0\boldsymbol{P}_0+p_1\boldsymbol{P}_1+p_2\boldsymbol{P}_2.$$

2.5 Coordinates for the transversal planes of $\mathbb{S}$

We determine the coordinates in $\mathrm{PG}(8,q^3)$ of the three transversal planes of the regular 2-spread $\mathbb{S}$ using the points P_0,P_1,P_2 defined above. Define the following constants $\mathsf{a}_0,\mathsf{a}_1,\mathsf{a}_2\in\mathbb{F}_{q^3}$ and the points $\mathsf{A}_0,\mathsf{A}_1,\mathsf{A}_2\in\mathrm{PG}(8,q^3)$, as

$$\begin{array}{ll}\mathsf{a}_0=t_1+t_2\tau-\tau^2=-\tau^q\tau^{q^2}, & \boldsymbol{\mathsf{A}}_0=(\mathsf{a}_0,\mathsf{a}_1,\mathsf{a}_2,\ 0,0,0,\ 0,0,0),\\ \mathsf{a}_1=t_2-\tau=\tau^q+\tau^{q^2}, & \boldsymbol{\mathsf{A}}_1=(0,0,0,\ \mathsf{a}_0,\mathsf{a}_1,\mathsf{a}_2,\ 0,0,0),\\ \mathsf{a}_2=-1, & \boldsymbol{\mathsf{A}}_2=(0,0,0,\ 0,0,0,\ \mathsf{a}_0,\mathsf{a}_1,\mathsf{a}_2).\end{array}$$

Lemma 2.3 *The three transversal planes of* $\mathbb{S}$ *are*

$$\mathsf{\Gamma} = \langle \mathsf{A}_0, \mathsf{A}_1, \mathsf{A}_2 \rangle, \quad \mathsf{\Gamma}^q = \langle \mathsf{A}_0^q, \mathsf{A}_1^q, \mathsf{A}_2^q \rangle, \quad \mathsf{\Gamma}^{q^2} = \langle \mathsf{A}_0^{q^2}, \mathsf{A}_1^{q^2}, \mathsf{A}_2^{q^2} \rangle.$$

Moreover, the point $\bar{\boldsymbol{P}} = (x, y, z) \in \mathrm{PG}(2, q^3)$ *corresponds to the point* P *in* $\mathsf{\Gamma}$ *where* $P = [\![P]\!]^{\star} \cap \mathsf{\Gamma}$ *and*

$$\boldsymbol{P} \;=\; \mathsf{a}_0\boldsymbol{P}_0 + \mathsf{a}_1\boldsymbol{P}_1 + \mathsf{a}_2\boldsymbol{P}_2 \;=\; x\mathbf{A}_0 + y\mathbf{A}_1 + z\mathbf{A}_2.$$

Proof Let $\bar{\boldsymbol{P}} = (x, y, z) \in \mathrm{PG}(2, q^3)$, so $[\![P]\!] = \langle P_0, P_1, P_2 \rangle$ with P_i as above. Straightforward calculations show that in $\mathrm{PG}(8, q^3)$ we have $\mathsf{a}_0\boldsymbol{P}_0 + \mathsf{a}_1\boldsymbol{P}_1 + \mathsf{a}_2\boldsymbol{P}_2 = x\mathbf{A}_0 + y\mathbf{A}_1 + z\mathbf{A}_2$. Hence these are the homogeneous coordinates of a point that lies in the plane $[\![P]\!]^{\star}$ and in the plane $\langle \mathsf{A}_0, \mathsf{A}_1, \mathsf{A}_2 \rangle$. Hence the extension of the $\mathbb{S}$-plane $[\![P]\!]$ to $\mathrm{PG}(8, q^3)$ meets the plane $\langle \mathsf{A}_0, \mathsf{A}_1, \mathsf{A}_2 \rangle$. As every extended $\mathbb{S}$-plane meets the plane $\langle \mathsf{A}_0, \mathsf{A}_1, \mathsf{A}_2 \rangle$, it is one of the transversal planes, which we denote by $\mathsf{\Gamma}$. The other two transversal planes are hence $\mathsf{\Gamma}^q$ and $\mathsf{\Gamma}^{q^2}$. □

Note that the points $\mathsf{A}_0, \mathsf{A}_1, \mathsf{A}_2$ of the transversal plane $\mathsf{\Gamma}$ correspond to the fundamental triangle of $\mathrm{PG}(2, q^2)$, namely $\bar{\mathbf{A}}_0 = (1, 0, 0)$, $\bar{\mathbf{A}}_1 = (0, 1, 0)$, $\bar{\mathbf{A}}_2 = (0, 0, 1)$.

We can now write $[\![P]\!]$ in terms of the point $P \in \mathsf{\Gamma}$, namely

$$[\![P]\!]^{\star} = \langle P, P^q, P^{q^2} \rangle,$$

where

$$\begin{array}{rcccccccc} \boldsymbol{P} & = & x\mathbf{A}_0 & + & y\mathbf{A}_1 & + & z\mathbf{A}_2 & \in & \mathsf{\Gamma} \\ \boldsymbol{P}^q & = & x^q\mathbf{A}_0{}^q & + & y^q\mathbf{A}_1{}^q & + & z^q\mathbf{A}_2{}^q & \in & \mathsf{\Gamma}^q \\ \boldsymbol{P}^{q^2} & = & x^{q^2}\mathbf{A}_0{}^{q^2} & + & y^{q^2}\mathbf{A}_1{}^{q^2} & + & z^{q^2}\mathbf{A}_2{}^{q^2} & \in & \mathsf{\Gamma}^{q^2}. \end{array}$$

In order to study $\mathbb{F}_q$-subplanes later in this article, we need to develop a description of the plane in $\mathrm{PG}(8, q^3)$ which contains the three points whose homogeneous coordinates are: $x\mathbf{A}_0 + y\mathbf{A}_1 + z\mathbf{A}_2$, $x\mathbf{A}_0{}^q + y\mathbf{A}_1{}^q + z\mathbf{A}_2{}^q$ and $x\mathbf{A}_0{}^{q^2} + y\mathbf{A}_1{}^{q^2} + z\mathbf{A}_2{}^{q^2}$. The next two subsections are devoted to carefully calculating a useful description of these points.

2.6 Conjugacy with respect to an $\mathbb{F}_q$-subplane in $\mathrm{PG}(2, q^3)$

An $\mathbb{F}_q$-subplane of $\mathrm{PG}(2, q^3)$ is a subplane which has order q, that is, a subplane isomorphic to $\mathrm{PG}(2, q)$. An $\mathbb{F}_q$-subline is a line of an $\mathbb{F}_q$-subplane, so is isomorphic to $\mathrm{PG}(1, q)$. We will define conjugacy with respect to an $\mathbb{F}_q$-subplane and $\mathbb{F}_q$-subline.

First consider the $\mathbb{F}_q$-subplane $\bar{\pi}_0 = \{(x, y, z) \mid x, y, z \in \mathbb{F}_q, \text{ not all } 0\}$ of $\mathrm{PG}(2, q^3)$. There are two collineations in $\mathrm{P\Gamma L}(3, q^3)$ which have order 3 and fix $\bar{\pi}_0$ pointwise, namely $\bar{\mathsf{c}}$ and $\bar{\mathsf{c}}^2$ where

$$\begin{array}{rcl} \bar{\mathsf{c}}\colon \mathrm{PG}(2, q^3) & \longrightarrow & \mathrm{PG}(2, q^3) \\ \bar{\boldsymbol{X}} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} & \longmapsto & \bar{\boldsymbol{X}}^q = \begin{pmatrix} x^q \\ y^q \\ z^q \end{pmatrix}. \end{array}$$

In fact, $\bar{G}_{\pi_0} = \langle \bar{\mathsf{c}} \rangle$ is the unique subgroup of $\mathrm{P\Gamma L}(3, q^3)$ which fixes $\bar{\pi}_0$ pointwise and has order 3. For $\bar{X} \in \mathrm{PG}(2, q^3) \backslash \bar{\pi}_0$, the three points $\bar{X}$, $\bar{X}^{\bar{\mathsf{c}}}$, $\bar{X}^{\bar{\mathsf{c}}^2}$ are called *conjugate with respect to* $\bar{\pi}_0$.

If we embed $\mathrm{PG}(2, q^3)$ in $\mathrm{PG}(2, q^6)$, then $\bar{\mathsf{c}}$ has a natural extension to acting on points of $\mathrm{PG}(2, q^6)$. We denote the extension of $\bar{\mathsf{c}}$ to $\mathrm{P\Gamma L}(3, q^6)$ by $\bar{\mathsf{c}}$ as well. That is, for a point $\bar{X} \in \mathrm{PG}(2, q^6)$, we have $\bar{\mathsf{c}}(\bar{X}) = \bar{X}^q$, and $\bar{\mathsf{c}}$ has order 6 when acting on $\mathrm{PG}(2, q^6)$. Under the collineation $\bar{\mathsf{c}}$, a point $\bar{X} \in \mathrm{PG}(2, q^6)$ lies in an orbit of size: 1 if $\bar{X} \in \bar{\pi}_0$; 3 if $\bar{X} \in \mathrm{PG}(2, q^3) \backslash \bar{\pi}_0$; 2 or 6 if $\bar{X} \in \mathrm{PG}(2, q^6) \backslash \mathrm{PG}(2, q^3)$, depending on whether $\bar{X}$ belongs to the $\mathbb{F}_{q^2}$-subplane $\mathrm{PG}(2, q^2)$ that contains $\bar{\pi}_0 = \mathrm{PG}(2, q)$, or not.

More generally, let $\bar{\pi}$ be any $\mathbb{F}_q$-subplane of $\mathrm{PG}(2, q^3)$. Acting on the points of $\mathrm{PG}(2, q^3)$ is a unique collineation group $\bar{G}_\pi \subseteq \mathrm{P\Gamma L}(3, q^3)$ which fixes $\bar{\pi}$ pointwise and has order 3. We wish to distinguish between the two non-identity collineations in $\bar{G}_\pi$, and do so as follows. Consider any homography that maps $\bar{\pi}$ to $\bar{\pi}_0$, and denote its 3×3 non-singular matrix over $\mathbb{F}_{q^3}$ by A, so if $\bar{X} \in \bar{\pi}$, then the point with coordinates $A\bar{\boldsymbol{X}}$ is in the $\mathbb{F}_q$-subplane $\bar{\pi}_0$. Let $\bar{\mathsf{c}}_\pi(\bar{\boldsymbol{X}}) = A^{-1}\, \bar{\mathsf{c}}(A\bar{\boldsymbol{X}})$. As $\bar{\mathsf{c}}_\pi$ has order 3 and fixes $\bar{\pi}$ pointwise, we have $\bar{G}_\pi = \langle \bar{\mathsf{c}}_\pi \rangle$. We expand $\bar{\mathsf{c}}_\pi$, and to avoid confusion use the following notation. For a 3×3 matrix $A = (a_{ij})$, $i, j = 1, 2, 3$, we let the matrix $A^\sigma = (a_{ij}^q)$, $i, j = 1, 2, 3$. Thus $\bar{\mathsf{c}}_\pi(\bar{\boldsymbol{X}}) = A^{-1} A^\sigma \bar{\boldsymbol{X}}^q$, or writing $B = A^{-1} A^\sigma$, we have $\bar{\mathsf{c}}_\pi(\bar{\boldsymbol{X}}) = B\bar{\boldsymbol{X}}^q$. That is, we can without loss of generality write $\bar{G}_\pi = \langle \bar{\mathsf{c}}_\pi \rangle$ with

$$\begin{array}{rcl} \bar{\mathsf{c}}_\pi \colon \mathrm{PG}(2, q^3) & \longrightarrow & \mathrm{PG}(2, q^3) \\ \bar{\boldsymbol{X}} & \longmapsto & B\bar{\boldsymbol{X}}^q \end{array} \tag{1}$$

with B a 3×3 non-singular matrix over $\mathbb{F}_{q^3}$. For $\bar{X} \in \mathrm{PG}(2, q^3) \backslash \bar{\pi}$, the three points $\bar{X}$, $\bar{X}^{\bar{\mathsf{c}}_\pi}$, $\bar{X}^{\bar{\mathsf{c}}_\pi^2}$ are called *conjugate with respect to* $\bar{\pi}$. Note that $\bar{X}$, $\bar{X}^{\bar{\mathsf{c}}_\pi}$, $\bar{X}^{\bar{\mathsf{c}}_\pi^2}$ are collinear if and only if $\bar{X}$ lies on an extended line of $\bar{\pi}$. As above, the collineation $\bar{\mathsf{c}}_\pi \in \mathrm{P\Gamma L}(3, q^3)$ has a natural extension to a collineation of $\mathrm{P\Gamma L}(3, q^6)$ acting on points of $\mathrm{PG}(2, q^6)$.

Similarly, if $\bar{b}$ is an $\mathbb{F}_q$-subline of a line $\bar{\ell}_b$ of $\mathrm{PG}(2, q^3)$, then acting on the points of $\bar{\ell}_b$ is a unique collineation group $\bar{G}_b \subseteq \mathrm{P\Gamma L}(2, q^3)$ of order 3 which fixes $\bar{b}$ pointwise. Moreover, $\bar{G}_\pi$ restricted to acting on $\bar{\ell}_b$ is isomorphic to $\bar{G}_b$ if and only if $\bar{b}$ is a line of $\bar{\pi}$. Without loss of generality we can write $\bar{G}_b = \langle \bar{\mathsf{c}}_b \rangle$ where for a point $\bar{X} \in \bar{\ell}_b$,

$$\bar{\mathsf{c}}_b(\bar{\boldsymbol{X}}) \;=\; D\bar{\boldsymbol{X}}^q \tag{2}$$

with D a non-singular matrix over $\mathbb{F}_{q^3}$. Further, we can extend $\bar{\mathsf{c}}_b$ to act on points of the quadratic extension of $\bar{\ell}_b$.

2.7 Conjugacy of $\mathrm{PG}(2, q^3)$ in the Bose representation

We now return to the $\mathbb{F}_q$-subplane $\bar{\pi}_0 = \mathrm{PG}(2, q)$ of $\mathrm{PG}(2, q^3)$ and look in more detail at the collineation $\bar{\mathsf{c}} \in \mathrm{P\Gamma L}(3, q^3)$ defined by

$$\begin{array}{rcl} \bar{\mathsf{c}}\colon \mathrm{PG}(2, q^3) & \longrightarrow & \mathrm{PG}(2, q^3) \\ \bar{\boldsymbol{X}} = (x, y, z) & \longmapsto & \bar{\boldsymbol{X}}^{\mathsf{c}} = \bar{\boldsymbol{X}}^q = (x^q, y^q, z^q). \end{array}$$

We have $\bar{G}_{\pi_0} = \{id, \bar{\mathsf{c}}, \bar{\mathsf{c}}^2\}$ where for $\bar{\boldsymbol{X}} = (x, y, z) \in \mathrm{PG}(2, q^3)$,

$$\bar{\boldsymbol{X}}^{\bar{\mathsf{c}}} = (x^q, y^q, z^q) \quad \text{and} \quad \bar{\boldsymbol{X}}^{\bar{\mathsf{c}}^2} = (x^{q^2}, y^{q^2}, z^{q^2}).$$

The collineation $\bar{\mathsf{c}}$ induces a map denoted c which acts on the points of the transversal plane Γ in $\mathrm{PG}(8, q^3)$. That is, π is an $\mathbb{F}_q$-subplane of Γ, and c fixes π pointwise and has orbit size 3 on the points of $\Gamma\backslash\pi$. Note that c is not a collineation of $\mathrm{PG}(8, q^3)$. By Lemma 2.3, a point $X \in \Gamma$ has coordinates

$$\boldsymbol{X} \;=\; x\mathbf{A}_0 + y\mathbf{A}_1 + z\mathbf{A}_2 \;\in\; \Gamma$$

for some $x, y, z \in \mathbb{F}_{q^3}$ not all zero. So

$$\begin{array}{rcl} \boldsymbol{X}^{\mathsf{c}} & = & x^q\mathbf{A}_0 + y^q\mathbf{A}_1 + z^q\mathbf{A}_2 \in \Gamma, \\ \boldsymbol{X}^{\mathsf{c}^2} & = & x^{q^2}\mathbf{A}_0 + y^{q^2}\mathbf{A}_1 + z^{q^2}\mathbf{A}_2 \in \Gamma. \end{array}$$

We will be interested in the images of these points under the map $X \mapsto X^q$ of $\mathrm{PG}(8, q^3)$, in particular, we work with the following points:

$$\begin{array}{rcl} (\boldsymbol{X}^{\mathsf{c}^2})^q & = & x{\mathbf{A}_0}^q + y{\mathbf{A}_1}^q + z{\mathbf{A}_2}^q \in \Gamma^q, \\ (\boldsymbol{X}^{\mathsf{c}})^{q^2} & = & x{\mathbf{A}_0}^{q^2} + y{\mathbf{A}_1}^{q^2} + z{\mathbf{A}_2}^{q^2} \in \Gamma^{q^2}. \end{array}$$

For $\boldsymbol{X} = x\mathbf{A}_0 + y\mathbf{A}_1 + z\mathbf{A}_2 \in \Gamma$, we will be interested in the plane

$$\begin{array}{l} \langle\, x\mathbf{A}_0 + y\mathbf{A}_1 + z\mathbf{A}_2,\; x{\mathbf{A}_0}^q + y{\mathbf{A}_1}^q + z{\mathbf{A}_2}^q,\; x{\mathbf{A}_0}^{q^2} + y{\mathbf{A}_1}^{q^2} + z{\mathbf{A}_2}^{q^2} \rangle \\ \qquad = \langle\, X,\; (X^{\mathsf{c}^2})^q,\; (X^{\mathsf{c}})^{q^2} \,\rangle. \end{array} \tag{3}$$

Consider the extension to $\mathrm{PG}(8, q^6)$. Let $\mathsf{e} \in \mathrm{P\Gamma L}(9, q^6)$ be the unique involution acting on points of $\mathrm{PG}(8, q^6)$ fixing $\mathrm{PG}(8, q^3)$ pointwise. We say the points X, X^{e} are conjugate with respect to the quadratic extension from $\mathrm{PG}(8, q^3)$ to $\mathrm{PG}(8, q^6)$, and have

$$\boldsymbol{X} = (x_0, \ldots, x_8) \quad \longmapsto \quad \boldsymbol{X}^{\mathsf{e}} = (x_0^{q^3}, \ldots, x_8^{q^3}).$$

As e fixes $\mathrm{PG}(8, q^3)$ pointwise, it fixes Γ pointwise. A point $X \in \Gamma^\star$ has coordinates $x\mathbf{A}_0 + y\mathbf{A}_1 + z\mathbf{A}_2$ for some $x, y, z \in \mathbb{F}_{q^6}$, not all zero, and

$$(x\mathbf{A}_0 + y\mathbf{A}_1 + z\mathbf{A}_2)^{\mathsf{e}} = x^{q^3}\mathbf{A}_0 + y^{q^3}\mathbf{A}_1 + z^{q^3}\mathbf{A}_2.$$

Hence for a point $X \in \Gamma^\star \backslash \Gamma$, we have $X^{\mathsf{e}} = X^{q^3} = X^{\mathsf{c}^3}$. It is straightforward to verify that

$$\begin{array}{ccccccccccc} (\boldsymbol{X}^{\mathsf{c}^2\mathsf{e}})^q & = & (\boldsymbol{X}^{\mathsf{c}^5})^q & = & x\mathbf{A_0}^q & + & y\mathbf{A_1}^q & + & z\mathbf{A_2}^q & \in & \Gamma^q \\ (\boldsymbol{X}^{\mathsf{ce}})^{q^2} & = & (\boldsymbol{X}^{\mathsf{c}^4})^{q^2} & = & x\mathbf{A_0}^{q^2} & + & y\mathbf{A_1}^{q^2} & + & z\mathbf{A_2}^{q^2} & \in & \Gamma^{q^2}. \end{array}$$

For a point $X \in \Gamma^\star$ with coordinates $\boldsymbol{X} = x\mathbf{A}_0 + y\mathbf{A}_1 + z\mathbf{A}_2$, we will be interested in the plane

$$\langle\, x\mathbf{A}_0 + y\mathbf{A}_1 + z\mathbf{A}_2,\ x\mathbf{A_0}^q + y\mathbf{A_1}^q + z\mathbf{A_2}^q,\ x\mathbf{A_0}^{q^2} + y\mathbf{A_1}^{q^2} + z\mathbf{A_2}^{q^2} \rangle = \langle\, X,\ (X^{\mathsf{c}^2\mathsf{e}})^q,\ (X^{\mathsf{ce}})^{q^2} \,\rangle. \tag{4}$$

Note that if $X \in \Gamma$, then this is the same plane as in (3).

2.8 Coordinates in the Bruck-Bose representation

We can construct the Bruck-Bose representation of $\mathrm{PG}(2, q^3)$ in $\mathrm{PG}(6, q)$ by intersecting the Bose representation with a 6-space $\Sigma_{6,q}$ of $\mathrm{PG}(8, q)$ which contains a unique 5-space that meets $\mathbb{S}$ in $q^3 + 1$ planes. To obtain the same coordinates for the Bruck-Bose representation as that used in [3, Section 2.2], we take the line at infinity in $\mathrm{PG}(2, q^3)$ to have equation $z = 0$, which contains the points $\bar{\boldsymbol{X}} = (1, 0, 0), \bar{\boldsymbol{Y}} = (0, 1, 0)$. In $\mathrm{PG}(8, q^3)$, this corresponds to the line g in the transversal plane Γ where $g = \langle \mathsf{A}_0, \mathsf{A}_1\rangle$. Take $\Sigma_{6,q}$ as the 6-space of $\mathrm{PG}(8, q)$ consisting of the points $(x_0, x_1, x_2, x_3, x_4, x_5, x_6, 0, 0)$, $x_i \in \mathbb{F}_q$, not all 0. Then $\Sigma^{\star}_\infty = \langle g, g^q, g^{q^2}\rangle$ and contains all points of form $(x_0, x_1, x_2, x_3, x_4, x_5, 0, 0, 0)$. Further, g, g^q, g^{q^2} are the transversal lines of the regular 2-spread $\mathcal{S}$ in Σ_∞. An affine point $\bar{P}$ with coordinates $\bar{\boldsymbol{P}} = (x, y, 1) = (x_0 + x_1\tau + x_2\tau^2,\ y_0 + y_1\tau + y_2\tau^2,\ 1)$ of $\mathrm{PG}(2, q^3)$ corresponds to the affine point $[\![P]\!] \cap \Sigma_{6,q}$ of $\Sigma_{6,q}\backslash\Sigma_\infty$ which has coordinates $(x_0, x_1, x_2,\ y_0, y_1, y_2,\ 1, 0, 0)$.

3 Varieties

3.1 Definitions

We define varieties over a finite field following the notation in [15, Section 2.7]. A *form* $f \in \mathbb{F}_q[x_0, \ldots, x_n]$ is a homogeneous polynomial $f(x_0, \ldots, x_n)$ in indeterminants $x_0, \ldots, x_n$ whose coefficients are in $\mathbb{F}_q$. Let $f_1, \ldots, f_k$ be k forms in $\mathbb{F}_q[x_0, \ldots, x_n]$. Let I be the ideal of $\mathbb{F}_q[x_0, \ldots, x_n]$ generated by $f_1, \ldots, f_k$, that is, $I = (f_1, \ldots, f_k) = \{a_1 f_1 + \cdots a_k f_k \mid a_i \in \mathbb{F}_q\}$. Let $V(f_1, \ldots, f_k)$ be the set of points P in $\mathrm{PG}(n, q)$ which satisfy $f_1(\boldsymbol{P}) = \cdots = f_k(\boldsymbol{P}) = 0$. Then the pair $\mathbf{v}(f_1, \ldots, f_k) = (V(f_1, \ldots, f_k), I)$ is a called a *variety*, or an $\mathbb{F}_q$*-variety of* $\mathrm{PG}(n, q)$.

A point P in $\mathrm{PG}(n, q)$ is called an $\mathbb{F}_q$*-rational point* of the variety $\mathbf{v}(f_1, \ldots, f_k)$ if $f_1(\boldsymbol{P}) = \cdots = f_k(\boldsymbol{P}) = 0$. More generally, a point P in $\mathrm{PG}(n, q^h)$ is called an $\mathbb{F}_{q^h}$*-rational point* of the variety $\mathbf{v}(f_1, \ldots, f_k)$ if $f_1(\boldsymbol{P}) = \cdots = f_k(\boldsymbol{P}) = 0$. Further, if $\mathbb{F}$ is the algebraic closure of $\mathbb{F}_q$, then an $\mathbb{F}$*-rational point* of $\mathbf{v}(f_1, \ldots, f_k)$ is a point P in $\mathrm{PG}(n, \mathbb{F})$ which satisfies $f_1(\boldsymbol{P}) = \cdots = f_k(\boldsymbol{P}) = 0$. Note that $V(f_1, \ldots, f_k) = V(f_1) \cap \cdots \cap V(f_k)$, and we use the notation $\mathbf{v}(f_1, \ldots, f_k) = \mathbf{v}(f_1) \cap \cdots \cap \mathbf{v}(f_k)$.

3.2 Varieties in the Bose representation

In this section we use coordinates to study varieties of $\mathrm{PG}(2,q^3)$ and their corresponding structure in $\mathrm{PG}(8,q)$. Note that the following Theorem 3.1 holds more generally, and can be used to study varieties of $\mathrm{PG}(2,q^h)$ and their corresponding structure in the Bose representation $\mathrm{PG}(3h-1,q)$. We are interested in the case $\mathrm{PG}(8,q)$, and the extensions to $\mathrm{PG}(8,q^3)$ and $\mathrm{PG}(8,q^6)$. In order to establish our notation, we state and prove the results in this setting. In particular, we will prove the following result.

Theorem 3.1 *Let $\bar{F}(x,y,z)$ be a homogeneous equation of degree k over $\mathbb{F}_{q^3}$, and let $\bar{\mathcal{K}} \subset \mathrm{PG}(2,q^3)$ be the set of $\mathbb{F}_{q^3}$-rational points of the variety $\mathbf{v}(\bar{F})$. Write $x = x_0 + \tau x_1 + \tau^2 x_2$, $y = y_0 + \tau y_1 + \tau^2 y_2$, $z = z_0 + \tau z_1 + \tau^2 z_2$ for indeterminants $x_i, y_i, z_i \in \mathbb{F}_q$. Expanding and simplifying yields*

$$\bar{F}(x,y,z) = G(x_0,x_1,x_2,y_0,y_1,y_2,z_0,z_1,z_2) = f_0 + \tau f_1 + \tau^2 f_2$$

where $f_i = f_i(x_0,x_1,x_2,y_0,y_1,y_2,z_0,z_1,z_2)$ is a homogeneous equation of degree k over $\mathbb{F}_q$, $i = 0,1,2$. Consider the Bose representation of $\mathrm{PG}(2,q^3)$.

1. *In $\mathrm{PG}(8,q)$, the pointset of $[\![\mathcal{K}]\!]$ coincides with the set of $\mathbb{F}_q$-rational points of the variety $\mathbf{v}(f_0,f_1,f_2)$.*
2. *In $\mathrm{PG}(8,q^3)$, the following three pointsets coincide.*
   - *(a) the set of $\mathbb{F}_{q^3}$-rational points of the $\mathbb{F}_q$-variety $\mathbf{v}(f_0,f_1,f_2)$,*
   - *(b) the set of $\mathbb{F}_{q^3}$-rational points of the $\mathbb{F}_{q^3}$-variety $\mathbf{v}(G,G^q,G^{q^2})$.*
   - *(c) the pointset of the planes $\{\langle X, Y^q, Z^{q^2}\rangle \mid X,Y,Z \in \mathcal{K}\}$.*

   Hence the set of $\mathbb{F}_{q^3}$-rational points of $\mathbf{v}(G,G^q,G^{q^2})$ which lie in Γ are precisely the points of $\mathcal{K}$.

The proof is given in a series of lemmas and proceeds as follows. First, Lemma 3.2 takes a form in $\mathbb{F}_{q^3}[x,y,z]$ and converts it to a form in $\mathbb{F}_q[x_0,\ldots,x_8]$. Next, in Lemma 3.3, we look at a set $\bar{\mathcal{K}}$ in $\mathrm{PG}(2,q^3)$ which is the set of $\mathbb{F}_{q^3}$-rational points of a variety. We use the calculations from Lemma 3.2 to show that the corresponding set of points in the Bose representation in $\mathrm{PG}(8,q)$ are the $\mathbb{F}_q$-rational points of a variety. We next determine the $\mathbb{F}_{q^3}$-rational points of this variety in Lemma 3.4. Lemma 3.5 is a stepping stone to Lemma 3.6 which gives a geometric description of the variety.

Lemma 3.2 *Let $\bar{F}(x,y,z)$ be a homogeneous equation of degree k over $\mathbb{F}_{q^3}$ in indeterminants x,y,z. Write $x = x_0+\tau x_1+\tau^2 x_2$, $y = y_0+\tau y_1+\tau^2 y_2$, $z = z_0+\tau z_1+\tau^2 z_2$ for indeterminants $x_i, y_i, z_i \in \mathbb{F}_q$. Expanding and simplifying yields*

$$\bar{F}(x,y,z) = G(x_0,x_1,x_2,y_0,y_1,y_2,z_0,z_1,z_2) = f_0 + \tau f_1 + \tau^2 f_2$$

where $f_i = f_i(x_0,x_1,x_2,y_0,y_1,y_2,z_0,z_1,z_2)$ is a homogeneous equation of degree k over $\mathbb{F}_q$, $i = 0,1,2$.

Proof Recall from Section 2.3, τ is a primitive element in $\mathbb{F}_{q^3}$ satisfying $\tau^3 = t_0 + t_1\tau + t_2\tau^2$ for some $t_0, t_1, t_2 \in \mathbb{F}_q$. Let $\bar{F} = \bar{F}(x, y, z)$ be a homogeneous form over $\mathbb{F}_{q^3}$ of degree k in indeterminants x, y, z. For indeterminants $x_0, x_1, x_2, y_0, y_1, y_2, z_0, z_1, z_2$, substitute $x = x_0 + x_1\tau + x_2\tau^2$, $y = y_0 + y_1\tau + y_2\tau^2$, $z = z_0 + z_1\tau + z_2\tau^2$ in $\bar{F}$ to obtain the homogeneous form $G(x_0, x_1, x_2, y_0, y_1, y_2, z_0, z_1, z_2)$ of degree k. For each coefficient $a \in \mathbb{F}_{q^3}$ of G, rewrite as $a = a_0 + a_1\tau + a_2\tau^2$ for unique $a_0, a_1, a_2 \in \mathbb{F}_q$. Then using $\tau^3 = t_0 + t_1\tau + t_2\tau^2$, we may write

$$G = f_0 + f_1\tau + f_2\tau^2 \tag{5}$$

for unique homogeneous forms $f_0, f_1, f_2 \in \mathbb{F}_q[x_0, x_1, x_2, y_0, y_1, y_2, z_0, z_1, z_2]$ of degree k. □

Now f_0, f_1 and f_2 are homogeneous equations with coefficients in $\mathbb{F}_q$, so $\mathbf{v}(f_0)$, $\mathbf{v}(f_1)$ and $\mathbf{v}(f_2)$ are $\mathbb{F}_q$-varieties of $\mathrm{PG}(8, q)$. Further, G is a homogeneous equation with coefficients in $\mathbb{F}_{q^3}$, so $\mathbf{v}(G)$ is an $\mathbb{F}_{q^3}$-variety. The natural geometric setting for the variety $\mathbf{v}(G)$ is $\mathrm{PG}(8, q^3)$, however, we can still determine whether any points of $\mathrm{PG}(8, q)$ satisfy the equation G. Formally, an $\mathbb{F}_q$-rational point of $\mathbf{v}(G)$ is an $\mathbb{F}_{q^3}$-rational point of $\mathbf{v}(G)$ that is fixed by the semilinear transformation $X \mapsto X^q$. The next result shows that a point $P \in \mathrm{PG}(8, q)$ is an $\mathbb{F}_q$-rational point of the variety $\mathbf{v}(G)$ if and only if P is an $\mathbb{F}_q$-rational point of the varieties $\mathbf{v}(f_0)$, $\mathbf{v}(f_1)$ and $\mathbf{v}(f_2)$.

Lemma 3.3 *Using the notation in Lemma 3.2, let $\bar{\mathcal{K}}$ be the set of $\mathbb{F}_{q^3}$-rational points of the variety $\mathbf{v}(\bar{F})$. Then the pointset of $[\![\mathcal{K}]\!]$ in $\mathrm{PG}(8, q)$ coincides with the set of $\mathbb{F}_q$-rational points of the variety $\mathbf{v}(f_0, f_1, f_2)$.*

Proof A point $Q \in \mathrm{PG}(8, q)$ with homogenous coordinates $\boldsymbol{Q} = (a_0, a_1, a_2, b_0, b_1, b_2, c_0, c_1, c_2)$ corresponds to the point $\bar{\boldsymbol{X}}_Q = (a_0 + a_1\tau + a_2\tau^2,\ b_0 + b_1\tau + b_2\tau^2,\ c_0 + c_1\tau + c_2\tau^2)$ of $\mathrm{PG}(2, q^3)$. By Lemma 3.2, $G(\boldsymbol{Q}) = \bar{F}(\bar{\boldsymbol{X}}_Q)$, so $G(\boldsymbol{Q}) = 0$ if and only if $\bar{F}(\bar{\boldsymbol{X}}_Q) = 0$. By (5), $G(\boldsymbol{Q}) = 0$ if and only if $f_0(\boldsymbol{Q}) = f_1(\boldsymbol{Q}) = f_2(\boldsymbol{Q}) = 0$. Hence a point $Q \in \mathrm{PG}(8, q)$ is an $\mathbb{F}_q$-rational point of the variety $\mathbf{v}(f_0, f_1, f_2)$ if and only if the corresponding point $\bar{X}_Q \in \mathrm{PG}(2, q^3)$ is an $\mathbb{F}_{q^3}$-rational point of the variety $\mathbf{v}(\bar{F})$.

We now consider the converse, a point $\bar{P}$ in $\mathrm{PG}(2, q^3)$ corresponds to an $\mathbb{S}$-plane $[\![P]\!]$, which contains $q^2 + q + 1$ points of $\mathrm{PG}(8, q)$. We want to show that if $\bar{P}$ is an $\mathbb{F}_{q^3}$-rational point of the variety $\mathbf{v}(\bar{F})$, then in $\mathrm{PG}(8, q)$, every point $Y \in [\![P]\!]$ is an $\mathbb{F}_q$-rational point of the variety $\mathbf{v}(f_0, f_1, f_2)$. Let $\bar{P}$ have homogeneous coordinates $\bar{\boldsymbol{P}} = (x, y, z) \equiv \rho(x, y, z)$, for any $\rho \in \mathbb{F}_{q^3}^*$, then as discussed in Section 2.3, the single point in $[\![P]\!]$ corresponding to the single coordinate representation $(\rho x, \rho y, \rho z)$ is the point Y_ρ with coordinates $\boldsymbol{Y}_\rho = \Theta(\rho x, \rho y, \rho z)$. Moreover, $[\![P]\!] = \{Y_\rho \mid \rho \in \mathbb{F}_{q^3}^*\}$. So $\bar{F}(\bar{\boldsymbol{P}}) = \bar{F}(\rho x, \rho y, \rho z) = 0$ for all $\rho \in \mathbb{F}_{q^3}^*$ if and only if $G(\boldsymbol{Y}_\rho) = 0$ for all $\rho \in \mathbb{F}_{q^3}^*$ if and only if $f_0(\boldsymbol{Y}_\rho) = f_1(\boldsymbol{Y}_\rho) = f_2(\boldsymbol{Y}_\rho) = 0$ for all $\rho \in \mathbb{F}_{q^3}^*$. This completes the proof. □

We have shown that the $\mathbb{F}_q$-rational points of the $\mathbb{F}_{q^3}$-variety $\mathbf{v}(G)$ and the $\mathbb{F}_q$-rational points of the $\mathbb{F}_q$-variety $\mathbf{v}(f_0, f_1, f_2)$ coincide. Next, we look at the $\mathbb{F}_{q^3}$-rational points of each variety, we show that they do not coincide, but determine how they are related. As G is a homogeneous equation with coefficients in $\mathbb{F}_{q^3}$, the homogeneous equations G^q and G^{q^2} have coefficients in $\mathbb{F}_{q^3}$. So $\mathbf{v}(G, G^q, G^{q^2}) = \mathbf{v}(G) \cap \mathbf{v}(G^q) \cap \mathbf{v}(G^{q^2})$ is an $\mathbb{F}_{q^3}$-variety.

Lemma 3.4 *The set of $\mathbb{F}_{q^3}$-rational points of the variety $\mathbf{v}(f_0, f_1, f_2)$ is precisely the set of $\mathbb{F}_{q^3}$-rational points of the variety $\mathbf{v}(G, G^q, G^{q^2})$.*

Proof The set of $\mathbb{F}_{q^3}$-rational points of the variety $\mathbf{v}(f_0, f_1, f_2)$ is the set of points P of $\mathrm{PG}(8, q^3)$ satisfying the three equations $f_0(\boldsymbol{P}) = 0$, $f_1(\boldsymbol{P}) = 0$, $f_2(\boldsymbol{P}) = 0$. This set is equivalent to the set of points satisfying any three linearly independent equations of form $\lambda_0 f_0 + \lambda_1 f_1 + \lambda_2 f_2 = 0$ where $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{F}_{q^3}$. Recalling that f_0, f_1, f_2 are equations over $\mathbb{F}_q$, we consider the three linearly independent equations

$$G = f_0 + \tau f_1 + \tau^2 f_2, \quad G^q = f_0 + \tau^q f_1 + \tau^{2q} f_2, \quad G^{q^2} = f_0 + \tau^{q^2} f_1 + \tau^{2q^2} f_2.$$

So a point $P \in \mathrm{PG}(8, q^3)$ satisfies $f_0(\boldsymbol{P}) = f_1(\boldsymbol{P}) = f_2(\boldsymbol{P}) = 0$ if and only if $G(\boldsymbol{P}) = G^q(\boldsymbol{P}) = G^{q^2}(\boldsymbol{P}) = 0$, as required. □

Lemma 3.5 *The set of $\mathbb{F}_{q^3}$-rational points of the variety $\mathbf{v}(G)$ form a cone with base $\mathcal{K}$ in the transversal plane $\mathsf{\Gamma}$ and vertex $\langle \mathsf{\Gamma}^q, \mathsf{\Gamma}^{q^2} \rangle$.*

Proof We first determine the set of $\mathbb{F}_{q^3}$-rational points of $\mathbf{v}(G)$ which lie in the transversal plane $\mathsf{\Gamma}$. By Lemma 2.3, a point Q in the transversal plane $\mathsf{\Gamma}$ has coordinates

$$\boldsymbol{Q} = x\mathbf{A}_0 + y\mathbf{A}_1 + z\mathbf{A}_2 = (x\mathsf{a}_0, x\mathsf{a}_1, x\mathsf{a}_2,\ y\mathsf{a}_0, y\mathsf{a}_1, y\mathsf{a}_2,\ z\mathsf{a}_0, z\mathsf{a}_1, z\mathsf{a}_2)$$

for some $x, y, z \in \mathbb{F}_{q^3}$. Moreover, the points of $\mathsf{\Gamma}$ are in one-to-one correspondence with the points of $\mathrm{PG}(2, q^3)$: the point $Q \in \mathsf{\Gamma}$ corresponds to the point $\bar{Q} \in \mathrm{PG}(2, q^3)$ where

$$\bar{\boldsymbol{Q}} = (x, y, z).$$

By Lemma 3.2, $G(\boldsymbol{Q}) = \bar{F}(\bar{\boldsymbol{Q}})$. So the point $Q \in \mathsf{\Gamma}$ is an $\mathbb{F}_{q^3}$-rational point of $\mathbf{v}(G)$ if and only if the point $\bar{Q} \in \mathrm{PG}(2, q^3)$ is an $\mathbb{F}_{q^3}$-rational point of the variety $\mathbf{v}(\bar{F})$, if and only if $\bar{Q} \in \bar{\mathcal{K}}$.

We now determine the singular space of the variety $\mathbf{v}(G)$. Let $\mathcal{V}$ denote the set of $\mathbb{F}_{q^3}$-rational points of $\mathbf{v}(G)$. As the plane $\mathsf{\Gamma}$ is not contained in $\mathcal{V}$, the maximum dimension of the singular space of $\mathbf{v}(G)$ is five. We show that the 5-space $\langle \mathsf{\Gamma}^q, \mathsf{\Gamma}^{q^2} \rangle$ is the singular space of $\mathbf{v}(G)$ by showing that every point in $\mathcal{V}$ lies on a line joining a point $Q \in \mathcal{K}$ to a point $R \in \langle \mathsf{\Gamma}^q, \mathsf{\Gamma}^{q^2} \rangle$. Let $Q \in \mathcal{K}$, $R \in \langle \mathsf{\Gamma}^q, \mathsf{\Gamma}^{q^2} \rangle$ and $P \in QR$. By Lemma 2.3, $\mathsf{\Gamma} = \langle \mathsf{A}_0, \mathsf{A}_1, \mathsf{A}_2 \rangle$, $\mathsf{\Gamma}^q = \langle \mathsf{A}_0^q, \mathsf{A}_1^q, \mathsf{A}_2^q \rangle$ and $\mathsf{\Gamma}^{q^2} = \langle \mathsf{A}_0^{q^2}, \mathsf{A}_1^{q^2}, \mathsf{A}_2^{q^2} \rangle$. So P has homogeneous coordinates of form

$$\boldsymbol{P} = x\mathbf{A}_0 + y\mathbf{A}_1 + z\mathbf{A}_2 + r\mathbf{A}_0{}^q + s\mathbf{A}_1{}^q + t\mathbf{A}_2{}^q + u\mathbf{A}_0{}^{q^2} + v\mathbf{A}_1{}^{q^2} + w\mathbf{A}_2{}^{q^2},$$

for some $x, y, z, r, s, t, u, v, w \in \mathbb{F}_{q^3}$. Simplifying $\boldsymbol{P}$ (using the coordinates for A_i and a_i as in Section 2.5) we calculate the first three coordinates of $\boldsymbol{P}$ are

$$(x\mathsf{a}_0 + r\mathsf{a}_0^q + u\mathsf{a}_0^{q^2},\ x\mathsf{a}_1 + r\mathsf{a}_1^q + u\mathsf{a}_1^{q^2},\ x\mathsf{a}_2 + r\mathsf{a}_2^q + u\mathsf{a}_2^{q^2}).$$

As in the proof of Lemma 3.3, recall that P corresponds to a unique point of $\mathrm{PG}(2, q^3)$ which we denote by $\bar{X}_P$, and the first coordinate of $\bar{\boldsymbol{X}}_P$ is

$$x\mathsf{a}_0 + r\mathsf{a}_0^q + u\mathsf{a}_0^{q^2} \ + \ (x\mathsf{a}_1 + r\mathsf{a}_1^q + u\mathsf{a}_1^{q^2})\tau \ + \ (x\mathsf{a}_2 + r\mathsf{a}_2^q + u\mathsf{a}_2^{q^2})\tau^2.$$

Straightforward manipulation shows that

$$\mathsf{a}_0^q + \tau\mathsf{a}_1^q + \tau^2\mathsf{a}_2^q \ = \ \mathsf{a}_0^{q^2} + \tau\mathsf{a}_1^{q^2} + \tau^2\mathsf{a}_2^{q^2} \ = \ 0,$$

and using this we simplify the first coordinate of $\bar{\boldsymbol{X}}_P$ to $(\mathsf{a}_0 + \tau\mathsf{a}_1 + \tau^2\mathsf{a}_2)\,x$. Similarly we calculate the other coordinates of P, and the coordinates of $\bar{\boldsymbol{X}}_P$ are

$$\bar{\boldsymbol{X}}_P = (\mathsf{a}_0 + \tau\mathsf{a}_1 + \tau^2\mathsf{a}_2)\,\big(x, y, z\big) \ \equiv \ (x, y, z) = \bar{\boldsymbol{Q}}.$$

By Lemma 3.2, $G(\boldsymbol{P}) = \bar{F}(\bar{\boldsymbol{X}}_P)$, so P lies in $\mathcal{V}$ if and only if the point $\bar{X}_P = \bar{Q}$ lies in $\bar{\mathcal{K}}$. Hence if P is on a line joining $Q \in \Gamma$ with a point R of $\langle \Gamma^q, \Gamma^{q^2} \rangle$, then $G(\boldsymbol{P}) = 0$ if and only if $\bar{F}(\bar{\boldsymbol{Q}}) = 0$. That is, $P \in \mathcal{V}$ if and only if $\bar{Q} \in \bar{\mathcal{K}}$ if and only if $Q \in \mathcal{K}$. Hence $\mathbf{v}(G)$ is a cone with base $\mathcal{K}$ and vertex $\langle \Gamma^q, \Gamma^{q^2} \rangle$. □

Lemma 3.6 *The set of $\mathbb{F}_{q^3}$-rational points of the variety $\mathbf{v}(f_0, f_1, f_2)$ is equivalent to the pointset of the planes in $\{\langle X, Y^q, Z^{q^2} \rangle \mid X, Y, Z \in \mathcal{K}\}$.*

Proof By Lemma 3.5, $\mathbf{v}(G)$ is a cone in $\mathrm{PG}(8, q^3)$ with base $\mathcal{K}$ in Γ and vertex $\langle \Gamma^q, \Gamma^{q^2} \rangle$. Hence $\mathbf{v}(G^q)$ is a cone with base $\mathcal{K}^{q^2}$ in Γ^{q^2} and vertex $\langle \Gamma^q, \Gamma \rangle$; and $\mathbf{v}(G^{q^2})$ is a cone with base $\mathcal{K}^q$ and vertex $\langle \Gamma, \Gamma^{q^2} \rangle$. Note that each of the three cones is a set of T-planes. Hence the intersection of the three cones $\mathbf{v}(G)$, $\mathbf{v}(G^q)$, $\mathbf{v}(G^{q^2})$ is the set of points lying on the T-planes

$$\{\langle X, Y^q, Z^{q^2} \rangle \mid X, Y, Z \in \mathcal{K}\}$$

as required. □

This completes the proof of Theorem 3.1. We will apply Theorem 3.1 to look at the following varieties of $\mathrm{PG}(2, q^3)$ in the Bose representation: we look at conics in Section 5; $\mathbb{F}_q$-sublines and $\mathbb{F}_q$-subplanes in Section 6; and $\mathbb{F}_q$-conics in Section 7. In each case, we determine the corresponding varieties of $\mathrm{PG}(8, q)$, then look at the extensions to $\mathrm{PG}(8, q^3)$ and $\mathrm{PG}(8, q^6)$, and determine their relationship with the transversal planes of the regular 2-spread $\mathbb{S}$. First, we discuss why it is important to carefully describe the variety of $\mathrm{PG}(8, q)$ that we extend, and highlight an important convention used in the literature.

3.3 A Bose representation convention regarding variety extensions

Using the notation of Theorem 3.1, let $\bar{\mathcal{K}}$ be the set of $\mathbb{F}_{q^3}$-rational points of a variety $\mathbf{v}(\bar{F})$ of $\mathrm{PG}(2,q^3)$, so the points of $[\![\mathcal{K}]\!]$ coincide with the $\mathbb{F}_q$-rational points of the variety $\mathbf{v}(f_0,f_1,f_2)$ of $\mathrm{PG}(8,q)$. The set of points in $[\![\mathcal{K}]\!]$ may be the set of $\mathbb{F}_q$-rational points of more than one variety of $\mathrm{PG}(8,q)$. In particular, as $x^q=x$ for all $x\in\mathbb{F}_q$, if the form f_i of degree d contains terms having an indeterminant raised to a power greater than or equal to q, then we can reduce these exponents by $q-1$ to yield a homogeneous form g_i of degree $d-q+1$; such that the $\mathbb{F}_q$-rational points of $\mathbf{v}(f_i)$ and $\mathbf{v}(g_i)$ coincide. That is, the $\mathbb{F}_q$-rational points of the two varieties $\mathbf{v}(f_0,f_1,f_2)$ and $\mathbf{v}(g_0,g_1,g_2)$ coincide. However, the $\mathbb{F}_{q^3}$-rational points of the two varieties may not coincide.

The Bruck-Bose and Bose representations of $\mathrm{PG}(2,q^2)$ have been well studied, and the literature has used this reduction of degree technique. As this is an important notion, we illustrate the convention the literature has used by looking at two examples in the Bose representation of $\mathrm{PG}(2,q^2)$ in $\mathrm{PG}(5,q)$. We use the following notation here. Let $\mathcal{S}$ denote the regular 1-spread in $\mathrm{PG}(5,q)$, and let Γ,Γ^q denote the transversal planes of $\mathcal{S}$ in $\mathrm{PG}(5,q^2)$. If $\bar{\mathcal{A}}$ is a set of points in $\mathrm{PG}(2,q^2)$, then the corresponding a set of points in the transversal plane Γ is denoted $\mathcal{A}$.

Example 3.7 Let $\bar{\mathcal{U}}$ be a classical unital of $\mathrm{PG}(2,q^2)$. So $\bar{\mathcal{U}}$ is projectively equivalent to the variety $\mathbf{v}(\bar{F})$ where $\bar{F}(x,y,z)=x^{q+1}+y^{q+1}+z^{q+1}$. Let $\tau\in\mathbb{F}_{q^2}$ have minimal polynomial $x^2-t_1x-t_0$. Let x_0,x_1,y_0,y_1,z_0,z_1 be indeterminants and substitute $x=x_0+x_1\tau, y=y_0+y_1\tau, z=z_0+z_1\tau$ in $\bar{F}$ to obtain the form

$$G(x_0,x_1,y_0,y_1,z_0,z_1)=(x_0+x_1\tau)^{q+1}+(y_0+y_1\tau)^{q+1}+(z_0+z_1\tau)^{q+1}.$$

As $\tau^q=t_1-\tau$ and $\tau\tau^q=-t_0$, this becomes $G(x_0,x_1,y_0,y_1,z_0,z_1)=x_0^{q+1}+\tau x_0^qx_1+(t_1-\tau)x_0x_1^q-t_0x_1^{q+1}+y_0^{q+1}+\tau y_0^qy_1+(t_1-\tau)y_0y_1^q)-t_0y_1^{q+1}+z_0^{q+1}+\tau z_0^qz_1+(t_1-\tau)z_0z_1^q)-t_0z_1^{q+1}$. It follows that

$$f_0(x_0,x_1,y_0,y_1,z_0,z_1)= x_0^{q+1}+t_1x_0x_1^q-t_0x_1^{q+1}+y_0^{q+1}+t_1y_0y_1^q-t_0y_1^{q+1}+z_0^{q+1}+t_1z_0z_1^q-t_0z_1^{q+1}$$

and

$$f_1(x_0,x_1,y_0,y_1,z_0,z_1)=-x_0x_1^q+x_0^qx_1-y_0y_1^q+y_0^qy_1-z_0z_1^q+z_0^qz_1.$$

We now consider the corresponding forms g_i where the exponents of the forms f_i are reduced by $q-1$, giving

$$g_0(x_0,x_1,y_0,y_1,z_0,z_1)=x_0^2+t_1x_0x_1-t_0x_1^2+y_0^2-t_0y_1^2+z_0^2-t_0z_1^2$$

and

$$g_1(x_0,x_1,y_0,y_1,z_0,z_1)=0.$$

The forms f_0,f_1 have degree $q+1$, while the form g_0 has degree 2 and g_1 is identically zero. The two varieties $\mathcal{Q}=\mathbf{v}(g_0,g_1)=\mathbf{v}(g_0)$ and $\mathcal{V}=\mathbf{v}(f_0,f_1)$ have the same set of $\mathbb{F}_q$-rational points, that is, they have the same pointset in $\mathrm{PG}(5,q)$. However, their $\mathbb{F}_{q^2}$-rational points do not coincide. The variety $\mathcal{Q}$ is an elliptic quadric whose extension to $\mathrm{PG}(5,q^2)$ is a hyperbolic quadric that contains the transversal planes Γ, Γ^q. Adapting Theorem 3.1 to this case, the set of $\mathbb{F}_{q^2}$-rational points of the variety $\mathcal{V} = \mathbf{v}(f_0, f_1)$ is the set of points of $\mathrm{PG}(5,q^2)$ which lie on the lines XY^q for points $X, Y \in \mathcal{U}$. Comparing these two varieties: the set of $\mathbb{F}_{q^2}$-rational points of $\mathbf{v}(g_0)$ contains Γ, and the set of $\mathbb{F}_{q^2}$-rational points of $\mathbf{v}(f_0, f_1)$ meets Γ in a unital. The standard convention used in the literature in this example is to reduce exponents in the forms, and define the variety of $\mathrm{PG}(5,q)$ corresponding to a unital to be the elliptic quadric $\mathcal{Q} = \mathbf{v}(g_0, g_1) = \mathbf{v}(g_0)$, not the variety $\mathcal{V} = \mathbf{v}(f_0, f_1)$; see [2].

Example 3.8 Consider the Bose representation of the Baer subplane $\bar{\pi}_0 = \mathrm{PG}(2,q)$ of $\mathrm{PG}(2,q^2)$. Let $\bar{F}_0(x,y,z) = xy^q - x^q y$, $\bar{F}_1(x,y,z) = yz^q - y^q z$, $\bar{F}_2(x,y,z) = zx^q - z^q x$, then $\bar{\pi}_0$ may be described as the variety $\mathbf{v}(\bar{F}_0, \bar{F}_1, \bar{F}_2)$. A similar analysis to Example 1 yields varieties $\mathbf{v}(f_{i,j})$, $i = 0,1,2$, $j = 0,1$, with $f_{0,0} = x_0 y_0^q - x_0^q y_0 + t_1(x_0 y_1^q - x_1 y_0^q) - t_0(x_1 y_1^q - x_1^q y_1)$, $f_{0,1} = x_0 y_1^q + x_1^q y_0 + x_1 y_0^q - x_0^q y_1$, and so on. Reducing the exponents gives varieties $\mathbf{v}(g_{i,j})$, $i = 0,1,2$, $j = 0,1$, with $g_{0,0} = t_1(x_0 y_1 - x_1 y_0)$, $g_{0,1} = -2(x_0 y_1 - x_1 y_0)$, and so on. The forms defining the variety $\mathcal{K} = \mathbf{v}(f_{0,0}, f_{0,1}, f_{1,0}, f_{1,1}, f_{2,0}, f_{2,1})$ have degree $q{+}1$, and the forms defining the variety $\mathcal{K}' = \mathbf{v}(g_{0,0}, g_{0,1}, g_{1,0}, g_{1,1}, g_{2,0}, g_{2,1})$ are quadrics. The $\mathbb{F}_q$-rational points of the two varieties coincide, but the $\mathbb{F}_{q^2}$-rational points do not. The $\mathbb{F}_{q^2}$-rational points of $\mathcal{K}$ form a set of points in $\mathrm{PG}(5,q^2)$ which meets the transversal plane Γ in the Baer subplane π_0. The $\mathbb{F}_{q^2}$-rational points of $\mathcal{K}'$ form a set of points in $\mathrm{PG}(5,q^2)$ that contains the transversal plane Γ. The standard convention used in the literature in this example is to reduce exponents in the forms and use the variety $\mathcal{K}'$ when considering the extension to $\mathrm{PG}(5,q^2)$; see [2].

4 Quadrics

4.1 Quadrics in $\mathrm{PG}(8,q)$ and their extension to $\mathrm{PG}(8,q^3)$

In the next three sections, we work with varieties which are the intersections of quadrics in $\mathrm{PG}(8,q)$. We want to look at their extensions to $\mathrm{PG}(8,q^3)$ and $\mathrm{PG}(8,q^6)$. Rather than continue with the formal variety notation, we use the following simpler notation, generalising the star notation we use for subspaces. A quadric $\mathcal{Q}$ in $\mathrm{PG}(8,q)$ is the set of $\mathbb{F}_q$-rational points of an $\mathbb{F}_q$-variety $\mathbf{v}(f)$ where $f \in \mathbb{F}_q[x_0,\ldots,x_8]$ is a homogeneous equation of degree two. The *extension of* $\mathcal{Q}$ *to* $\mathrm{PG}(8,q^3)$ is denoted $\mathcal{Q}^{\star}$ and is the set of $\mathbb{F}_{q^3}$-rational points of $\mathbf{v}(f)$, that is, $\mathcal{Q}^{\star}$ is the set of points P in $\mathrm{PG}(8,q^3)$ satisfying $f(\boldsymbol{P}) = 0$. Similarly, the extension of $\mathcal{Q}$ to $\mathrm{PG}(8,q^6)$ is denoted $\mathcal{Q}^{\bigstar}$ and is the set of points P in $\mathrm{PG}(8,q^6)$ that satisfy $f(\boldsymbol{P}) = 0$.

More generally, for $i = 1,\ldots,k$, let $\mathcal{Q}_i$ be a quadric in $\mathrm{PG}(8,q)$ with homogeneous equation $f_i \in \mathbb{F}_q[x_0,\ldots,x_8]$ of degree two. Let $\mathcal{V} = \mathcal{Q}_1 \cap \cdots \cap \mathcal{Q}_k$ be the intersection of these k quadrics, so $\mathcal{V}$ is the set of points of $P \in \mathrm{PG}(8,q)$ which satisfy $f_1(\boldsymbol{P}) = \cdots = f_k(\boldsymbol{P}) = 0$. We use the notation $\mathcal{V}^{\star}$ to refer to the set of points in the cubic extension PG$(8, q^3)$ which satisfy $f_1, \dots, f_k$, that is, $\mathcal{V}^{\star} = \mathcal{Q}_1^{\star} \cap \dots \cap \mathcal{Q}_k^{\star}$. Similarly, $\mathcal{V}^{\bigstar} = \mathcal{Q}_1^{\bigstar} \cap \cdots \cap \mathcal{Q}_k^{\bigstar}$ denotes the extension to PG$(8, q^6)$, so is the set of points $P \in$ PG$(8, q^6)$ with $f_1(\boldsymbol{P}) = \cdots = f_k(\boldsymbol{P}) = 0$.

4.2 Segre varieties

We define the Segre variety $\mathcal{S}_{2;2}$ following [16, Section 25.5]. Consider the projective spaces $\mathcal{P}_1 = \{(X_0, X_1, X_2) \mid X_i \in \mathbb{F}_q\}$, $\mathcal{P}_2 = \{(Y_0, Y_1, Y_2) \mid Y_i \in \mathbb{F}_q\}$. Then the Segre variety $\mathcal{S}_{2;2}$ of $\mathcal{P}_1$ and $\mathcal{P}_2$ is the set of points with coordinates

$$(x_0, \dots, x_8) = (X_0Y_0,\ X_0Y_1,\ X_0Y_2,\ X_1Y_0,\ X_1Y_1,\ X_1Y_2,\ X_2Y_0,\ X_2Y_1,\ X_2Y_2)$$

in PG$(8, q)$. The variety $\mathcal{S}_{2;2}$ contains two maximal systems of subspace which are planes, denote these $\mathcal{R}, \mathcal{R}'$. The system $\mathcal{R}$ contains q^2+q+1 pairwise disjoint planes, and $\mathcal{R}'$ contains q^2+q+1 pairwise disjoint planes. Each point of $\mathcal{S}_{2;2}$ lies in a unique plane in $\mathcal{R}$ and a unique plane in $\mathcal{R}'$.

The variety $\mathcal{S}_{2;2}$ is the intersection of the nine quadrics with equations $x_0x_4 = x_1x_3$, $x_0x_5 = x_2x_3$, $x_0x_7 = x_1x_6$, $x_0x_8 = x_2x_6$, $x_1x_5 = x_2x_4$, $x_1x_8 = x_2x_7$, $x_3x_7 = x_4x_6$, $x_3x_8 = x_5x_6$, and $x_4x_8 = x_5x_7$. In fact, these nine quadrics are not linearly independent, and a suitable choice of six quadrics suffices to uniquely determine $\mathcal{S}_{2;2}$. That is, we can define the Segre variety $\mathcal{S}_{2;2}$ in PG$(8, q)$ as the intersection of six quadrics $\mathcal{Q}_1 \cap \cdots \cap \mathcal{Q}_6$ with associated forms $f_1, \dots, f_6 \in \mathbb{F}_q[x_0, \dots, x_8]$. We use these quadrics to define the extension of $\mathcal{S}_{2;2}$ as follows. Let $\mathcal{V}$ be the set of points on the Segre variety in PG$(8, q)$, so $\mathcal{V} = \mathcal{Q}_1 \cap \cdots \cap \mathcal{Q}_6$. Let $\mathcal{V}^{\star} = \mathcal{Q}_1^{\star} \cap \cdots \cap \mathcal{Q}_6^{\star}$ denote the set of points in the extension of $\mathcal{V}$ to PG$(8, q^3)$, that is, the set of points of PG$(8, q^3)$ which satisfy $f_1, \dots, f_6$. Similarly, $\mathcal{V}^{\bigstar} = \mathcal{Q}_1^{\bigstar} \cap \cdots \cap \mathcal{Q}_6^{\bigstar}$ denotes the set of points of $\mathcal{V}$ lying in the extension of $\mathcal{V}$ to PG$(8, q^6)$.

4.3 Scrolls in PG$(8, q)$

We define scrolls in PG$(8, q)$. For a more general definition of a rational normal k-fold scroll in PG(n, q), see [14, p93]. Let π_1, π_2, π_3 be three planes in PG$(8, q)$ which are pairwise disjoint, and together span PG$(8, q)$. Let $\mathcal{C}_i$ be a non-degenerate conic in π_i, $i = 1, 2, 3$. Let $\phi_i \in$ PGL$(2, q)$ be a projectivity mapping $\mathcal{C}_1$ to $\mathcal{C}_i$, $i = 2, 3$. Then the set of planes

$$\mathcal{S}(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3) = \{\langle P, P^{\phi_2}, P^{\phi_3}\rangle \mid P \in \mathcal{C}_1\}$$

form a *rational normal* 3*-fold scroll.* Any two scrolls of PG$(8, q)$ constructed from three conics is this way are projectively equivalent. Without loss of generality, we can coordinatise as follows:

$$\begin{aligned}
\mathcal{C}_1 &= \{\boldsymbol{P}_{r,s} = (r^2, rs, s^2, 0, 0, 0, 0, 0, 0) \mid r, s \in \mathbb{F}_q,\ \text{not both } 0\},\\
\mathcal{C}_2 &= \{\boldsymbol{P}'_{r,s} = (0, 0, 0, r^2, rs, s^2, 0, 0, 0) \mid r, s \in \mathbb{F}_q,\ \text{not both } 0\},\\
\mathcal{C}_3 &= \{\boldsymbol{P}''_{r,s} = (0, 0, 0, 0, 0, 0, r^2, rs, s^2) \mid r, s \in \mathbb{F}_q,\ \text{not both } 0\},
\end{aligned}$$

so the homographies ϕ_2, ϕ_3 are essentially the identity and the scroll $\mathcal{S}(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3)$ consists of the $q+1$ planes $\langle P_{r,s},\ P'_{r,s},\ P''_{r,s}\rangle$.

The pointset of the scroll $\mathcal{S}(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3)$ coincides with the $\mathbb{F}_q$-rational points of a variety $\mathcal{V}_3^6$ of dimension 3 and degree 6, see [14, p93, p256]. We consider two ways to look at the extension of the scroll $\mathcal{S}(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3)$ to $\mathrm{PG}(8, q^3)$. Firstly, in the variety setting, the *variety extension* is the set of $\mathbb{F}_{q^3}$-rational points of the variety $\mathcal{V}_3^6$. Secondly, embed $\mathrm{PG}(8, q)$ in $\mathrm{PG}(8, q^3)$, so π_i has a natural extension to a plane $\pi_i^{\star}$ of $\mathrm{PG}(8, q^3)$, $\mathcal{C}_i$ has a natural extension to a conic $\mathcal{C}_i^{\star}$ of $\pi_i^{\star}$, and ϕ_i has a natural extension to the projectivity $\phi_i^{\star}$ acting on the points of $\mathcal{C}_1^{\star}$. We define the *scroll-extension* of $\mathcal{S}(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3)$ to $\mathrm{PG}(8, q^3)$ to be the set of q^3+1 planes $\langle P, P^{\phi_2^{\star}}, P^{\phi_3^{\star}}\rangle$ for $P \in \mathcal{C}_1^{\star}$. It is not immediately obvious whether the pointsets of these two extensions coincide, we prove in Theorem 7.1 that they do coincide.

We generalise this to define a scroll which rules the three planes π_1, π_2, π_3 in $\mathrm{PG}(8, q)$. Let $\phi_i \in \mathrm{PGL}(3, q)$ map π_1 to π_i, $i = 2, 3$. Then a *plane 3-fold scroll* is the set of planes

$$\mathcal{S}(\pi_1, \pi_2, \pi_3) = \{\langle P, P^{\phi_2}, P^{\phi_3}\rangle \mid P \in \pi_1\}.$$

It is well known that this set of planes is one system of maximal subspaces of the Segre variety $\mathcal{S}_{2;2}$. Further, each system of maximal subspaces of $\mathcal{S}_{2;2}$ form a scroll.

5 Conics of $\mathrm{PG}(2, q^3)$ in the Bose representation

In this section, we use Theorem 3.1 to show that a non-degenerate conic of $\mathrm{PG}(2, q^3)$ corresponds to the intersection of three quadrics in $\mathrm{PG}(8, q)$. Further, we describe the extension to $\mathrm{PG}(8, q^3)$ and $\mathrm{PG}(8, q^6)$. We note that Gill [13] looks at conics of $\mathrm{PG}(2, q^3)$ using field reduction, by taking the polarity of a conic and generating one related form in $\mathrm{PG}(8, q)$. Our approach is different, we take the equation of a conic and generate three related forms in $\mathrm{PG}(8, q)$.

Theorem 5.1 *Let $\bar{\mathcal{O}}$ be a non-degenerate conic in $\mathrm{PG}(2, q^3)$. Consider the Bose representation.*

1. *In $\mathrm{PG}(8, q)$, the pointset of $[\![\mathcal{O}]\!]$ coincides with the pointset of a variety $\mathcal{Q}_0 \cap \mathcal{Q}_1 \cap \mathcal{Q}_2$, where $\mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3$ are quadrics of $\mathrm{PG}(8, q)$.*
2. *In $\mathrm{PG}(8, q^3)$, the extension $\mathcal{Q}_0^{\star} \cap \mathcal{Q}_1^{\star} \cap \mathcal{Q}_2^{\star}$ has pointset which coincides with the points on the planes $\{\langle X, Y^q, Z^{q^2}\rangle \mid X, Y, Z \in \mathcal{O}\}$.*
3. *In $\mathrm{PG}(8, q^6)$, the extension $\mathcal{Q}_0^{\star} \cap \mathcal{Q}_1^{\star} \cap \mathcal{Q}_2^{\star}$ has pointset which coincides with the points on the planes $\{\langle X, Y^q, Z^{q^2}\rangle \mid X, Y, Z \in \mathcal{O}^{\star}\}$, where $\mathcal{O}^{\star}$ is the quadratic extension of the conic $\mathcal{O}$ to $\Gamma^{\star} \subset \mathrm{PG}(8, q^6)$.*

Proof Let $\bar{\mathcal{O}}$ be a non-degenerate conic in $\mathrm{PG}(2, q^3)$, so $\bar{\mathcal{O}}$ is the set of points satisfying a homogeneous equation $F(x, y, z) = 0$ of degree 2 over $\mathbb{F}_{q^3}$. As in Theorem 3.1, we can write $F(x, y, z) = f(x_0, x_1, x_2, y_0, y_1, y_2, z_0, z_1, z_2) = f_0 + f_1\tau + f_2\tau^2$

where f_0, f_1, f_2 are homogeneous equations of degree 2 over $\mathbb{F}_q$. The set of points of $\mathrm{PG}(8,q)$ satisfying $f_i = 0$ form a quadric denoted $\mathcal{Q}_i$, $i = 0, 1, 2$, and the pointset of $[\![\mathcal{O}]\!]$ is the intersection of the three quadrics $\mathcal{Q}_0 \cap \mathcal{Q}_1 \cap \mathcal{Q}_2$.

The extension to $\mathrm{PG}(8,q^3)$ is the set of points $P \in \mathrm{PG}(8,q^3)$ with $f_i(\boldsymbol{P}) = 0$, $i = 0, 1, 2$; this is denoted $\mathcal{Q}_0^{\star} \cap \mathcal{Q}_1^{\star} \cap \mathcal{Q}_2^{\star}$. We can equivalently define this as the set of points on the intersection of three quadrics with linearly independent polynomials $\lambda_0 f_0 + \lambda_1 f_1 + \lambda_2 f_2$ for some $\lambda_0, \lambda_1, \lambda_2 \in \mathbb{F}_{q^3}$. Let $\mathcal{T}_0$ be the quadric of $\mathrm{PG}(8,q^3)$ with homogeneous polynomials $h = f_0 + \tau f_1 + \tau^2 f_2$. Let $\mathcal{T}_1$ be the quadric with homogeneous equation $h^q = 0$ and $\mathcal{T}_2$ be the quadric with homogeneous equation $h^{q^2} = 0$. Then $\mathcal{Q}_0^{\star} \cap \mathcal{Q}_1^{\star} \cap \mathcal{Q}_2^{\star} = \mathcal{T}_0 \cap \mathcal{T}_1 \cap \mathcal{T}_2$. By Theorem 3.1, $\mathcal{T}_0$ is a cone with base $\mathcal{O}$ in Γ and vertex $\langle \Gamma^q, \Gamma^{q^2} \rangle$. Similarly $\mathcal{T}_1$ is a cone with base $\mathcal{O}^q$ and vertex $\langle \Gamma, \Gamma^{q^2} \rangle$ and $\mathcal{T}_2$ is a cone with base $\mathcal{O}^{q^2}$ and vertex $\langle \Gamma, \Gamma^q \rangle$. The intersection of these three cones is the set of T-planes that contain a point of $\mathcal{O}$, a point of $\mathcal{O}^q$ and a point of $\mathcal{O}^{q^2}$, proving part 2.

In $\mathrm{PG}(8,q^3)$, we have $\mathcal{Q}_0^{\star} \cap \mathcal{Q}_1^{\star} \cap \mathcal{Q}_2^{\star} = \mathcal{T}_0 \cap \mathcal{T}_1 \cap \mathcal{T}_2$, so in $\mathrm{PG}(8,q^6)$ we have $\mathcal{Q}_0^{\star} \cap \mathcal{Q}_1^{\star} \cap \mathcal{Q}_2^{\star} = \mathcal{T}_0{}^{\star} \cap \mathcal{T}_1{}^{\star} \cap \mathcal{T}_2{}^{\star}$. In $\mathrm{PG}(8,q^3)$, the quadric $\mathcal{T}_0$ is a cone with base $\mathcal{O}$ in Γ and vertex $\langle \Gamma^q, \Gamma^{q^2} \rangle$. Hence in $\mathrm{PG}(8,q^6)$, $\mathcal{T}_0{}^{\star}$ is a cone with base $\mathcal{O}^{\star}$ in $\Gamma^{\star}$ and vertex $\langle \Gamma^q, \Gamma^{q^2} \rangle^{\star} = \langle (\Gamma^q)^{\star}, (\Gamma^{q^2})^{\star} \rangle$. We can similarly describe the quadrics $\mathcal{T}_1{}^{\star}, \mathcal{T}_2{}^{\star}$ as cones. The intersection of these three cones is the set of T-planes that contain a point of $\mathcal{O}^{\star}$, a point of $(\mathcal{O}^q)^{\star} = (\mathcal{O}^{\star})^q$ and a point of $(\mathcal{O}^{q^2})^{\star} = (\mathcal{O}^{\star})^{q^2}$. That is, the set of planes of form $\langle X, Y^q, Z^{q^2} \rangle$ where $X, Y, Z \in \mathcal{O}^{\star}$, proving part 3. □

6 Sublines and subplanes of $\mathrm{PG}(2,q^3)$

The representation of $\mathbb{F}_q$-subplanes and $\mathbb{F}_q$-sublines of $\mathrm{PG}(2,q^3)$ in the Bose representation in $\mathrm{PG}(8,q)$ is known. For example, it is proved in [19] and also in [18, Theorem 2.6] using field reduction techniques.

Result 6.1 *1. Let $\bar{\pi}$ be an $\mathbb{F}_q$-subplane of $\mathrm{PG}(2,q^3)$ then in $\mathrm{PG}(8,q)$, the planes $\{[\![X]\!] \mid X \in \pi\}$ of the Bose representation $[\![\pi]\!]$ form one system of maximal spaces of a Segre variety $\mathcal{S}_{2;2}$.*

2. Let $\bar{b}$ be an $\mathbb{F}_q$-subline lying on the line $\bar{\ell}_b$ of $\mathrm{PG}(2,q^3)$, then the planes of $[\![b]\!]$ form a 2-regulus of the 5-space $\Pi_b = \langle \ell_b, \ell_b^q, \ell_b^{q^2} \rangle \cap \mathrm{PG}(8,q)$.

Remark 6.2 Note that this result follows the convention described in Example 3.8. Let $\bar{\pi}$ be an $\mathbb{F}_q$-subplane of $\mathrm{PG}(2,q^3)$, so π is a variety whose forms in $\mathbb{F}_{q^3}[x,y,z]$ have degree $q+1$. Using Theorem 3.1, this corresponds to a variety $\mathbf{v}(f_{i,j})$ with forms in $f_{i,j} \in \mathbb{F}_q[x_0,\ldots,x_8]$ of degree $q+1$. As in Example 3.8, we reduce these exponents to get forms $g_{i,j} \in \mathbb{F}_q[x_0,\ldots,x_8]$ of degree 2. In $\mathrm{PG}(8,q)$, the intersection of the quadrics $\mathbf{v}(g_{i,j})$ is a Segre variety $\mathcal{S}_{2;2}$, as described in Result 6.1.

Let $\bar{\pi}$ be an $\mathbb{F}_q$-subplane of $\mathrm{PG}(2,q^3)$, so by Result 6.1, $[\![\pi]\!]$ is a Segre variety. As discussed in Section 4.2, this is the intersection of six quadrics $\mathcal{Q}_1 \cap \cdots \cap \mathcal{Q}_6$ where

$\mathcal{Q}_i$ has form $f_i \in \mathbb{F}_q[x_0, \ldots, x_8]$ of degree two, $i = 1, \ldots, 6$. We use the notation $\mathcal{V}_\pi = \mathcal{Q}_1 \cap \cdots \cap \mathcal{Q}_6$ and look at the extension of this to $\mathrm{PG}(8, q^3)$ and $\mathrm{PG}(8, q^6)$. We use the two collineations of $\mathrm{PG}(8, q^6)$ defined in Section 2.7, namely $\boldsymbol{X} = (x_0, \ldots, x_8) \longmapsto \boldsymbol{X}^q = (x_0^q, \ldots, x_8^q)$ and $\mathsf{e}\colon \boldsymbol{X} = (x_0, \ldots, x_8) \mapsto \boldsymbol{X}^{\mathsf{e}} = (x_0^{q^3}, \ldots, x_8^{q^3})$.

Theorem 6.3 *Let $\bar{\pi}$ be an $\mathbb{F}_q$-subplane of $\mathrm{PG}(2, q^3)$. In the Bose representation, let c_π be the collineation of order 3 acting on the points of Γ which fixes π pointwise as defined in (1). Let $\mathcal{V}_\pi = \mathcal{Q}_1 \cap \cdots \cap \mathcal{Q}_6$ denote the Segre variety $\mathcal{S}_{2;2}$ whose pointset coincides with the pointset of $[\![\pi]\!]$.*

1. *In $\mathrm{PG}(8, q^3)$, $\mathcal{V}_\pi^{\star} = \mathcal{Q}_1^{\star} \cap \cdots \cap \mathcal{Q}_6^{\star}$ is a Segre variety $\mathcal{S}_{2;2}$, with one system of maximal spaces the planes $\{(\!|X|\!)_\pi \mid X \in \Gamma\}$ where*

$$(\!|X|\!)_\pi = \langle\, X,\ (X^{\mathsf{c}_\pi^2})^q,\ (X^{\mathsf{c}_\pi})^{q^2}\,\rangle.$$

2. *In $\mathrm{PG}(8, q^6)$, $\mathcal{V}_\pi^{\bigstar} = \mathcal{Q}_1^{\bigstar} \cap \cdots \cap \mathcal{Q}_6^{\bigstar}$ is a Segre variety $\mathcal{S}_{2;2}$, with one system of maximal spaces the planes $\{(\!|X|\!)_\pi \mid X \in \Gamma^{\bigstar}\}$ where*

$$(\!|X|\!)_\pi = \big\langle\, X, (X^{\mathsf{c}_\pi^2\mathsf{e}})^q, (X^{\mathsf{c}_\pi\mathsf{e}})^{q^2}\,\big\rangle = \big\langle\, X, (X^{\mathsf{c}_\pi^5})^q, (X^{\mathsf{c}_\pi^4})^{q^2}\,\big\rangle.$$

Proof Let $\bar{\pi}$ be an $\mathbb{F}_q$-subplane of $\mathrm{PG}(2, q^3)$, so $[\![\pi]\!]$ is a Segre variety $\mathcal{V}_\pi = \mathcal{Q}_1 \cap \cdots \cap \mathcal{Q}_6$ where $\mathcal{Q}_i$ has form $f_i \in \mathbb{F}_q[x_0, \ldots, x_8]$ of degree two, $i = 1, \ldots, 6$. The points of $\mathcal{V}_\pi$ lie on two systems of planes denoted $\mathcal{R} = \{\alpha_i,\ i = 0, \ldots, q^2 + q\}$, $\mathcal{R}' = \{\alpha_i',\ i = 0, \ldots, q^2 + q\}$. The extension of this variety is $\mathcal{V}_\pi^{\star} = \mathcal{Q}_1^{\star} \cap \cdots \cap \mathcal{Q}_6^{\star}$, that is, $\mathcal{V}_\pi^{\star}$ is the set of points P of $\mathrm{PG}(8, q^3)$ which satisfy $f_1(\boldsymbol{P}) = \cdots = f_6(\boldsymbol{P}) = 0$. Further, $\mathcal{V}_\pi^{\star}$ is a Segre variety $\mathcal{S}_{2;2}$, so the points of $\mathcal{V}_\pi^{\star}$ lie on two systems of planes $\mathcal{T}, \mathcal{T}'$ each of size $q^6 + q^3 + 1$. Note that ${\alpha_i}^{\star} \in \mathcal{T}$, $i = 0, \ldots, q^2 + q$ and ${\alpha_i'}^{\star} \in \mathcal{T}'$, $i = 0, \ldots, q^2 + q$.

As discussed in Section 4.3, the planes of $\mathcal{R}$ form a scroll. That is, without loss of generality suppose that the three planes $\alpha_0', \alpha_1', \alpha_2'$ of $\mathcal{R}'$ do not lie in a 5-space. Then there are homographies $\phi_i \in \mathrm{PGL}(3, q)$, $\phi_i : \alpha_0' \to \alpha_i'$, $i = 1, 2$, and $\mathcal{S}(\alpha_0', \alpha_1', \alpha_2') = \mathcal{R} = \{\langle P, P^{\phi_1}, P^{\phi_2}\rangle \mid P \in \alpha_0'\}$. The scroll-extension is the set of $q^6 + q^3 + 1$ planes $\{\langle P, P^{\phi_1^{\star}}, P^{\phi_2^{\star}}\rangle \mid P \in {\alpha_0'}^{\star}\}$. The planes of the Segre variety $\mathcal{V}_\pi^{\star}$ in $\mathcal{T}$ also form a scroll. As a homography is uniquely determine by the image of a quadrangle, the planes in the scroll-extension of $\mathcal{S}(\alpha_0', \alpha_1', \alpha_2')$ coincides with the planes of $\mathcal{T}$. Hence to describe the planes of $\mathcal{T}$, we determine the planes of the scroll.

Without loss of generality, let $\bar{\pi} = \mathrm{PG}(2, q) = \{(x, y, z) \mid x, y, z \in \mathbb{F}_q,\ \text{not all } 0\}$. In the Bose representation,

$$\pi = \{x\mathbf{A}_0 + y\mathbf{A}_1 + z\mathbf{A}_2 \mid x, y, z \in \mathbb{F}_q,\ \text{not all } 0\}$$

is an $\mathbb{F}_q$-subplane of the transversal plane Γ. The planes of $[\![\pi]\!]$ have form $\{[\![X]\!] \mid X \in \pi\}$. Using the notation of Section 2.3, for a point $\boldsymbol{X} = x\mathbf{A}_0 + y\mathbf{A}_1 + z\mathbf{A}_2 \in \pi$, (so $x, y, z \in \mathbb{F}_q$) we have $[\![X]\!] = [\![X]\!]^{\star} \cap \mathrm{PG}(8, q)$ where

$$[\![X]\!]^{\star} = \langle X,\ X^q,\ X^{q^2}\rangle = \langle x\mathbf{A}_0 + y\mathbf{A}_1 + z\mathbf{A}_2,\ x{\mathbf{A}_0}^q + y{\mathbf{A}_1}^q + z{\mathbf{A}_2}^q,\ x{\mathbf{A}_0}^{q^2} + y{\mathbf{A}_1}^{q^2} + z{\mathbf{A}_2}^{q^2}\rangle.$$

Note that $\phi_i : x\mathbf{A}_0 + y\mathbf{A}_1 + z\mathbf{A}_2 \mapsto x{\mathbf{A}_0}^{q^i} + y{\mathbf{A}_1}^{q^i} + z{\mathbf{A}_2}^{q^i}$, $i = 1, 2$, so ϕ_i is essentially the identity in $\mathrm{PGL}(3, q)$. The scroll-extension to $\mathrm{PG}(8, q^3)$ is the set of planes

$$\{\langle x\mathbf{A}_0{+}y\mathbf{A}_1{+}z\mathbf{A}_2,\ x{\mathbf{A}_0}^q{+}y{\mathbf{A}_1}^q{+}z{\mathbf{A}_2}^q,\ x{\mathbf{A}_0}^{q^2}{+}y{\mathbf{A}_1}^{q^2}{+}z{\mathbf{A}_2}^{q^2}\rangle \mid x, y, z \in \mathbb{F}_{q^3}, \text{ not all } 0\}.$$

Using the calculations from (3) in Section 2.6, this is the set of planes

$$\{\langle\, X,\ (X^{\mathsf{c}_\pi^2})^q,\ (X^{\mathsf{c}_\pi})^{q^2}\,\rangle \mid X \in \Gamma\}.$$

Hence the Segre variety $\mathcal{V}_\pi^{\star}$ has as one system of maximal spaces the planes $\{(\!|X|\!)_\pi \mid X \in \Gamma\}$, with $(\!|X|\!)_\pi = \langle\, X,\ (X^{\mathsf{c}_\pi^2})^q,\ (X^{\mathsf{c}_\pi})^{q^2}\,\rangle$, proving part 1.

The proof of part 2 is similar. The scroll-extension of $\mathcal{S}_\pi$ to $\mathrm{PG}(8, q^6)$ is the set of planes

$$\{\langle x\mathbf{A}_0{+}y\mathbf{A}_1{+}z\mathbf{A}_2,\ x{\mathbf{A}_0}^q{+}y{\mathbf{A}_1}^q{+}z{\mathbf{A}_2}^q,\ x{\mathbf{A}_0}^{q^2}{+}y{\mathbf{A}_1}^{q^2}{+}z{\mathbf{A}_2}^{q^2}\rangle \mid x, y, z \in \mathbb{F}_{q^6} \text{ not all } 0\}.$$

Using the calculations from (4), this is the set of planes

$$\{\langle\, X, (X^{\mathsf{c}_\pi^2\mathsf{e}})^q, (X^{\mathsf{c}_\pi\mathsf{e}})^{q^2}\,\rangle \mid X \in \Gamma^\star\}.$$

Hence the Segre variety $\mathcal{V}_\pi^\star$ has as one system of maximal spaces the planes $\{(\!|X|\!)_\pi \mid X \in \Gamma^\star\}$ with $(\!|X|\!)_\pi = \langle\, X, (X^{\mathsf{c}_\pi^2\mathsf{e}})^q, (X^{\mathsf{c}_\pi\mathsf{e}})^{q^2}\,\rangle$, proving part 2. □

Note that for a point $X \in \pi$, $X^{\mathsf{c}_\pi} = X$ and $(\!|X|\!)_\pi = [\![X]\!]^{\star}$, which meets $\mathrm{PG}(8, q)$ in a plane $[\![X]\!]$. However, if $X \in \Gamma\backslash\pi$, then by Corollary 2.2, the plane $(\!|X|\!)_\pi$ does not meet any plane of $\{[\![X]\!] \mid X \in \Gamma\}$. As the planes in the set $\{[\![X]\!] \mid X \in \Gamma\}$ partition the points of $\mathrm{PG}(8, q)$, it follows that $(\!|X|\!)_\pi$ is disjoint from $\mathrm{PG}(8, q)$ for $X \in \Gamma\backslash\pi$.

We use Theorem 6.3 to look at an $\mathbb{F}_q$-subline $\bar{b}$, and describe the planes of the extension of the 2-regulus $[\![b]\!]$ to $\mathrm{PG}(8, q^3)$ and $\mathrm{PG}(8, q^6)$.

Corollary 6.4 *Let $\bar{b}$ be an $\mathbb{F}_q$-subline lying on the line $\bar{\ell}_b$ of $\mathrm{PG}(2, q^3)$. In the Bose representation, let c_b be the collineation of order 3 acting on the points of ℓ_b which fixes b pointwise as defined in (2). The 2-regulus $[\![b]\!]$ can be extended to a unique 2-regulus of $\mathrm{PG}(8, q^3)$ with planes $\{(\!|X|\!)_b \mid X \in \ell_b\}$; and to a unique 2-regulus of $\mathrm{PG}(8, q^6)$ with planes $\{(\!|X|\!)_b \mid X \in \ell_b^\star\}$ where*

$$(\!|X|\!)_b = \big\langle\, X, (X^{\mathsf{c}_b^5})^q, (X^{\mathsf{c}_b^4})^{q^2}\,\big\rangle.$$

Proof Note that if $X \in \ell_b$, then $X^{\mathsf{c}_b^5} = X^{\mathsf{c}_b^2}$ and $X^{\mathsf{c}_b^4} = X^{\mathsf{c}_b}$. Further, if $X \in b$, then $X^{\mathsf{c}_b^5} = X^{\mathsf{c}_b^4} = X$. If π is an $\mathbb{F}_q$-subplane of Γ, then c_b coincides with c_π restricted to ℓ_b if and only if b is a line of π. Hence letting π be an $\mathbb{F}_q$-subplane of Γ such that b is a line of π, we can intersect the results of Theorem 6.3 with the 5-space $\Pi_b = \langle \ell_b, \ell_b^q, \ell_b^{q^2}\rangle$ to obtain the required result. □

7 $\mathbb{F}_q$-conics of $\mathrm{PG}(2,q^3)$ in the Bose representation

We define an $\mathbb{F}_q$-conic of $\mathrm{PG}(2,q^3)$ to be a non-degenerate conic in an $\mathbb{F}_q$-subplane of $\mathrm{PG}(2,q^3)$. That is, an $\mathbb{F}_q$-conic is projectively equivalent to a set of points in $\mathrm{PG}(2,q)$ that satisfy a non-degenerate homogeneous quadratic equation over $\mathbb{F}_q$. We determine the Bose representation of an $\mathbb{F}_q$-conic $\bar{\mathcal{C}}$ of $\mathrm{PG}(2,q^3)$. An $\mathbb{F}_q$-conic $\bar{\mathcal{C}}$ of $\mathrm{PG}(2,q^3)$ corresponds to an $\mathbb{F}_q$-conic in the transversal plane Γ denoted $\mathcal{C}$. Let $\mathcal{C}^+$ denote the unique $\mathbb{F}_{q^3}$-conic of Γ containing $\mathcal{C}$. The quadratic extension of the non-degenerate conic $\mathcal{C}^+ \subset \Gamma$ to the extended transversal plane $\Gamma^\star \cong \mathrm{PG}(2,q^6)$ is a non-degenerate conic which we denote by $\mathcal{C}^{++}$.

Theorem 7.1 *Let $\bar{\mathcal{C}}$ be an $\mathbb{F}_q$-conic in the $\mathbb{F}_q$-subplane $\bar{\pi}$ of $\mathrm{PG}(2,q^3)$, and consider the Bose representation of $\mathrm{PG}(2,q^3)$ in $\mathrm{PG}(8,q)$.*

1. *In $\mathrm{PG}(8,q)$, the planes of $[\![\mathcal{C}]\!]$ form a scroll of $\mathrm{PG}(8,q)$, and the pointset of $[\![\mathcal{C}]\!]$ forms a variety $\mathcal{V}_\mathcal{C}$ of dimension 3 and degree 6 which is the intersection of nine quadrics, $\mathcal{V}_\mathcal{C} = \mathcal{Q}_1 \cap \cdots \mathcal{Q}_9$.*
2. *In $\mathrm{PG}(8,q^3)$, the points of the variety $\mathcal{V}^{\star}_\mathcal{C} = \mathcal{Q}^{\star}_1 \cap \cdots \cap \mathcal{Q}^{\star}_9$ coincide with the points on the planes $\{(\!|X|\!)_\pi \mid X \in \mathcal{C}^+\}$, which form a scroll.*
3. *In $\mathrm{PG}(8,q^6)$, the points of the variety $\mathcal{V}^\star_\mathcal{C} = \mathcal{Q}^\star_1 \cap \cdots \cap \mathcal{Q}^\star_9$ coincide with the points on the planes $\{(\!|X|\!)_\pi \mid X \in \mathcal{C}^{++}\}$, which form a scroll.*

Proof Let $\mathcal{C}$ be an $\mathbb{F}_q$-conic in an $\mathbb{F}_q$-subplane π of Γ, so $\mathcal{C} = \pi \cap \mathcal{C}^+$. By definition, in $\mathrm{PG}(8,q)$, $[\![\mathcal{C}]\!] = \{\langle X, X^q, X^{q^2}\rangle \cap \mathrm{PG}(8,q) \mid X \in \mathcal{C}\}$, $[\![\pi]\!] = \{\langle X, X^q, X^{q^2}\rangle \cap \mathrm{PG}(8,q) \mid X \in \pi\}$ and $[\![\mathcal{C}^+]\!] = \{\langle X, X^q, X^{q^2}\rangle \cap \mathrm{PG}(8,q) \mid X \in \mathcal{C}^+\}$, so

$$[\![\mathcal{C}]\!] = [\![\mathcal{C}^+]\!] \cap [\![\pi]\!]. \tag{6}$$

By Result 6.1, the planes of $[\![\pi]\!]$ form one system of maximal subspaces of a Segre variety $\mathcal{S}_{2;2}$, and so form a scroll. As the planes of $[\![\mathcal{C}]\!]$ are a subset of the planes of $[\![\pi]\!]$, the planes of $[\![\mathcal{C}]\!]$ form a scroll, ruled by the same homography as for the scroll $[\![\pi]\!]$. Recall from Section 4.3, the pointset of $[\![\mathcal{C}]\!]$ forms a variety $\mathcal{V}^6_3$. By Theorem 5.1, the pointset of $[\![\mathcal{C}^+]\!]$ forms a variety $\mathcal{V}_{\mathcal{C}^+} = \mathcal{Q}_1 \cap \mathcal{Q}_2 \cap \mathcal{Q}_3$ where $\mathcal{Q}_i$ is a quadric with homogenous equation $f_i = 0$ of degree two over $\mathbb{F}_q$, $i = 1,2,3$. As in Section 4.2, the Segre variety $\mathcal{S}_{2;2}$ is the intersection of six quadrics, so by Result 6.1, the pointset of $[\![\pi]\!]$ forms a variety $\mathcal{V}_\pi = \mathcal{Q}_4 \cap \cdots \cap \mathcal{Q}_9$ where $\mathcal{Q}_i$ is a quadric with homogenous equation $f_i = 0$ of degree two over $\mathbb{F}_q$, $i = 4,\dots,9$. So by (6), the pointset of $[\![\mathcal{C}]\!]$ coincides with the pointset of a variety $\mathcal{V}_\mathcal{C}$ which is the intersection of nine quadrics, namely $\mathcal{V}_\mathcal{C} = (\mathcal{Q}_1 \cap \mathcal{Q}_2 \cap \mathcal{Q}_3) \cap (\mathcal{Q}_4 \cap \cdots \cap \mathcal{Q}_9)$.

The set of points P in $\mathrm{PG}(8,q^3)$ which satisfy $f_i(\boldsymbol{P}) = 0$, $i = 1,\dots,9$ is denoted $\mathcal{V}^{\star}_\mathcal{C} = \mathcal{Q}^{\star}_1 \cap \cdots \cap \mathcal{Q}^{\star}_9$, so in particular,

$$\mathcal{V}^{\star}_\mathcal{C} = \mathcal{V}^{\star}_{\mathcal{C}^+} \cap \mathcal{V}^{\star}_\pi. \tag{7}$$

We now determine the points of $\mathcal{V}^{\star}_{\mathcal{C}}$. By Theorem 5.1, the points of $\mathcal{V}^{\star}_{\mathcal{C}^+}$ are the points of $\mathrm{PG}(8,q^3)$ on the planes

$$\{\langle X, Y^q, Z^{q^2}\rangle \mid X, Y, Z \in \mathcal{C}^+\}. \tag{8}$$

By Theorem 6.3, the points of $\mathcal{V}^{\star}_{\pi}$ are the points of $\mathrm{PG}(8,q^3)$ on the planes

$$\{(\!|X|\!)_\pi \mid X \in \pi\}. \tag{9}$$

The planes in (8) and (9) are T-planes, so by Corollary 2.2, two planes in (8) and (9) either coincide, are disjoint, or meet in a T-point or a T-line. Thus by (7), $\mathcal{V}^{\star}_{\mathcal{C}}$ consists of points on the set of planes which are in both (8) and (9). That is, $\mathcal{V}^{\star}_{\mathcal{C}}$ consists of the points of $\mathrm{PG}(8,q^3)$ on the planes $\{(\!|X|\!)_\pi \mid X \in \mathcal{C}^+\}$. Further, as in the proof of Theorem 6.3, the planes $\{(\!|X|\!)_\pi \mid X \in \mathcal{C}^+\}$ form a scroll. This completes the proof of part 2. Part 3 is similar. □

Acknowledgements

Thanks to the anonymous referees for their comments which helped improve the exposition of this article.

References

[1] J. André, Über nicht-Desarguessche Ebenen mit transitiver Translationgruppe, *Math. Z.* **60** (1954), 156–186.

[2] S. G. Barwick and G. L. Ebert, *Unitals in Projective Planes*, Springer Monographs in Mathematics. Springer, New York, 2008.

[3] S. G. Barwick and W.-A. Jackson, Sublines and subplanes of $\mathrm{PG}(2,q^3)$ in the Bruck-Bose representation in $\mathrm{PG}(6,q)$, *Finite Fields Appl.* 18 (2012), 93–107.

[4] S. G. Barwick and W.-A. Jackson, A Characterisation of Tangent Subplanes of $\mathrm{PG}(2,q^3)$, *Des. Codes Crypto.* **71** (2014), 541–545.

[5] S. G. Barwick and W.-A. Jackson, The exterior splash in $\mathrm{PG}(6,q)$: transversal lines, *Innov. Incidence Geom*, **17** (2019), 1–24.

[6] S. G. Barwick, W.-A. Jackson and P. Wild, 2-special normal rational curves in $\mathrm{PG}(4,q)$, *J. Geom.* 111 (2020), 3.

[7] S. G. Barwick, W.-A. Jackson and P. Wild, A characterisation of $\mathbb{F}_q$-conics of $\mathrm{PG}(2,q^3)$, Preprint. http://arxiv.org/abs/1907.02629

[8] R. C. Bose, On a representation of the Baer subplanes of the Desarguesian plane $\mathrm{PG}(2,q^2)$ in a projective five dimensional space, *Teorie Combinatorie*, vol. I, Accad. Naz. dei Lincei, Rome, 1976, (Rome, 1973), pp. 381–391.

[9] R. H. Bruck and R. C. Bose, The construction of translation planes from projective spaces, *J. Algebra* **1** (1964), 85–102.

[10] R. H. Bruck and R. C. Bose, Linear representations of projective planes in projective spaces, *J. Algebra* **4** (1966), 117–172.

[11] L. R. A. Casse and C. M. O'Keefe, Indicator sets for t-spreads of $\mathrm{PG}((s+1)(t+1)-1,q)$, *Bollettino U.M.I.* **4** (1990), 13–33.

[12] R. Casse and C. T. Quinn, Ruled cubic surfaces in $\mathrm{PG}(4,q)$, Baer subplanes of $\mathrm{PG}(2,q^2)$ and Hermitian curves, *Discrete Math.* **248** (2002), 17–25.

[13] N. Gill, Polar spaces and embeddings of classical groups, *New Zealand J. Math.* **36** (2007), 175–184.

[14] J. Harris, *Algebraic geometry, a first course*, Springer-Verlag, New York, 1992.

[15] J. W. P. Hirschfeld, *Projective Geometries over Finite Fields,* Second Edition, Oxford University Press, 1998.

[16] J. W. P. Hirschfeld and J. A. Thas, *General Galois Geometries,* Oxford University Press, 1991.

[17] M. Lavrauw and G. Van de Voorde, On $\mathbb{F}_q$-linear sets on a projective line, *Des. Codes Crypto.* **56** (2010), 89–104.

[18] M. Lavrauw and G. Van de Voorde, Field reduction and linear sets in finite geometry, *Contemp. Math.* **632** (2015), 271–293.

[19] G. Lunardon, Normal Spreads, *Geom. Ded.* **75** (1999), 245–261.

[20] K. Metsch, A note on Buekenhout-Metz unitals, Geometry, Combinatorial Designs and Related Structures (Spetses, 1996), *London Math. Soc. Lec. Note Ser.* **245** (1997), 177–180.

[21] B. Segre, Teoria di Galois, fibrazioni proiettive e geometrie non desarguesiane, *Ann. Mat. Pura Appl.* **64** (1964), 1–64.

[22] J. G. Semple and L. Roth, *Introduction to Algebraic Geometry*, Oxford University Press, 1949.

(Received 26 Apr 2019; revised 28 Apr 2020)