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Classes of cubic graphs containing cycles of integer-power lengths

P. J. Couch B. D. Daniel W. Paul Wright

Lamar University, Department of Mathematics
Beaumont, Texas 77710
U.S.A.

pjcouch@lamar.edu dale.daniel@lamar.edu research@weierstrass.org

Abstract

Erdős and Gyárfás conjectured in 1995 that every graph with minimum degree three has a cycle of length 2^k for some integer $k > 1$. Y. Caro has asked the related question of whether every such graph has a cycle whose length is a non-trivial power of some natural number. There have been numerous related questions and conjectures, including questions by various authors. We address a special case of the question of Caro, as well as others, by showing that every graph G of minimum degree 3, such that the set of centers of induced claws of G is independent, contains a cycle of length a^k for some integers $a \geq 2$ and $k \geq 2$.

1 Introduction

Erdős and Gyárfás conjectured ([5], [8]) that every graph with minimum degree three has a cycle of length 2^k for some integer $k > 1$.

Debose, Erdős, and Hobbs [4] narrowed the question by asking if each claw-free graph with minimum degree two, maximum degree three, and at most two vertices of degree 2 contains a cycle of length 2^k for some non-negative integer k. In [3], Shauger and the second-named author proved the result for planar claw-free graphs. In [11], the result is proved for cubic claw-free graphs of genus at most six. More recently, Verstraete has related results concerning unavoidable cycle lengths ([12], [14]), Heckman and Krakovski [7] have shown that each 3-connected 3-regular planar graph contains some 2^j cycle for $2 \leq j \leq 7$, and Bensmail [1] showed that there exist arbitrarily large cubic graphs all of whose 2-power cycles have length 4 only, or 8 only.

Herein, we study a related result. West [15] relates that Caro suggests the weaker question of whether every such graph has a cycle whose length is a non-trivial power of some natural number. We address a special case of the question of Caro, as well

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as others, by showing that every graph G of minimum degree 3, and such that the set of centers of induced claws of G is independent, contains a cycle of length a^k for some integers $a \geq 2$ and $k \geq 2$.

2 Preliminaries

We use the terminology of Bondy and Murty [2]. All graphs are finite, simple, and undirected. In particular, for a graph G, we let $\nu(G)$ denote the number of vertices of G. A graph in which each vertex has degree 3 is said to be cubic (or 3-regular). A *triangle* is an isomorphic copy of K_3. A vertex v of G is said to be *contained in a triangle of* G if and only if there exists a triangle T that is a subgraph of G and $v \in V(T)$.

An isomorphic copy of $K_{1,3}$ is said to be a *claw*. For graphs G and H, G is said to be H*-free* if G has no induced subgraph isomorphic to H. Our emphasis is on claw-free and almost claw-free cubic graphs. The reader is referred to the excellent survey [6] of such graphs by Faudree, Flandrin and Ryjáček.

3 Main Results

An important result in our Theorem 1 below is the following of Paz (Theorem 7.3 of [9]).

Lemma 1. *If m is a positive integer then for every positive integer n such that $n > \frac{14.4}{|\sqrt[m]{1.5}-1|^m}$ there is at least one positive integer a such that $n < a^m < \frac{3}{2}n$.*

We begin with a specific case of the more general results that follow, because the ideas and techniques used throughout are exemplified in the simpler case.

Theorem 1. *Suppose that G is a graph containing a cycle D such that:*

1. *D is not of length 10;*
2. *each vertex of D is of degree 3 in G;*
3. *each vertex of D is contained in precisely one triangle of G; and*
4. *if D meets a triangle T in G then D contains at most one edge of T.*

Then G has a cycle of length a^k for some positive integers $a \geq 2$ and $k \geq 2$.

Proof. The reader may verify that in the result of Paz above, n must be chosen greater than or equal to 286 in order that m be greater than or equal to 2. The following table (Figure 1) notes that there is at least one power a^k for some positive integers $a \geq 2$ and $k \geq 2$ such that $2n \leq a^k \leq 3n$ for each $n = 2, 3, 4, \ldots, 286$ with the exception of $n = 5$.

By contracting each triangle T of the cycle D in G to an associated unique vertex of degree 3 in the corresponding graph G', D gives rise to a cycle D' in G'. If D' has

n	a^k in $[2n, 3n]$
2	4
$3-4$	9
5	none
$6-8$	16
$9-13$	27
$14-16$	32
$17-28$	49
$29-81$	81
$82-112$	225
$113-121$	243
$122-171$	343
$172-256$	512
$257-286$	625

Figure 1: Integer-powers in $[2n, 3n]$.

length less than or equal ito 286 then the neighborhood of cycle D in G contains a cycle of each length $2n, \ldots, 3n$ and therefore a cycle of length a^k for some positive integers $a \geq 2$ and $k \geq 2$. Since D does not have length 10, D' does not have length 5.

We may therefore assume that cycle D in G has length $\nu(D) \geq \frac{3}{2} \cdot 286 = 429$. We may suppress one-third (or more) of the vertices of D to single vertices to note that the subgraph of G induced by the neighborhood $N(D)$ of D gives rise to every cycle length L in G such that $286 \leq \frac{2}{3}\nu(D) \leq L \leq \nu(D)$. By the result of Paz above, there are integers $a \geq 2$ and $k \geq 2$ such that $n < a^k < \frac{3}{2}n$. With $n = \frac{2}{3}\nu(D) \geq 286$, we have $\frac{2}{3}\nu(D) \leq a^k \leq \nu(D)$. □

Each vertex v of a cubic claw-free graph G is contained in exactly one triangle of G. Straightforward calculation ensures that such a graph G' must contain cycle lengths other than $n = 10$. The following corollary then follows immediately.

Corollary 1. *Suppose that G is a claw-free graph with $\delta \geq 3$. Then G has a cycle of length a^k for some positive integers $a \geq 2$ and $k \geq 2$.*

Corollary 2. *Suppose that G is a claw-free graph with minimum degree $\delta \geq 2$, maximum degree $\Delta = 3$, and the collection $V_2(G) = \{v \in V(G) : d(v) = 2\}$ has two or fewer elements. Then G has a cycle of length a^k for some positive integer $a \geq 2$ and some integer $k \geq 2$.*

Proof. Suppose that $V_2(G)$ consists of a single vertex v with neighborhood $N(v) = \{a, b\}$.

Assume that v is contained in no triangle and let G' denote $(G - v) \cup \{e = ab\}$. Then G' has a cycle E' of length a^k for some positive integers $a \geq 2$ and $k \geq 2$ by Corollary 1. It then follows that G has a cycle E of length $1 + a^k$ for some positive integer $a \geq 2$ and some integer $k \geq 2$. We may assume that E contains vertex v. If E contains only one edge of each triangle that it meets then its length is $2t+1$, where t is the number of triangles meeting E. Then G contains cycles of all lengths from $2t+1$ to $3t+1$, inclusive. With $2t+1$ playing the role of n in Lemma 1, there is a cycle of length a^k where $n = 2t+1 < a^m \leq 3t+1 < \frac{3}{2}(2t+1)$. As above, smaller cases of t may be calculated computationally. We may therefore assume that E contains two edges of some triangle that it meets. The length $1+a^k$ of E may then be reduced by one.

If v is contained in triangle T then two copies of G are joined by an edge whose end vertices are the copies of v. The resulting graph is cubic and claw-free and we then apply Corollary 1.

We therefore assume that $V_2(G) = \{u, v\}$. As a first case, assume that u and v are adjacent. We may assume that neither u nor v is contained in a triangle. Assume the other vertex adjacent to u is w. By replacing the path wuv by a single edge wv and applying the case above, we conclude that G has a cycle F of length $1+a^k$ for some positive integer $a \geq 2$ and some integer $k \geq 2$ and such that F contains vertex u and v. If F contains only one edge of each triangle that it meets then its length is $2t+2$, where t is the number of triangles meeting E. Then G contains cycles of all lengths from $2t+2$ to $3t+2$, inclusive. As above, there is a cycle of length b^j where $n = 2t+2 < b^j \leq 3t+2 < \frac{3}{2}(2t+2)$, with the smaller cases completed computationally. We may therefore assume that F contains two edges of some triangle that it meets. The length $1+a^k$ of F may then be reduced by one.

As a next case, we assume $V_2(G) = \{u, v\}$, that u and v are not adjacent, and that u is contained in triangle T_u and v is contained in triangle T_v. The proof of this is almost identical to the preceding case. As a final case, we assume $V_2(G) = \{u, v\}$, that u and v are not adjacent, and that u is contained in triangle T_u and v is contained in no triangle. The proof of this case is as in the case above that $V_2(G)$ consists of a single vertex and that vertex is contained in no triangle. □

Corollary 3. *Suppose that G is a graph containing a cycle D such that:*

1. *the minimum degree of vertices in D is 2 and there are at most two vertices of degree 2;*
2. *each vertex of D which is of degree 3 is contained in precisely one triangle of G;*
3. *if D meets a triangle T in G then D contains at most one edge of T; and*
4. *the length of D is not 5 if $V_2(D) = \{u\}$ and the length of D is not 10 if $V_2(D) = \{u, v\}$.*

Then G has a cycle of length a^k for some positive integers $a \geq 2$ and $k \geq 2$.

Theorem 2. *Suppose that a graph G has minimum degree 3 and that the set of centers of induced claws of G is independent. Then G contains a cycle of length a^k for some integers $a \geq 2$ and $k \geq 2$.*

The proof of this is very similar and is left to the reader; the only nuance is dealing with the potential for triangles that share vertices, which can be tedious and repetitive.

Ryjáček [10] has defined a graph G as *almost claw-free* if the set of centers of induced claws is independent and for every vertex x, the domination number of $G[N(x)]$ is at most two. It is straightforward to see that every claw-free graph is almost claw-free. Let G be an almost claw-free cubic graph. Suppose that D is a cycle in G such that the length $\nu(D)$ of D is greater than or equal to 6 and if D meets a triangle T in G then D contains at most one edge of T. The condition that the centers of induced claws are independent guarantees that such a cycle meets $\frac{1}{3}\lceil \nu(D) \rceil$ or more triangles in G, where $\lceil x \rceil$ is the usual ceiling function.

Theorem 3. *Suppose that G is an almost claw-free graph with minimum degree* 3. *Then G has a cycle of length a^k for some integers $a \geq 2$ and $k \geq 2$.*

Proof. We may assume that G is non-planar by [7]. As a result (e.g., [13]), G contains a cycle E with three pair-wise crossing chords. It is straightforward that such a cycle must have length greater than or equal to 20. We may therefore assume that the circumference of G is greater than or equal to 20. Let D be a cycle in G of length 20 or more such that if D meets a triangle T in G then D contains at most one edge of T. As in Theorem 1, we contract each of $\frac{1}{3}\lceil \nu(D) \rceil$ triangles of D in G to an associated unique vertex of degree 3 in the corresponding graph G', yielding a cycle D' in G'. If $\nu(D') \geq 286$ then the proof proceeds exactly as in Theorem 1. It therefore suffices to note that there is at least one power a^k for some positive integers $a \geq 2$ and $k \geq 2$ such that $n \leq a^k \leq \frac{4}{3}n$ for the smaller possible values of $n = \nu(D)$. A table may be easily constructed as in Theorem 1 demonstrating that such an a^k exists for all $20 \leq n \leq 286$. □

4 Remaining Questions

We close with some questions related to the results of this work. Some of these we have not studied, but they are clearly related and are of interest; some are new, while others are quite old.

1. If a graph G has minimum degree at least three, then does G contain a cycle whose length is a power of two? [5]
2. If a graph G is claw-free and is of minimum degree three, then does G contain a cycle whose length is a power of two? [3]
3. If a graph G has minimum degree three, then does G contain a cycle whose length is a power (of two or more) of some integer greater than or equal to 2? [15]
4. As a special case of the previous questions, what if G is assumed to be Hamiltonian?

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