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Counting asymmetric weighted pyramids in non-decreasing Dyck paths

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Abstract

A Dyck path is *non-decreasing* if the y-coordinates of its valleys form a non-decreasing sequence. A pyramid is *asymmetric* if the valleys determining the maximal pyramid are at distinct levels. In this paper we count non-decreasing Dyck paths having asymmetric pyramids of a fixed height at a fixed level in the path. These paths are counted using generating functions (by the symbolic method) and recurrence relations. We parameterize the results found here using Riordan arrays. We have found some relations between the asymmetric pyramids with the p-ascent sequences, the asymmetric Delannoy paths, the q-Catalan numbers, and the Fibonacci polynomials.

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1 Introduction

A *Dyck path* is a lattice path in the first quadrant of the xy-plane that starts at the origin and ends on the x-axis, composed of North-East (X) and South-East (Y) steps. We say that a Dyck path P has length $2n$, if P has exactly n North-East steps. A *valley* is a sub-path of the form YX and a *peak* is a sub-path of the form XY. The height of a valley is the y-coordinate of its lowest point. A Dyck path is called *non-decreasing* if the heights of its valleys form a non-decreasing sequence when we consider them from left to right. For example, in Figure 1 we show a non-decreasing Dyck path of length 32, where the valleys are the points $(5,1)$, $(7,1)$, $(11,1)$, $(14,2)$, $(20,4)$, and $(24,4)$.

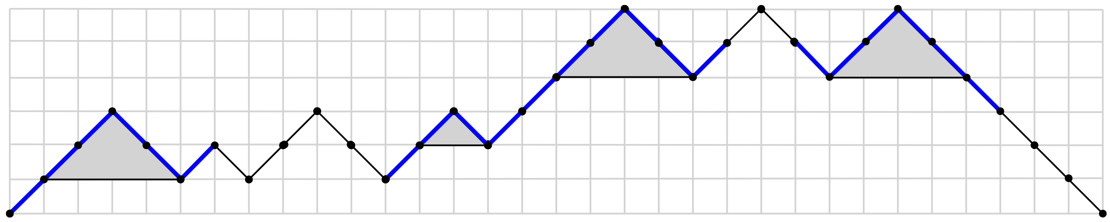

Figure 1: A non-decreasing Dyck path of length 32.

Following the notation from [4, 5, 7, 10], we denote by $\mathcal{D}$ the set of all non-decreasing Dyck paths, and by $\mathcal{D}_n$ the set of all non-decreasing Dyck paths of length $2n$. This gives $\mathcal{D} = \uplus_{n\geq 1}\mathcal{D}_n$ ($\uplus$ means disjoint union). A *pyramid* of semilength (height) $h \geq 1$ is a sub-path of the form X^hY^h, and it is *maximal* if it cannot be extended to a pyramid $X^{h+1}Y^{h+1}$. We use Δ_ℓ to denote a maximal pyramid of the form $X^\ell Y^\ell$. The *weight* of Δ_ℓ is defined as ℓ. Note that for simplicity we use the term weight instead of height of maximal pyramid. A maximal pyramid Δ_ℓ is *symmetric* if Δ_ℓ is not preceded by an X and is not followed by a Y; and it is *asymmetric* if Δ_ℓ is either preceded by an X or is followed by a Y. In Figure 1, the asymmetric pyramids are within the configurations represented with thick blue lines. Geometrically, Δ_ℓ is *symmetric* if it is at a ground level or if two valleys bounding Δ_ℓ are at the same height, and it is *asymmetric* otherwise (see [7, 8, 11]). For example, in Figure 1, the second, the third, and the sixth pyramids are symmetric. The first, the fourth, the fifth, and the seventh pyramids are asymmetric (see the shaded pyramids).

Let C be the set of all configurations of the form $Y^r\Delta_\ell Y$ and $X^r\Delta_\ell X^t$, where $\ell, r, t \geq 1$ and Δ_ℓ is maximal. Thus, C is the set of configurations containing asymmetric pyramids. These configurations are depicted in Figure 1. The *feature* of a peak p is the pair (h, w) where h is the height of p and w is the weight of the maximal pyramid containing p (clearly $h \geq w > 0$). For example, the peak in the fifth pyramid in Figure 1 has feature $(6, 2)$. If $c \in C$ and p is a peak in c with feature (h, w), then we use $f_n(c; h)$ to denote the number of paths in $\mathcal{D}_n$ containing p. For instance, in Figure 2 we can see that $f_5(Y\Delta_1Y; 2) = 8$. Note that the configuration $Y^r\Delta_\ell X^t$ (symmetric pyramid) was not considered here. It gives rise to a different problem that does not fit to the nature of this paper.

In this paper we give recursive relations and closed formulas to evaluate $f_n(c;h)$. These formulas are found using generating functions (by the symbolic method). We have found that these formulas are related with the asymmetric Delannoy paths, the q-Catalan numbers, and the Fibonacci polynomials. The numerical values resulting from $f_n(c;h)$ are encoded using Riordan arrays.

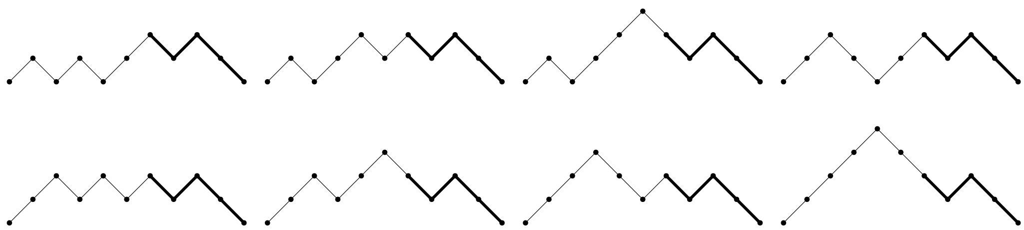

Figure 2: The number of paths in $\mathcal{D}_5$, $f_5(Y\Delta_1 Y;2) = 8$, containing $(Y\Delta_1 Y;2)$.

The non-decreasing Dyck paths can now be considered as a classic topic of research, some classic papers are [1, 6, 19]. However, recently the authors have found that this concept still has many new questions to investigate (see [4, 5, 10]). The authors have also found some new relations with other areas of combinatorics. For instance, questions studied in others type of lattice paths have been extended to non-decreasing paths, and sequences that have been used to count other combinatorial objects are also used to count aspects in non-decreacreasing lattice paths (some of them are in bijective relation, for example families of polyominoes, and elena trees). In this paper we give some connections between non-decreasing Dyck paths, q-Catalan numbers, p-ascent sequences, asymmetric Delannoy paths, and Fibonacci polynomials. Whoever is interested in Fibonacci numbers or Riordan arrays may find it interesting that most of the sequences from non-decreasing lattice paths can be expressed using Fibonacci numbers and/or Riordan arrays.

The non-decreasing concept has been extended to other types of lattice paths. For example, non-decreasing Motzkin paths [13] and t-Dyck Paths [12].

2 Counting asymmetric pyramids in a configuration of the form $Y^r\Delta_\ell Y$

In this section we count the number of paths having a maximal pyramid Δ_ℓ at a fixed height and lying in a configuration $Y^r\Delta_\ell Y$, where ℓ and r are fixed positive integers. We give a closed formula for $f_n(Y^r\Delta_\ell Y; s+\ell)$, where s is the height at which the pyramid Δ_ℓ is located. Note that by the nature of non-decreasing Dyck paths the configuration $Y^r\Delta_\ell Y$ always contains the last valley of the path. Finally we give a relation between these counts and the p-ascent sequences, the asymmetric Delannoy paths, the q-Catalan numbers, and the Fibonacci polynomials.

In [10] the authors prove that the generating function that counts Dyck paths having only valleys of height zero is given by $V(x) = (1-x)/(1-2x)$. We now define the generating function

$$T^{(1)}_{s,\ell,r}(x) := \sum_{n\geq r+\ell+s} f_n(Y^r\Delta_\ell Y; s+\ell)x^n,$$

where $r, \ell \geq 1$, and $s \geq 0$.

Theorem 2.1. *The generating function $T^{(1)}_{s,\ell,r}(x)$ is given by*

$$T^{(1)}_{s,\ell,r}(x) = \frac{x^{s+\ell+r}}{1-x}\left(\frac{1-x}{1-2x}\right)^{s+1}.$$

Proof. First of all, we note that the configuration $Y^r\Delta_\ell Y$, where its associated peak has feature $(s+\ell, \ell)$, contains the last valley of the path. Therefore, any non-decreasing Dyck path in $\mathcal{D}_n$ with the above configuration may be decomposed as

$$\overbrace{TXTX\cdots TX}^{s}\, T\Delta_{\geq r}\Delta_\ell Y^s, \tag{1}$$

where $T \in \mathcal{D}$ contains only valleys of height zero, Δ_ℓ is a maximal pyramid and $\Delta_{\geq r}$ is a pyramid of weight at least r, see Figure 3.

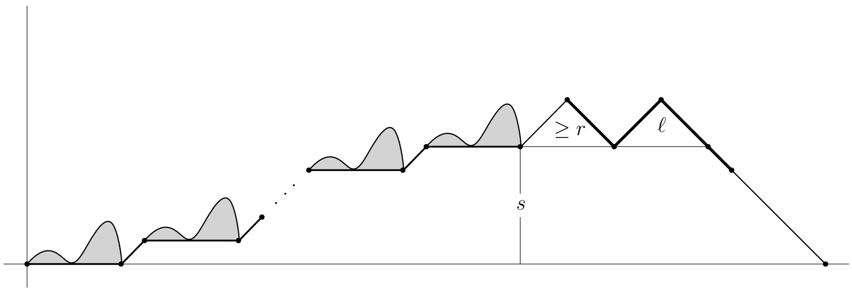

Figure 3: Factoring the path in (1).

From the symbolic method (see [9]) we obtain the equation

$$\begin{aligned} T^{(1)}_{s,\ell,r}(x) &= \overbrace{(V(x)x)\,(V(x)x)\cdots(V(x)x)}^{s}\, V(x)\frac{x^r}{1-x}x^\ell \\ &= \left(x\left(\frac{1-x}{1-2x}\right)\right)^s\left(\frac{1-x}{1-2x}\right)\frac{x^r}{1-x}x^\ell \\ &= \frac{x^{s+\ell+r}}{1-x}\left(\frac{1-x}{1-2x}\right)^{s+1}. \end{aligned}$$

This completes the proof. $\square$

The set $\mathcal{D}_n$ can be partitioned into two disjoint sets $\mathcal{A}_n$ and $\mathcal{B}_n$, where $\mathcal{A}_n$ contains the paths that have at least one valley of height 0, and $\mathcal{B}_n = \mathcal{D}_n \setminus \mathcal{A}_n$. Thus, a path

Q in $\mathcal{D}$ has one of these two forms: $Q = XPY$ or $Q = X^kY^kP$, where $k \geq 1$ and $P \in \mathcal{D}$ is non-empty. Note that

$$\mathcal{D}_n = \mathcal{A}_n \mathbin{\dot\cup} \mathcal{B}_n, \quad \text{and} \quad \mathcal{A}_n = \dot{\bigcup_{i=1}^{n-1}} \mathcal{C}_{n,i}, \tag{2}$$

where $\mathcal{C}_{n,i}$ is formed by paths whose first valley point is at $(2i, 0)$ ($\dot\cup$ means disjoint union). The set $\mathcal{C}_{n,i}$ can be mapped bijectively into $\mathcal{D}_{n-i}$ by removing the first pyramid Δ_i from all paths in $\mathcal{C}_{n,i}$. Thus,

$$\text{if } E_{n-i} = \{P \setminus \Delta_i \mid P \in \mathcal{C}_{n,i} \text{ and } \Delta_i \text{ is the first pyramid in } P\}, \text{ then } E_{n-i} = \mathcal{D}_{n-i}. \tag{3}$$

The set $\mathcal{B}_n$ can be mapped bijectively into $\mathcal{D}_{n-1}$ by removing the first up-step and last down-step from all paths in $\mathcal{B}_n$.

Theorem 2.2. *Let $\ell, s, r \geq 1$ and $n \geq \ell + s + r$. If $c_1 = Y^r \Delta_\ell Y$, then $f_n(c_1; s + \ell)$ satisfies the recurrence relation*

$$f_n(c_1; s+\ell) = 2f_{n-1}(c_1; s+\ell) + f_{n-1}(c_1; s-1+\ell) - f_{n-2}(c_1; s-1+\ell),$$

where $f_{\ell+r}(c_1; \ell) = 1$ and $f_n(c_1; s+\ell) = 0$ if $n < r + \ell$.

Proof. Let p be a peak within the configuration c_1 and with feature $(s + \ell, \ell)$. From (2) we know that $\mathcal{A}_n = \dot\cup_{i=1}^{n-1} \mathcal{C}_{n,i}$. From (3) we know that $\mathcal{C}_{n,i}$ maps bijectively into $\mathcal{D}_{n-i} = E_{n-i}$. This and the definition of $f_n(c; h)$ imply that the number of paths in $\mathcal{A}_n$ containing p is given by $F_{\mathcal{A}_n} := \sum_{i=\ell+r}^{n-1} f_i(c_1; s + \ell)$.

We now find the number of paths in $\mathcal{B}_n$ containing p. Since $\mathcal{B}_n$ maps bijectively into $\mathcal{D}_{n-1}$, this number is counted by the number of paths in $\mathcal{D}_{n-1}$ having c_1 which is given by $f_{n-1}(c_1; s-1+\ell)$ (recall that c_1 holds only in the end of the path). Adding this last result with $F_{\mathcal{A}_n}$ we have

$$f_n(c_1; s+\ell) = f_{n-1}(c_1; s-1+\ell) + \sum_{i=m}^{n-1} f_i(c_1; s+\ell),$$

where $f_{\ell+r}(c_1; \ell) = 1$ and $f_n(c_1; s+\ell) = 0$ for $n < r + l$. It is easy to see that

$$f_{n+1}(c_1; s+\ell) - f_n(c_1; s+\ell) = f_n(c_1; s+\ell) + f_n(c_1; s-1+\ell) - f_{n-1}(c_1; s-1+\ell),$$

which gives the expected result

$$f_n(c_1; s+\ell) = 2f_{n-1}(c_1; s+\ell) + f_{n-1}(c_1; s-1+\ell) - f_{n-2}(c_1; s-1+\ell). \qquad \square$$

2.1 A relation with the Riordan arrays

We now give some background of Riordan arrays [21]. The same background was given in [5, 10, 13]. An infinite lower triangular matrix is called a Riordan array if its kth column satisfies the generating function $g(x)\left(f(x)\right)^k$ for $k \geq 0$, where $g(x)$ and $f(x)$ are formal power series with $g(0) \neq 0$, $f(0) = 0$ and $f'(0) \neq 0$ (where $f'(x)$ is the formal derivative of $f(x)$). The matrix corresponding to the pair $f(x), g(x)$ is denoted by $(g(x), f(x))$. If we multiply (g, f) by a column vector $(c_0, c_1, \dots)^T$ associated to the generating function $h(x)$, then the resulting column vector has generating function $g(x)h(f(x))$. This property is known as the fundamental theorem of Riordan arrays or summation property.

The product of two Riordan arrays $(g(x), f(x))$ and $(h(x), l(x))$ is defined by

$$(g(x), f(x)) * (h(x), l(x)) = (g(x)h\left(f(x)\right), l\left(f(x)\right)).$$

We recall that the set of all Riordan arrays is a group under the operator " $*$ " [21]. The identity element is $I = (1, x)$, and the inverse of $(g(x), f(x))$ is

$$(g(x), f(x))^{-1} = \left(1/\left(g \circ \overline{f}\right)(x), \overline{f}(x)\right), \tag{4}$$

where $\overline{f}(x)$ is the compositional inverse of $f(x)$.

Let $\mathcal{A}_1 = [a_1(n,k)]_{n,k\geq 0}$ be a Riordan array defined by

$$\mathcal{A}_1 = \left(\frac{1}{1-x}, \frac{x(1-x)}{1-2x}\right).$$

We now show the first few rows of $\mathcal{A}_1$. So,

$$\mathcal{A}_1 = [a_1(n,k)]_{n,k\geq 0} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 4 & 3 & 1 & 0 & 0 & 0 & 0 \\ 1 & 8 & 8 & 4 & 1 & 0 & 0 & 0 \\ 1 & 16 & 20 & 13 & 5 & 1 & 0 & 0 \\ 1 & 32 & 48 & 38 & 19 & 6 & 1 & 0 \\ 1 & 64 & 112 & 104 & 63 & 26 & 7 & 1 \\ \vdots & & & & \vdots & & & \vdots \end{pmatrix}.$$

From the generating function given in Theorem 2.1 and the definition of Riordan array $\mathcal{A}_1$ we obtain the following theorem, and in Theorem 2.4 we give a combinatorial formula for the entries of the matrix $\mathcal{A}_1$.

Theorem 2.3. *If $n, s \geq 0$, $r, \ell \geq 1$, and $a_1(n,k)$ is the (n,k)-th entry of the Riordan array $\mathcal{A}_1$, then $f_n(Y^r\Delta_\ell Y; s+\ell) = a_1(n-\ell-r+1, s+1)$.*

Theorem 2.4. *If $n, k \geq 0$ and $a_1(n,k)$ is the (n,k)-th entry of the Riordan array $\mathcal{A}_1$, then*

$$a_1(n,k) = \sum_{j=0}^{n} \binom{k-1}{j}\binom{n-j-1}{k-1}(-1)^j 2^{n-j-k}.$$

Proof. From the definition of the Riordan array $\mathcal{A}_1$ we have

$$\begin{aligned} a_1(n,k) &= [x^n]\,\frac{1}{1-x}\left(\frac{x(1-x)}{1-2x}\right)^k \\ &= \left[x^{n-k}\right]\frac{(1-x)^{k-1}}{(1-2x)^k} \\ &= \left[x^{n-k}\right]\sum_{n\ge 0}\sum_{\ell=0}^{n}\binom{k-1}{\ell}\binom{k+(n-\ell)-1}{n-\ell}(-1)^\ell 2^{n-\ell}x^n. \end{aligned}$$

Therefore, comparing coefficients we obtain the desired result. $\square$

Let the Pascal matrix be $\mathbb{P} = \left[\binom{n}{k}\right]_{n,k\ge 0}$ and let $\mathbb{P}^{(\ell)} = \left[p_{n,k}^{(\ell)}\right]_{n,k\ge 0}$ be the matrix obtained from the partial row sums of $\mathbb{P}^{(\ell-1)} = \left[p_{n,k}^{(\ell-1)}\right]_{n,k\ge 0}$, with the initial value $\mathbb{P}^{(0)} = \mathbb{P}$. Note that $p_{n,k}^{(0)} = \binom{n}{k}$ and $p_{n,k}^{(\ell)} = \sum_{j=k}^{n} p_{n,j}^{(\ell-1)}$, for $\ell \ge 1$. For example,

$$\mathbb{P} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 3 & 3 & 1 & 0 & 0 & 0 \\ 1 & 4 & 6 & 4 & 1 & 0 & 0 \\ 1 & 5 & 10 & 10 & 5 & 1 & 0 \\ 1 & 6 & 15 & 20 & 15 & 6 & 1 \\ \vdots & & & & & & \vdots \end{pmatrix}, \qquad \mathbb{P}^{(1)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 4 & 3 & 1 & 0 & 0 & 0 & 0 \\ 8 & 7 & 4 & 1 & 0 & 0 & 0 \\ 16 & 15 & 11 & 5 & 1 & 0 & 0 \\ 32 & 31 & 26 & 16 & 6 & 1 & 0 \\ 64 & 63 & 57 & 42 & 22 & 7 & 1 \\ \vdots & & & & & & \vdots \end{pmatrix},$$

$$\mathbb{P}^{(2)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 & 0 & 0 \\ 8 & 4 & 1 & 0 & 0 & 0 & 0 \\ 20 & 12 & 5 & 1 & 0 & 0 & 0 \\ 48 & 32 & 17 & 6 & 1 & 0 & 0 \\ 112 & 80 & 49 & 23 & 7 & 1 & 0 \\ 256 & 192 & 129 & 72 & 30 & 8 & 1 \\ \vdots & & & & & & \vdots \end{pmatrix}.$$

We define the matrix $U = [u(n,k)]_{n,k\ge 0}$, where

$$u(n,k) := \begin{cases} 1, & n \ge k; \\ 0, & \text{otherwise.} \end{cases}$$

Note that U is the Riordan array given by $U = (1/(1-x), x)$. Therefore, the matrix $\mathbb{P}^{(\ell)}$ is the Riordan array

$$\mathbb{P}^{(\ell)} = \mathbb{P} * U^\ell = \left(\frac{1}{1-x}, \frac{x}{1-x}\right) * \left(\frac{1}{(1-x)^\ell}, x\right) = \left(\frac{1}{1-x}\left(\frac{1-x}{1-2x}\right)^\ell, \frac{x}{1-x}\right).$$

So, the first column of the matrix $\mathbb{P}^{(\ell)}$ coincides with the ℓ-th column of the Riordan array $\mathcal{A}_1$ ($\ell \geq 0$). So, we obtain the following theorem.

Theorem 2.5. *If $n \geq 0$, $k \geq 1$ and $a_1(n,k)$ is the (n,k)-th entry of the Riordan array $\mathcal{A}_1$, then $a_1(n,0) = 1$ and*

$$a_1(n,k) = \sum_{\ell_k=0}^{n-k} \sum_{\ell_{k-1}=0}^{\ell_k} \cdots \sum_{\ell_1=0}^{\ell_2} \binom{n-k}{\ell_1}.$$

From this theorem it is easy to verify that $a_1(n+1,1) = 2^n$. In [16] Hirschhorn proved that $\sum_{k=0}^{n} \sum_{j=0}^{k} \binom{n}{k} = (n+2)2^{n-1}$. So, $a_1(n+2,2) = (n+2)2^{n-1}$.

Theorem 2.6. *If $n, k \geq 0$ and $a_1(n,k)$ is the (n,k)-th entry of the Riordan array $\mathcal{A}_1$, then*

$$a_1(n+k,k) = \sum_{\ell=0}^{n} \binom{n}{\ell} \binom{k+\ell-1}{\ell}.$$

Proof. Since the ℓ-th column of the Riordan array $\mathcal{A}_1$ is equal to the first column of the Riordan array $\mathbb{P}^{(\ell)}$, we have

$$A_1(x) := \sum_{n\geq 0} a_1(n+k,k)x^n = \frac{1}{1-x}\left(\frac{1-x}{1-2x}\right)^k = \frac{1}{1-x} f\left(\frac{1}{1-x}\right), \tag{5}$$

where $f(x) = 1/(1-x)^\ell$. Since

$$\frac{1}{(1-x)^k} = \sum_{n\geq 0} \binom{k+n-1}{n} x^n,$$

by the Cauchy product we have

$$A_1(x) = \sum_{n\geq 0} \left(\sum_{\ell=0}^{n} \binom{n}{\ell} \binom{k+\ell-1}{\ell} \right) x^n.$$

Comparing the n-th coefficient of this last equation with the n-th coefficient of (5) we obtain the desired result. □

If $n, s \geq 0$ and $r, \ell \geq 1$, then from Theorem 2.3 we have this combinatorial formula

$$f_n(Y^r \Delta_\ell Y; s+\ell) = \sum_{k=0}^{n-\ell-r-s} \binom{n-\ell-r-s}{k} \binom{s+k}{k}. \tag{6}$$

We now find an hypergeometric expression for the sequences $\{f_n(Y^r\Delta_\ell Y; s+\ell)\}_n$. First we give some needed background. The rising factorial (also known as Pochhammer symbol) is defined by

$$(x)_n = \begin{cases} x(x+1)(x+2)\cdots(x+n-1), & \text{if } n \geq 1; \\ 1, & \text{if } n = 0. \end{cases} \tag{7}$$

Clearly $(1)_n = n!$. The hypergeometric function (or hypergeometric series) is defined by

$${}_pF_q\left(\begin{matrix} a_1, & a_2, & \dots, & a_p \\ b_1, & b_2, & \dots, & b_q \end{matrix} \middle| t \right) = \sum_{k=0}^{\infty} \frac{(a_1)_k(a_2)_k\cdots(a_p)_k}{(b_1)_k(b_2)_k\cdots(b_q)_k}\frac{t^k}{k!}.$$

Theorem 2.7. *If $n, s \geq 0$ and $r, \ell \geq 1$ are integers, then*

$$f_n(Y^r\Delta_\ell Y; s+\ell) = {}_2F_1\left(\begin{matrix} \ell+r+s-n, & s+1 \\ 1 & \end{matrix} \middle| -1 \right).$$

Proof. The Equation (6) and the relation

$$\binom{-x}{n} = \frac{(-1)^n}{n!}(x)_n$$

imply that

$$\begin{aligned} f_n(Y^r\Delta_\ell Y; s+\ell) &= \sum_{k\geq 0}\binom{n-\ell-r-s}{k}\binom{s+k}{k} \\ &= \sum_{k\geq 0}\frac{(-1)^k}{k!}(\ell+r+s-n)_k\frac{(-1)^k}{k!}(-s-k)_k \\ &= \sum_{k\geq 0}\frac{(\ell+r+s-n)_k(s+1)_k}{(1)_k}\frac{(-1)^k}{k!} \\ &= {}_2F_1\left(\begin{matrix} \ell+r+s-n, & s+1 \\ 1 & \end{matrix} \middle| -1 \right). \end{aligned}$$

This completes the proof. $\square$

These are some particular closed formulas:

$$\begin{aligned} &f_n(Y\Delta_1 Y; 2) = (n-1)2^{n-4},\ n \geq 3, \\ &f_n(Y\Delta_1 Y; 3) = f_n(Y\Delta_1 YY; 3) = (n^2-n-4)2^{n-7},\ n \geq 4, \\ &f_n(Y\Delta_1 Y; 4) = \frac{1}{3}(n^3-19n+18)2^{n-9},\ n \geq 5, \\ &f_n(Y\Delta_2 Y; 3) = (n-2)2^{n-5},\ n \geq 4, \\ &f_n(Y^2\Delta_2 Y; 7) = \frac{1}{3}(n^4-10n^3-13n^2+286n-408)2^{n-16},\ n \geq 9. \end{aligned}$$

3 A relation with other combinatorial objects

In this section we give a brief connection between the sequences associated to asymmetric pyramids and other combinatorial objects; such as Delannoy paths, p-ascent sequences, q-Catalan numbers, and Fibonacci polynomials.

3.1 A relation with the p-ascent sequences

The number of *ascents* of the sequence $(a_1, \dots, a_n)$ of non-negative integers is defined as

$$\operatorname{asc}(a_1, \dots, a_n) := |\{j : a_j < a_{j+1}, 1 \leqslant j < n\}|.$$

A sequence $(a_1, \dots, a_n)$ of non-negative integers is a *p-ascent sequence* of length n if $a_1 = 0$ and $a_i \in [0, p + \operatorname{asc}(a_1, \dots, a_{i-1})]$ for all $2 \le i \le n$. For example, $(0, 2, 3, 2, 0, 1, 2)$ is a 2-ascent sequences of length 7, with 4 ascents. For the case $p = 1$ this concept was studied by Bousquet-Mélou et al. [2] in the context of $(\mathbf{2}+\mathbf{2})$-free posets of size n. The general case was studied by Kitaev and Remmel [17].

Let $\{a_{n,p,\mathsf{10}}\}_n$ be the number of p-ascent sequence of length n avoiding the pattern $\mathsf{10}$. For example, $a_{3,2,\mathsf{10}} = 8$, with the sequences being

$$(0,0,0), \quad (0,0,1), \quad (0,0,2), \quad (0,1,1), \quad (0,1,2), \quad (0,1,3), \quad (0,2,2), \quad (0,2,3).$$

Notice that $(0, 1, 0), (0, 2, 0), (0, 2, 1)$ are 2-ascent sequences but they are not avoiding the pattern $\mathsf{10}$. Kitaev and Remmel [17] proved the combinatorial formula:

$$a_{n,p,\mathsf{10}} = \sum_{s=0}^{n} \binom{n-1}{s}\binom{p+s-1}{s}.$$

From Equation (6) we have $f_n(Y^r \Delta_\ell Y; s + \ell) = a_{n-\ell-r-s+1,s+1,\mathsf{10}}$.

3.2 A relation with the asymmetric Delannoy paths

Let $\mathcal{A}_2 = [a_2(n, k)]_{n,k\ge 0}$ be the Riordan array defined by

$$\mathcal{A}_2 = \left(\frac{1}{1-x}, \frac{x(2-x)}{1-x}\right).$$

The first few rows of $\mathcal{A}_2$ are

$$\mathcal{A}_2 = [a_2(n,k)]_{n,k\ge 0} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 4 & 0 & 0 & 0 & 0 & 0 \\ 1 & 4 & 8 & 8 & 0 & 0 & 0 & 0 \\ 1 & 5 & 13 & 20 & 16 & 0 & 0 & 0 \\ 1 & 6 & 19 & 38 & 48 & 32 & 0 & 0 \\ 1 & 7 & 26 & 63 & 104 & 112 & 64 & 0 \\ 1 & 8 & 34 & 96 & 192 & 272 & 256 & 128 \\ \vdots & & & & \vdots & & & \vdots \end{pmatrix}.$$

Note that the entries of the Riordan arrays $\mathcal{A}_1$ and $\mathcal{A}_2$ are related by $a_1(n+k,k) = a_2(n+k-1,n)$ for all $n \geq 0$ and $k \geq 1$. Therefore, we have the combinatorial identity, $a_2(n,k) = a_1(n+1, n-k+1) = \sum_{\ell=0}^{k} \binom{k}{\ell}\binom{n-k+\ell}{\ell}$, where $0 \leq k \leq n$. From Theorem 2.1 and the definition of the Riordan array $\mathcal{A}_2$, we have the following theorem.

Theorem 3.1. *If $n, s \geq 0$, $r, \ell \geq 1$, and $a_2(n,k)$ is the (n,k)-th entry of the Riordan array $\mathcal{A}_2$, then $f_n(Y^r \Delta_\ell Y; s+\ell) = a_2(n-\ell-r, n-s-\ell-r)$.*

The entries of the matrix $\mathcal{A}_2$ are related with the number of asymmetric Delannoy paths. Let $d(n,k)$ be the number of lattice paths from the point $(0,0)$ to the point $(n,k+1)$, with step set $S = \{(x,y) : x \geq 0, y \geq 1\}$. The sequence $d(n,k)$ is called *asymmetric Delannoy numbers* (see Hetyei [15]). In particular, the sequence $d(n,n)$ coincides with the central Delannoy numbers (cf. [22]). For example, $d(1,2) = 8$ where the paths associated to $(1,2)$ are shown in Figure 4. From [15, Lemma 3.2] we have that $a_2(n,k) = d(n-k,k)$.

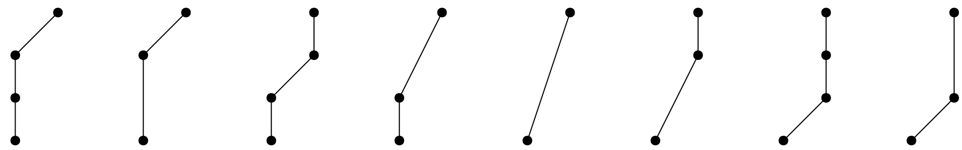

Figure 4: Asymmetric Delannoy paths to the point $(1,3)$.

3.3 A relation with the q-Catalan numbers

The q-Catalan numbers $C_n(q)$ were introduced by Carlitz and Riordan in [3]. They can be defined combinatorially by

$$C_n(q) := \sum_{D \in \mathbb{D}_n} q^{\mathsf{area}(D)},$$

where $\mathbb{D}_n$ is the set of Dyck paths of length $2n$ and $\mathsf{area}(D)$ is the area of the Dyck path D. In Figure 5 we show the Dyck paths counted by $C_3(q) = q^3 + q^2 + 2q + 1$ and their areas.

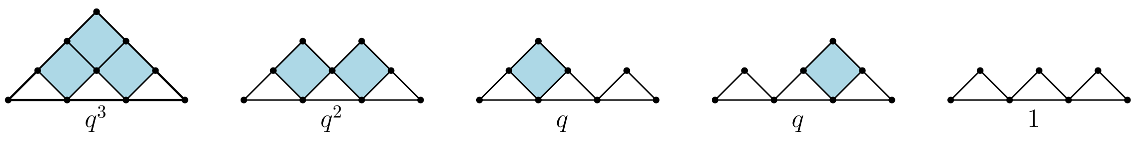

Figure 5: The area of the Dyck paths of length 6.

The number $C_n(q)$ satisfies the recurrence relation

$$C_n(q) = \sum_{k=1}^{n} q^{k-1} C_{k-1}(q) C_{n-k}(q),$$

with $n \geq 1$ and $C_0(q) = 1$.

In this subsection we show a relation between the Riordan array $\mathcal{A}_2 = [a_2(n,k)]_{n,k\geq 0}$ and the q-Catalan numbers. In Theorem 3.4 we show how to calculate the entries of $\mathcal{A}_2$ using the q-Catalan numbers. First we introduce some additional results for Riordan arrays. Rogers [20] introduced the concept of the A-sequence. Specifically, Rogers observed that every element $d_{n+1,k+1}$ of a Riordan matrix (not belonging to row 0 or column 0) can be expressed as a linear combination of the elements in the preceding row. Merlini et al. [18] introduced the Z-sequence which characterizes column 0, except for the element $d_{0,0}$. Therefore, a Riordan array is completely characterized by the A-sequence, the Z-sequence and the element $d_{0,0}$. Summarizing, we have the following theorem:

Theorem 3.2 ([18]). *An infinite lower triangular array $D = [d_{n,k}]_{n,k\in\mathbb{N}}$ is a Riordan array if and only if $d_{0,0} \neq 0$ and there are two sequences $A = (a_0 \neq 0, a_1, a_2, \dots)$ and $Z = (z_0, z_1, z_2, \dots)$ such that*

$$\begin{aligned} d_{n+1,k+1} &= a_0 d_{n,k} + a_1 d_{n,k+1} + a_2 d_{n,k+2} + \cdots, \quad n,k = 0,1,\dots \\ d_{n+1,0} &= z_0 d_{n,0} + z_1 d_{n,1} + z_2 d_{n,2} + \cdots, \quad n = 0,1,\dots. \end{aligned}$$

Theorem 3.3 ([14, 18]). *Let $D = (g(x), f(x))$ be a Riordan array with inverse $D^{-1} = (d(x), h(x))$. Then the A-sequence and Z-sequence of D are*

$$A(x) = \frac{x}{h(x)}; \qquad Z(x) = \frac{1}{h(x)}\left(1 - d_{0,0}d(x)\right).$$

Theorem 3.4. *If $n,k \geq 0$, $a_2(n,k)$ is the (n,k)-th entry of the Riordan array $\mathcal{A}_2$, and $C_n(q)$ is the q-Catalan number, then*

$$a_2(n+1,k+1) = a_2(n,k) + \sum_{k\geq 1} C_k(-1)a_2(n,k),$$

and $a_2(n+1,0) = a_2(n,0)$ with initial value $a_2(0,0) = 1$.

Proof. The inverse of the Riordan array $\mathcal{A}_2 = (1/(1-x), x(2-x)/(1-x))$ is given by

$$\mathcal{A}_2^{-1} = \left(\frac{1}{2}\left(1 - 2x + \sqrt{1+4x^2}\right), \frac{1}{2}\left(1 + 2x - \sqrt{1+4x^2}\right)\right).$$

Therefore, from Theorem 3.3 we have that the generating functions of the A sequence and the Z sequence for the Riordan array $\mathcal{A}_2$ are given by

$$A(x) = \frac{2x}{1 + 2x - \sqrt{1+4x^2}}; \qquad Z(x) = 1.$$

If

$$Ca(x) := \sum_{n\geq 0} C_n(1)x^n = \frac{1 - \sqrt{1-4x}}{2x}$$

is the generating function of the Catalan numbers, then $A(x) = 1 + x + x^2Ca(-x^2)$. It is easy to show that

$$G(x) := \sum_{n\geq 0} C_n(-1)x^n = 1 + xCa(-x^2).$$

Therefore, $A(x) = 1 + xG(x)$. This and Theorem 3.2 prove the first equality. Since $Z(x) = 1$, the second equality clearly holds. □

The first few values of the sequence $C_n(-1)$ are

$$1, \quad 1, \quad 0, \quad -1, \quad 0, \quad 2, \quad 0, \quad -5, \quad 0, \quad 14, \quad 0, \quad -42, \quad 0, \quad 132, \ldots.$$

So, $a_2(n+1, k+1)$ is equal to

$$a_2(n,k)+a_2(n,k+1)+a_2(n,k+2)-a_2(n,k+4)+2a_2(n,k+6)-5a_2(n,k+8)+\cdots.$$

3.4 A relation with the Fibonacci polynomials

The *Fibonacci polynomials* $F_n(x)$ are defined recursively by $F_n(x) = xF_{n-1}(x) + F_{n-2}(x)$ for $n \geq 2$, with initial values $F_0(x) = 0$ and $F_1(x) = 1$. The generating function of these polynomials is

$$F(z) := \sum_{n\geq 0} F_n(x)z^n = \frac{z}{1 - xz - z^2}.$$

From this we obtain the generating function of the binomial transform of the Fibonacci polynomials

$$\frac{1}{1-z}F\left(\frac{z}{1-z}\right) = \sum_{n\geq 0}\sum_{k=0}^{n}\binom{n}{k}F_k(x)z^n = \frac{z}{1-(2+x)z+xz^2}. \tag{8}$$

We use $H_n(x)$ to denote the polynomial

$$H_n(x) := \sum_{k=0}^{n}\binom{n}{k}F_k(x).$$

The first few polynomials are

$$H_1(x) = 1, \quad H_2(x) = x + 2, \quad H_3(x) = x^2 + 3x + 4, \quad H_4(x) = x^3 + 4x^2 + 8x + 8,$$
$$H_5(x) = x^4 + 5x^3 + 13x^2 + 20x + 16, \ldots$$

In Theorem 3.5 we show that the coefficients of $H_n(x)$ coincide with the entries of the matrix $\mathcal{A}_2$.

Theorem 3.5. *If $0 \leq k \leq n$ and $a_2(n,k)$ is the (n,k)-th entry of the Riordan array $\mathcal{A}_2$, then $a_2(n,k) = [x^k]x^{n-1}H_n(1/x)$.*

Proof. From the summation property of the Riordan arrays we have

$$\sum_{n\geq 0}\sum_{k=0}^{n} a_2(n,k)x^k z^n = \mathcal{A}_2 * \frac{1}{1-xz} = \left(\frac{1}{1-z}, \frac{z(2-z)}{1-z}\right) * \frac{1}{1-xz}$$
$$= \frac{1}{1-z}\left(\frac{1}{1-x\frac{z(2-z)}{1-z}}\right) = \frac{1}{1-(2+x)z+xz^2}.$$

Note that $1/(1-(2+x)z+xz^2)$ is equal to (8), so comparing coefficients we obtain the desired result. □

4 Counting asymmetric pyramids in a configuration of the form $X^r\Delta_\ell X^t$

In this section we count the number of paths having a maximal pyramid Δ_ℓ, at a fixed height and lying in a configuration $X^r\Delta_\ell X^t$, where ℓ, r, and t are fixed positive integers. We give a closed formula for $f_n(X^r\Delta_\ell X^t; s+\ell)$, where s is the height at where the pyramid Δ_ℓ is located. Note that by the nature of non-decreasing Dyck paths the configuration $X^r\Delta_\ell X^t$ holds at most once in a path.

The generating function that counts the total number of non-decreasing Dyck paths of length $2n$ is given by $D(x) = (x(1-x))/(1-3x+x^2) = \sum_{n=1}^{\infty} F_{2n-1}x^n$, where F_n is a Fibonacci number (see [1]). Now we define the generating function

$$T^{(2)}_{s,\ell,r,t}(x) := \sum_{n\geq s+\ell+t} f_n(X^r\Delta_\ell X^t; s+\ell)x^n.$$

Theorem 4.1. *The generating function $T^{(2)}_{s,\ell,r,t}(x)$ is given by*

$$T^{(2)}_{s,\ell,r,t}(x) = \frac{(1-x)x^{s+\ell+t}}{1-3x+x^2}\left(\frac{1-x}{1-2x}\right)^{s-r+1}.$$

Proof. Since the configuration $X^r\Delta_\ell X^t$, where the associated peak has feature $(s+\ell,\ell)$ holds at most once in a non-decreasing Dyck path, we have that any non-decreasing Dyck path in $\mathcal{D}$ with this configuration may be decomposed as either

$$\overbrace{TXTX\cdots TX}^{s-r}TX^r\Delta_\ell\Delta_{\geq t}DY^s \quad \text{or} \quad \overbrace{TXTX\cdots TX}^{s-r}TX^r\Delta_\ell X^t\,(D\cup\lambda)\,Y^tY^s, \tag{9}$$

where D is a non-empty non-decreasing Dyck path, λ is the empty path, $T\in\mathcal{D}$ contains only valleys of height zero. See Figure 6.

Therefore, we obtain the following generating function

$$T^{(2)}_{s,\ell,r,t}(x) = (xV(x))^{s-r}\,V(x)x^r x^\ell \frac{x^t}{1-x}D(x) + (xV(x))^{s-r}\,V(x)x^r x^\ell x^t(D(x)+1)$$
$$= x^{s+\ell+t}\left(\frac{1-x}{1-2x}\right)^{s-r+1}\left(\frac{2-x}{1-x}D(x)+1\right).$$

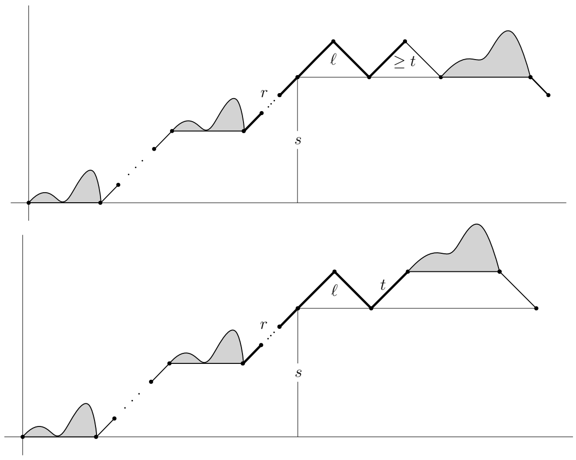

Figure 6: Factoring the paths in (9).

So,

$$T^{(2)}_{s,\ell,r,t}(x) = x^{s+\ell+t}\left(\frac{1-x}{1-2x}\right)^{s-r+1}\left(\frac{1-x}{1-3x+x^2}\right). \qquad \square$$

Let $\mathcal{B} = [b(n,k)]_{n,k\geq 0}$ be the Riordan array defined by

$$\mathcal{B} = \left(\frac{1-x}{1-3x+x^2}, \frac{x(1-x)}{1-2x}\right).$$

The first few rows of $\mathcal{B}$ are

$$\mathcal{B}1 = [b(n,k)]_{n,k\geq 0} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 5 & 3 & 1 & 0 & 0 & 0 & 0 & 0 \\ 13 & 9 & 4 & 1 & 0 & 0 & 0 & 0 \\ 34 & 26 & 14 & 5 & 1 & 0 & 0 & 0 \\ 89 & 73 & 45 & 20 & 6 & 1 & 0 & 0 \\ 233 & 201 & 137 & 71 & 27 & 7 & 1 & 0 \\ 610 & 546 & 402 & 234 & 105 & 35 & 8 & 1 \\ \vdots & & & & \vdots & & & \vdots \end{pmatrix}.$$

The proof of the following theorem is similar to the proof of Theorem 2.2, so it is omitted.

Theorem 4.2. *Let $\ell, s, r, t \geq 1$ and $n \geq \ell + r + t$. If $c_2 = X^r \Delta_\ell X^t$, then $f_n(c_2; s+\ell)$ satisfies the recurrence relation*

$$f_n(c_2; s+\ell) = 2f_{n-1}(c_2; s+\ell) + f_{n-1}(c_2; s-1+\ell) - f_{n-2}(c_2; s-1+\ell),$$

where $f_n(c_2; \ell) = F_{2(n-\ell-r-t+1)+1} - 2^{n-\ell-r-t}$ and $f_n(c_2; s+\ell) = 0$ if $n < r+\ell+t$.

Note that by the generating function given in Theorem 4.1 and the definition of the Riordan array $\mathcal{B}$ we have the following theorem.

Theorem 4.3. *If $n \geq 0, t \geq 1$, $1 \leq r \leq s$ and $b(n,k)$ is the (n,k)-th entry of the Riordan array $\mathcal{B}$, then $f_n(X^r\Delta_\ell X^t; s+\ell) = b(n-\ell-r-t+1, s-r+1)$.*

The following theorem shows that the entries of the matrix $\mathcal{B}$ can be expressed as a linear combination of the entries of the matrix $\mathcal{A}_1$ and Fibonacci numbers.

Theorem 4.4. *Let n and k be nonnegative integers. If $a_1(n,k)$ and $b(n,k)$ are the (n,k)-th entry of the Riordan arrays $\mathcal{A}_1$ and $\mathcal{B}$, respectively, and F_m is a Fibonacci number, then*

$$b(n,k) = \sum_{i=0}^{n-1} F_{2(n-i)} a_1(i,k) + a_1(n,k).$$

Proof. From the product of Riordan arrays we have

$$\mathcal{B} = \left(\frac{(1-x)^2}{1-3x+x^2}, x\right)\mathcal{A}_1. \tag{10}$$

Since

$$\frac{(1-x)^2}{1-3x+x^2} = 1 + \frac{x}{1-3x+x^2} = 1 + \sum_{k\geq 0} F_{2k}x^k,$$

we conclude that

$$\left(\frac{(1-x)^2}{1-3x+x^2}, x\right) = [f(n,k)]_{n,k\geq 0},$$

where

$$f(n,k) := \begin{cases} 1, & \text{if } n = k; \\ F_{2(n-k)}, & \text{if } n > k; \\ 0, & \text{otherwise.} \end{cases}$$

From the product in (10) we have

$$b(n,k) = \sum_{i=0}^{n} f(n,i)a_1(i,k) = \sum_{i=0}^{n-1} F_{2(n-i)}a_1(i,k) + a_1(n,k). \qquad \square$$

For some particular values of s, ℓ, r, we obtain these two examples,

$$f_n(X\Delta_1 X; 2) = F_{2n-3} - 2^{n-3},\ n \geq 3,$$
$$f_n(X^2\Delta_1 X; 3) = f_n(X\Delta_1 X; 3) = F_{2n-5} - 2^{n-4},\ n \geq 4.$$

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