AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **79(2)** (2021), Pages 215–233

Decompositions of some classes of regular graphs and digraphs into cycles of length $4p$

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Abstract

In this paper, we prove the existence of a $4p$-cycle decomposition of the graph $K_m \times K_n$ and a directed $4p$-cycle decomposition of the symmetric digraph $(K_m \circ \overline{K}_n)^*$, where $\circ$ and $\times$ denote the wreath product and tensor product of graphs, respectively, and p is an odd prime. It is proved that, for integers $m \geq 3$ and $n \geq 3$, the obvious necessary conditions for the existence of a $4p$-cycle decomposition of $K_m \times K_n$ are sufficient, where p is an odd prime. Also, it is shown that the necessary conditions for the existence of a directed $4p$-cycle decomposition of the symmetric digraph $(K_m \circ \overline{K}_n)^*$ are sufficient, where p is an odd prime. Recently, the same type of results are obtained for $2p$; see [S. Ganesamurthy and P. Paulraja, *Discrete Math.* 341 (2018), 2197–2210].

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1 Introduction

All graphs (respectively, digraphs) considered here are loopless and finite. Let C_k (respectively, $\vec{C}_k$) and P_k (respectively, $\vec{P}_k$) denote a cycle (respectively, directed cycle) and a path (respectively, directed path) on k vertices. For a graph G, $G(\lambda)$ denotes the multigraph obtained from G by replacing each edge of G by λ edges. The complete graph on n vertices is denoted by K_n and its complement is denoted by $\overline{K}_n$. For an integer $k \geq 2$, kH denotes k vertex disjoint copies of H. For a graph G, G^* denotes the *symmetric digraph* of G and it is obtained from G by replacing every edge by a symmetric pair of arcs. If $H_1, H_2, \ldots, H_\ell$ are edge-disjoint subgraphs of a graph G such that $E(G) = E(H_1) \cup E(H_2) \cup \cdots \cup E(H_\ell)$, then we say that $H_1, H_2, \ldots, H_\ell$ *decompose* G and we write this as $G = H_1 \oplus H_2 \oplus \cdots \oplus H_\ell$, where $\oplus$ denotes the edge disjoint union of graphs. If each $H_i \simeq H$, $1 \leq i \leq \ell$, then we say that H decomposes G and we denote this by $H \,|\, G$. Similarly, if $\vec{H}_1, \vec{H}_2, \ldots, \vec{H}_\ell$ are arc-disjoint subdigraphs of a digraph $\vec{D}$ such that $A(\vec{D}) = A(\vec{H}_1) \cup A(\vec{H}_2) \cup \cdots \cup A(\vec{H}_\ell)$, then we say that $\vec{H}_1, \vec{H}_2, \ldots, \vec{H}_\ell$ *decompose* $\vec{D}$ and we write this as $\vec{D} = \vec{H}_1 \oplus \vec{H}_2 \oplus \cdots \oplus \vec{H}_\ell$. If each $\vec{H}_i \simeq \vec{H}$, $1 \leq i \leq \ell$, then we say that $\vec{H}$ decomposes $\vec{D}$ and we denote this by $\vec{H} \,|\, \vec{D}$. If $H_i \simeq C_k$ (respectively, $\vec{H}_i \simeq \vec{C}_k$), $1 \leq i \leq \ell$ and $k \geq 3$, then we write $C_k \,|\, G$ (respectively, $\vec{C}_k \,|\, \vec{D}$) and in this case we say that G (respectively, $\vec{D}$) has a *C_k-decomposition* (respectively, *$\vec{C}_k$-decomposition*) or *a k-cycle decomposition* (respectively, *directed k-cycle decomposition*). A *C_k-factor* of a graph G is a spanning subgraph H of G such that each component of H is a k-cycle. A partition of the edge set of G into C_k-factors is called a *C_k-factorization* of G, that is, a 2-factorization in which each of its factors contains only cycles of length k as its components. A k-regular graph G is said to be *Hamilton cycle decomposable* if its edge set can be partitioned into Hamilton cycles or Hamilton cycles plus a perfect matching if k is even or odd, respectively.

For two graphs (respectively, digraphs) G and H, their *tensor product*, denoted by $G \times H$, is the graph with vertex set $V(G) \times V(H)$ in which $(g_1, h_1)(g_2, h_2)$ is an edge (respectively, arc) whenever g_1g_2 is an edge (respectively, arc) in G and h_1h_2 is an edge (respectively, arc) in H. Similarly, the *wreath product* of graphs (respectively, digraphs) G and H, denoted by $G \circ H$, is the graph with vertex set $V(G) \times V(H)$ in which $(g_1, h_1)(g_2, h_2)$ is an edge (respectively, arc) whenever g_1g_2 is an edge (respectively, arc) in G or, $g_1 = g_2$ and h_1h_2 is an edge (respectively, arc) in H; see Figure 1. It can be easily seen that $K_m \circ \overline{K}_n$ is the complete m-partite graph in which each partite set has n vertices. Moreover, $K_m \circ \overline{K}_n - E(nK_m) \cong K_m \times K_n$. The complete multipartite graph with partite sets having sizes $m_1, m_2, \ldots, m_k$ is denoted by $K_{m_1, m_2, \ldots, m_k}$. It is well-known that the tensor product is commutative and distributive over edge-disjoint union of graphs, that is, if $G = H_1 \oplus H_2 \oplus \cdots \oplus H_k$, then $G \times H = (H_1 \times H) \oplus (H_2 \times H) \oplus \cdots \oplus (H_k \times H)$. If G and H are two graphs with vertex sets $\{x_0, x_1, \ldots, x_r\}$ and $\{y_0, y_1, \ldots, y_s\}$, respectively, then $V(G \times H) = V(G) \times V(H) = \{(x_i, y_j) \,|\, 0 \leq i \leq r \text{ and } 0 \leq j \leq s\}$. For $x_i \in V(G)$ we define $X_i = x_i \times V(H) = \{(x_i, y_0), (x_i, y_1), \ldots, (x_i, y_s)\}$ and we call this set of vertices the i^{th} *row* of $G \times H$. Similarly, for $y_j \in V(H)$ we define $Y_j = V(G) \times y_j = \{(x_0, y_j), (x_1, y_j), \ldots, (x_r, y_j)\}$ and we call this set of vertices the j^{th} *column* of $G \times H$.

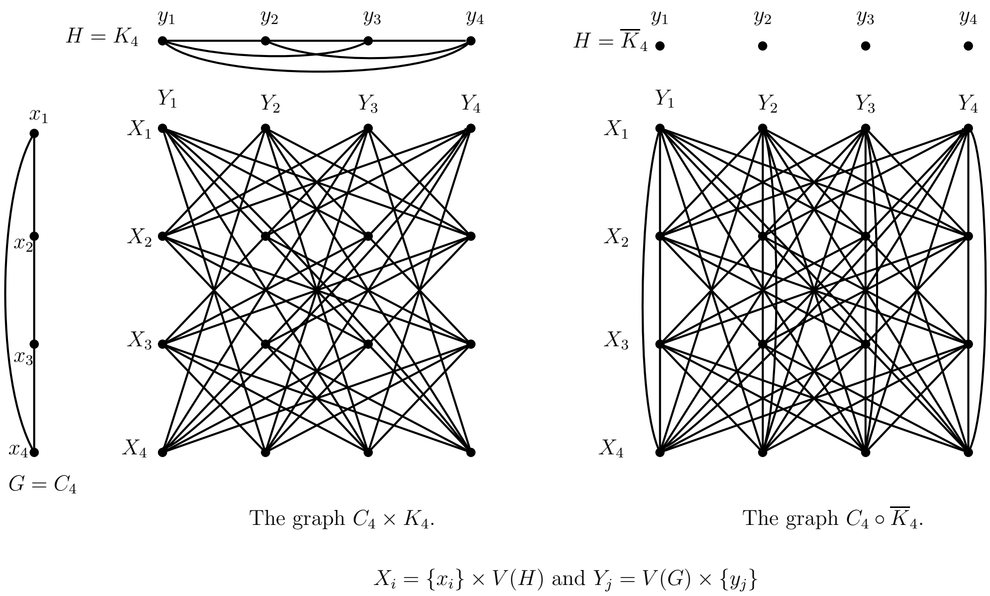

Figure 1: The graphs $C_4 \times K_4$ and $C_4 \circ \overline{K}_4$.

Let G be a bipartite graph with bipartition (X, Y), where $X = \{x_0, x_1, \ldots, x_{r-1}\}$, $Y = \{y_0, y_1, \ldots, y_{r-1}\}$. For some i, $1 \le i \le r-1$, if G contains the set of edges $F_i(X, Y) = \{x_j y_{i+j} \mid 0 \le j \le r-1\}$, where addition in the subscript is taken modulo r, then we say that G has the *1-factor of jump i from X to Y* and each edge of $F_i(X, Y)$ is called an edge of jump i from X to Y. Note that $F_i(Y, X) = F_{r-i}(X, Y)$, $0 \le i \le r-1$. Clearly, if $G = K_{r,r}$, then $E(G) = \bigcup_{i=0}^{r-1} F_i(X, Y)$. Definitions which are not given here can be found in [6].

The problem of decomposing regular graphs into cycles is not new. The obvious necessary conditions for the existence of an m-cycle decomposition of K_n (respectively, $K_n - I$, where I is a perfect matching) when n is odd (respectively, even) are proved to be sufficient; see [2, 14, 28]. In 2003, Buratti [10] obtained a short proof for the existence of an odd cycle decomposition of K_n. Recently, Bryant et al. have proved that the complete graph K_n (respectively, $K_n - I$, where I is a perfect matching) can be decomposed into cycles of lengths $m_1, m_2, \ldots, m_k$, where $\sum_{i=1}^{k} m_i = \binom{n}{2}$ (respectively, $\sum_{i=1}^{k} m_i = \binom{n}{2} - \frac{n}{2}$) and n is odd (respectively, even); see [9].

Necessary and sufficient conditions for the existence of a k-cycle decomposition of $K_m \circ \overline{K}_n$, $k \in \{mn, p, 2p, 3p, p^2\}$, are given in [7, 17, 20, 21, 23, 29, 30, 31], where p is a prime. The existence of an even cycle decomposition of $(K_m \circ \overline{K}_n)(\lambda)$ has been proved by Muthusamy and Shanmuga Vadivu; see [26]. Very recently, regardless of the parity of k, the authors of [11] actually solved the existence problem for a C_k-decomposition of $(K_m \circ \overline{K}_n)(\lambda)$ whose cycle-set can be partitioned into 2-regular graphs containing all the vertices except those belonging to one part.

The graph $K_m \times K_n$ is a proper spanning regular subgraph of $K_m \circ \overline{K}_n$ (in fact, $K_m \times K_n \cong (K_m \circ \overline{K}_n) - E(nK_m))$; the existence of a k-cycle decomposition of $K_m \times K_n$ is not a straightforward consequence of the existence of a k-cycle decomposition of $K_m \circ \overline{K}_n$. Assaf [4] proved that $C_3 \mid (K_m \times K_n)(\lambda)$ whenever the necessary conditions are sufficient. Manikandan and Paulraja proved that the necessary conditions for the existence of a C_p-decomposition of $K_m \times K_n$ are also sufficient whenever $p \geq 5$ is prime; see [20, 21, 23]. Further, in [13], Ganesamurthy and Paulraja proved that the necessary conditions are sufficient for the existence of a C_k-decomposition of $K_m \times K_n$, where $k \in \{2^\ell, 2p\}$, $\ell \geq 2$ and $p \geq 3$ is a prime. Recently, Manikandan et al. [24] proved the existence of a p^2-cycle decomposition of $K_m \times K_n$ whenever the necessary conditions are satisfied. Balakrishnan et al. [5] obtained a Hamilton cycle decomposition of $K_m \times K_n$.

Directed k-cycle decompositions of $(K_n(\lambda))^*$ are studied in [3, 32]. Furthermore, directed p-cycle and $2p$-cycle decompositions of $(K_m \circ \overline{K}_n)^*$ are obtained in [13, 22].

Besides other results, we prove the following theorems.

Theorem 1.1. *If the integers m and n are at least 3 and $p \geq 3$ is prime, then $C_{4p} \mid K_m \times K_n$ if and only if either m or n is odd, $4p \leq mn$ and $\binom{m}{2}\binom{n}{2} \equiv 0 \pmod{2p}$.*

Theorem 1.2. *If the integers m and n are at least 3 and $p \geq 3$ is prime, then $\vec{C}_{4p} \mid (K_m \circ \overline{K}_n)^*$ if and only if $4p \leq mn$ and $m(m-1)n^2 \equiv 0 \pmod{4p}$.*

2 Some known theorems and lemmas

We quote the following theorems for our future reference.

Theorem 2.1. [2] *For odd integers $3 \leq k \leq m$, $C_k \mid K_m$ if and only if $m(m-1) \equiv 0 \pmod{2k}$.*

Theorem 2.2. [34] *For positive integers k, m and λ, $P_{k+1} \mid K_m(\lambda)$ if and only if $2 \leq k+1 \leq m$ and $\lambda m(m-1) \equiv 0 \pmod{2k}$.*

Theorem 2.3. [33] *For positive integers m, n and k, $C_k \mid K_{m,n}$ if and only if m, n and k are all even with $\frac{k}{2} \leq m$, $\frac{k}{2} \leq n$ and $k \mid mn$.*

Theorem 2.4. [19] *Let $m \geq 3$ be an odd integer and let $k \geq 4$ be an even integer. Then $C_k \mid (K_{m,m} - I)$ if and only if $k \leq 2m$ and $k \mid m(m-1)$, where I is a perfect matching of $K_{m,m}$.*

Theorem 2.5. [25] *If $n \mid m$, then $C_k \times K_m$ admits a C_{kn}-factorization except possibly when k is an odd integer and $m \equiv 2 \pmod 4$.*

Theorem 2.6. [5] *For m, $n \geq 3$, the graph $K_m \times K_n$ is Hamilton cycle decomposable.*

Theorem 2.7. [15] *Let $m \geq 3$ be an odd integer and let $n \geq 3$ be an integer. Then $C_m \times C_n$ is Hamilton cycle decomposable.*

Theorem 2.8. [16] *For $k \geq 3$ and $n \geq 2$, the graph $C_k \circ \overline{K}_n$ is Hamilton cycle decomposable.*

Theorem 2.9. [3] *For positive integers k and n, with $2 \leq k \leq n$, $\vec{C}_k \mid K_n^*$ if and only if $n(n-1) \equiv 0 \pmod k$ and $(k,n) \neq (3,6),(4,4),(6,6)$.*

Theorem 2.10. [27] *For positive integers $m \geq 2$ and n, $(K_m \circ \overline{K}_n)^*$ is directed Hamilton cycle decomposable except when $(m,n) = (4,1)$ or $(6,1)$.*

Theorem 2.11. [12] *Let λ, m, n be positive integers with $m, n \geq 3$, and $p \geq 2$ prime. Then $C_{4p} \mid K_m(\lambda) \circ \overline{K}_n$ if and only if (1) $mn \geq 4p$, (2) $\lambda(m-1)n$ is even, and (3) $4p \mid \lambda\binom{m}{2}n^2$.*

Lemma 2.12. [13] *If $P_{k+1} \mid K_n$, then $C_{2k} \mid K_m \times K_n$ when $k \geq 3$ and for all odd integers $m \geq 3$.*

Lemma 2.13. [13] *Let $k \geq 2$, $m \geq 5$ and $m \equiv 1 \pmod 4$. If $P_{k+1} \mid K_n$, then $C_{4k} \mid K_m \times K_n$.*

Lemma 2.14. [13] *If $k \geq 2$, then $C_{4k} \mid P_{k+1} \times K_{4,4}$.*

3 Building blocks

In this section we prove some lemmas which are used in the proof of the main Theorem 1.1.

Lemma 3.1. *If $m \geq 2$ is an integer and n, $k \geq 3$ are odd integers with $n \equiv 1 \pmod{4k}$, then $C_{4k} \mid K_m \times K_n$.*

Proof. Clearly, $K_m \times K_n = (K_2 \times K_n) \oplus \cdots \oplus (K_2 \times K_n)$. The graph $K_2 \times K_n \cong K_{n,n} - I$, where I is a perfect matching of $K_{n,n}$. Since $n \equiv 1 \pmod{4k}$, $4k \mid n(n-1)$ and hence $C_{4k} \mid K_{n,n} - I$, by Theorem 2.4. Thus $C_{4k} \mid K_m \times K_n$. □

Lemma 3.2. *If $k \geq 3$ is an odd integer, then $C_{4k} \mid K_5 \times C_k$.*

Proof. Let $V(K_5) = \{v, w, x, y, z\}$ and $C_k = (a_1, a_2, \ldots, a_k)$. Then $V(G) = \{(v,a_1), (v,a_2), \ldots, (v,a_k)\} \cup \{(w,a_1),(w,a_2),\ldots,(w,a_k)\} \cup \{(x,a_1),(x,a_2),\ldots,(x,a_k)\} \cup \{(y,a_1),(y,a_2),\ldots,(y,a_k)\} \cup \{(z,a_1),(z,a_2),\ldots,(z,a_k)\}$. For our convenience, we denote $(v,a_i),(w,a_i),(x,a_i),(y,a_i)$ and (z,a_i) by v_i, w_i, x_i, y_i and z_i, respectively. Now we construct a base cycle C of length $4k$ in $K_5 \times C_k$ as follows; see Figure 2. Let $C = (v_1, w_2, v_3, w_4, v_5, \ldots, w_{k-1}, v_k, x_1, z_2, x_3, \ldots, z_{k-1}, x_k, w_1, v_2, w_3, \ldots, v_{k-1}, w_k, z_1, x_2, z_3, \ldots, x_{k-1}, z_k)$.

Consider the permutation $\rho = Z_1 Z_2 \ldots Z_k$, where $Z_i = (v_i\, w_i\, x_i\, y_i\, z_i)$, $1 \leq i \leq k$, on the set $V(K_5 \times C_k)$. Then $\{C, \rho(C), \rho^2(C), \rho^3(C), \rho^4(C)\}$ is a C_{4k}-decomposition of $K_5 \times C_k$. This completes the proof.

□

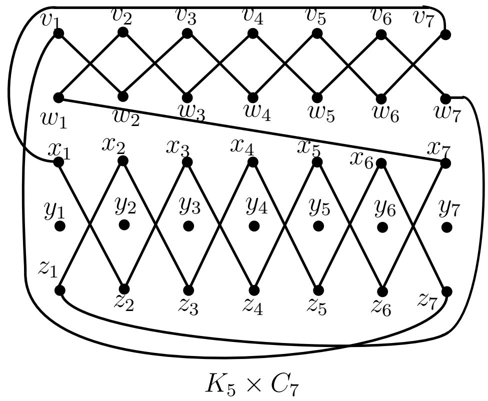

Figure 2: A base cycle C of $K_5 \times C_7$ for a C_{28}-decomposition of $K_5 \times C_7$ is shown above.

Lemma 3.3. *Let k and m be odd integers with $3 \leq k \leq m$. If $C_k \mid K_m$, then $C_{4k} \mid K_{4,4} \times K_m$.*

Proof. As $C_k \mid K_m$, $K_{4,4} \times K_m = K_{4,4} \times C_k \oplus \cdots \oplus K_{4,4} \times C_k = C_4 \times C_k \oplus \cdots \oplus C_4 \times C_k$, since $C_4 \mid K_{4,4}$, by Theorem 2.3. The graph $C_4 \times C_k$ admits a C_{4k}-decomposition, by Theorem 2.7. Thus $C_{4k} \mid K_{4,4} \times K_m$. □

Lemma 3.4. *Let $k \geq 3$ be an odd integer, n be an integer with $k \leq n$ and $k \mid \binom{n}{2}$. If $m \geq 5$ and $m \equiv 1 \pmod 4$, then $C_{4k} \mid K_m \times K_n$.*

Proof. Let $m = 4t + 1,\ t \geq 1$.

Case 1. n is odd.
Since n is odd and $k \mid \binom{n}{2}$, $K_n = C_k \oplus \cdots \oplus C_k$, by Theorem 2.1. If $t = 1$, $C_{4k} \mid K_5 \times K_n$, by Lemma 3.2, because $K_5 \times K_n = K_5 \times C_k \oplus \cdots \oplus K_5 \times C_k$. For all $t \geq 2$, the edges of K_{4t+1} can be decomposed into t copies of K_5 which each share a common vertex and $\binom{t}{2}$-copies of $K_{4,4}$; see Figure 3.

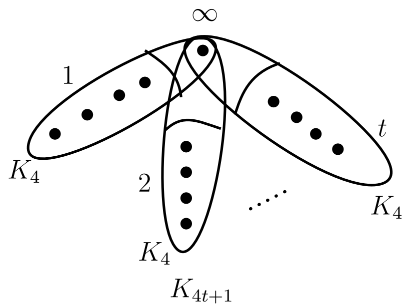

Figure 3: $K_{4t+1} = K_5 \oplus K_5 \oplus \cdots \oplus K_5 \oplus K_{4,4} \oplus K_{4,4} \oplus \cdots \oplus K_{4,4}$. A copy of K_4 and ∞ induce a K_5 and the edges between any two K_4's yield a $K_{4,4}$.

Thus, for all $t \geq 2$, we have

$$\begin{aligned} K_m \times K_n &= (K_5 \oplus \cdots \oplus K_5 \oplus K_{4,4} \oplus \cdots \oplus K_{4,4}) \times K_n \\ &= (K_5 \times K_n \oplus \cdots \oplus K_5 \times K_n) \oplus (K_{4,4} \times K_n \oplus \cdots \oplus K_{4,4} \times K_n). \end{aligned}$$

The graphs $K_5 \times K_n$ and $K_{4,4} \times K_n$ admit C_{4k}-decompositions, by the above argument and Lemma 3.3, respectively. This completes the proof of this case.

Case 2. n is even.
Since $k \mid \binom{n}{2}$, $2k \mid n(n-1)$. As n is even and k is odd with $k < n$, it easily follows that $k+1 \leq n$. Thus $P_{k+1} \mid K_n$, by Theorem 2.2. Hence, by Lemma 2.13, $C_{4k} \mid K_m \times K_n$. This completes the proof of the lemma. □

Lemma 3.5. *If p is prime and $p \equiv 1 \pmod 4$, then $C_{4p} \mid K_6 \times K_p$.*

Proof. Let $G = K_6 \times K_p$ and let $\{x_0, x_1, \ldots, x_5\}$ and $\{0, 1, \ldots, p-1\}$ be the vertex sets of K_6 and K_p, respectively. Then $V(G) = V(K_6) \times V(K_p) = \bigcup_{i=0}^{5} X_i$, where $X_i = x_i \times V(K_p) = \{(x_i, 0), (x_i, 1), \ldots, (x_i, p-1)\}$. For each i, $1 \leq i \leq \frac{p-1}{4}$, we obtain three C_{4p}-cycles in the graph G as follows; see Figure 4.

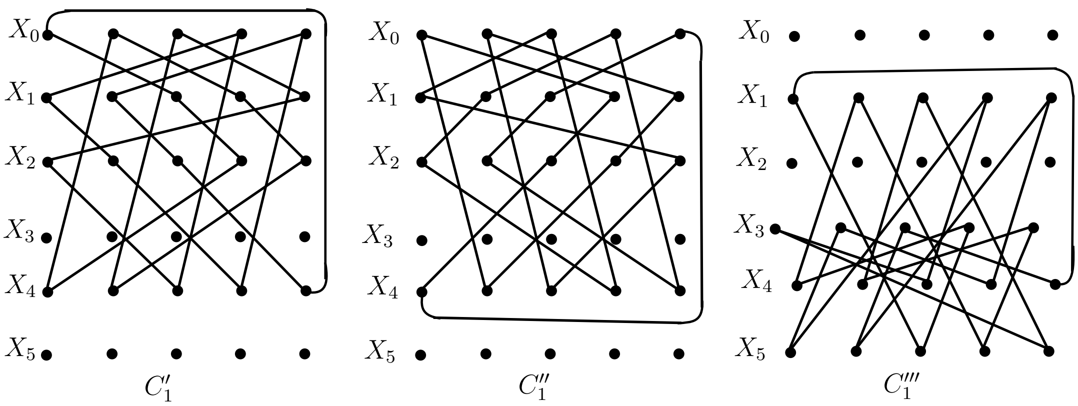

Figure 4: Three base cycles C_1', C_1'' and C_1''' of $K_6 \times K_5$ for a C_{20}-decomposition of $K_6 \times K_5$ are shown above.

$C_i' = F_{2i}(X_0, X_1) \oplus F_{2i-1}(X_1, X_2) \oplus F_{2i}(X_2, X_4) \oplus F_{2i-1}(X_4, X_0)$,

$C_i'' = F_{2i-1}(X_0, X_4) \oplus F_{2i}(X_4, X_2) \oplus F_{2i-1}(X_2, X_1) \oplus F_{2i}(X_1, X_0)$ and

$C_i''' = F_{2i}(X_1, X_5) \oplus F_{2i-1}(X_5, X_3) \oplus F_{2i}(X_3, X_4) \oplus F_{2i-1}(X_4, X_1)$,
where $F_k(X_i, X_j)$ stands for the 1-factor of jump k from X_i to X_j.

The sum of jumps of the 1-factors between the partite sets, that appear in C_i', of $K_6 \times K_p$, is $2i + (2i-1) + 2i + (2i-1) = 4i - 2$. Clearly, $\gcd(4i-2, p) = 1$, since $i \leq \frac{p-1}{4}$ implies $4i - 2 < p$. Hence, C_i' is a cycle of length $4p$; similarly, C_i'' and C_i''' are cycles of length $4p$. Consider the permutation $\rho = (X_0)(X_1\, X_2\, X_3\, X_4\, X_5)$ on the set $\{X_0, X_1, X_2, X_3, X_4, X_5\}$; then

$$\{C_i', \rho(C_i'), \ldots, \rho^4(C_i'), C_i'', \rho(C_i''), \ldots, \rho^4(C_i''), C_i''', \rho(C_i'''), \ldots, \rho^4(C_i''')\},\ 1 \leq i \leq \tfrac{p-1}{4}$$

is a C_{4p}-decomposition of G, where

$$\begin{aligned} \rho(C_i') &= F_{2i}(\rho(X_0), \rho(X_1)) \oplus F_{2i-1}(\rho(X_1), \rho(X_2)) \oplus F_{2i}(\rho(X_2), \rho(X_4)) \oplus \\ &\qquad F_{2i-1}(\rho(X_4), \rho(X_0)) \\ &= F_{2i}(X_0, X_2) \oplus F_{2i-1}(X_2, X_3) \oplus F_{2i}(X_3, X_5) \oplus F_{2i-1}(X_5, X_0). \end{aligned}$$

Similarly,

$$\begin{aligned}\rho^j(C_i') &= F_{2i}(\rho^j(X_0),\rho^j(X_1)) \oplus F_{2i-1}(\rho^j(X_1),\rho^j(X_2)) \oplus F_{2i}(\rho^j(X_2),\rho^j(X_4)) \oplus \\ &\quad F_{2i-1}(\rho^j(X_4),\rho^j(X_0)),\\ \rho^j(C_i'') &= F_{2i-1}(\rho^j(X_0),\rho^j(X_4)) \oplus F_{2i}(\rho^j(X_4),\rho^j(X_2)) \oplus F_{2i-1}(\rho^j(X_2),\rho^j(X_1)) \oplus \\ &\quad F_{2i}(\rho^j(X_1),\rho^j(X_0)) \text{ and}\\ \rho^j(C_i''') &= F_{2i}(\rho^j(X_1),\rho^j(X_5)) \oplus F_{2i-1}(\rho^j(X_5),\rho^j(X_3)) \oplus F_{2i}(\rho^j(X_3),\rho^j(X_4)) \oplus \\ &\quad F_{2i-1}(\rho^j(X_4),\rho^j(X_1)).\end{aligned}$$

□

Lemma 3.6. *If* $m \equiv 1 \pmod 4$ *and* $m \geq 5$*, then* $C_{4m} \mid K_m \times K_7$.

Proof. Let $V(K_m) = \{x_\infty, x_0, x_1, \ldots, x_{m-2}\}$ and $V(K_7) = \{1, 2, \ldots, 7\}$. Then $V(K_m \times K_7) = X_\infty \cup X_0 \cup X_1 \cup \cdots \cup X_{m-2}$, where $X_\infty = x_\infty \times V(K_7) = \{(x_\infty, 1), (x_\infty, 2), \ldots, (x_\infty, 7)\}$ and $X_i = x_i \times V(K_7) = \{(x_i, 1), (x_i, 2), \ldots, (x_i, 7)\}$, for $0 \leq i \leq m-2$. For our convenience, we denote (x_∞, i) by x_∞^i and (x_i, j) by x_i^j.

Let $m = 2t+1$, for an even integer $t \geq 2$. Since m is odd, by Walecki's Hamilton cycle decomposition (see [1]), $K_m = \bigoplus_{i=0}^{t-1} H_i$, where

$$H_i = (x_\infty, x_i, x_{i+1}, x_{i-1}, x_{i+2}, x_{i-2}, \ldots, x_{i+t-2}, x_{i-t+2}, x_{i+t-1}, x_{i-t+1}, x_{i+t})$$

is the Hamilton cycle and addition in the subscripts is taken modulo $m-1$. Let $H = H_0 \oplus H_1$, where H_0 and H_1 are the Hamilton cycles of K_m obtained above. Let $\sigma = (x_\infty)(x_0\, x_2\, x_4 \ldots x_{m-3})(x_1\, x_3\, x_5 \ldots x_{m-2})$ be a permutation on $V(K_m)$. Then $H, \sigma(H), \ldots, \sigma^k(H), k = \frac{t}{2} - 1$, decompose K_m into isomorphic copies of H. Clearly, $K_m \times K_7 = H \times K_7 \oplus H \times K_7 \oplus \cdots \oplus H \times K_7$. Hence it is enough to obtain a C_{4m}-decomposition of $H \times K_7$.

Consider the permutation $\rho = (1\,2\,3\,4\,5\,6\,7)$ on $V(K_7)$. Then F, $\rho(F)$, $\rho^2(F)$, $\ldots$, $\rho^6(F)$ is a near 1-factorization of K_7, where $F = \{12, 37, 46\}$ and $\rho^\ell(F) = \{\rho^\ell(1)\,\rho^\ell(2),\ \rho^\ell(3)\,\rho^\ell(7),\ \rho^\ell(4)\,\rho^\ell(6)\}$, so for example $\rho(F) = \{23, 41, 57\}$. Let A_0 (respectively, A_1) denote the path $H_0 \setminus \{x_t x_\infty\}$ (respectively, $H_1 \setminus \{x_{t+1} x_\infty\}$) obtained by deleting the edge $x_\infty x_t$ (respectively, $x_\infty x_{t+1}$) from H_0 (respectively, H_1), see Figure 5(a) (respectively, 5(c)). Observe that A_0 and A_1 are Hamilton paths of K_m. For each edge $ij \in E(K_7)$, $A_0 \times ij\ (\cong A_0 \times K_2)$ is a pair of disjoint paths $A_{0(1)}^{ij}$ and $A_{0(2)}^{ij}$, each of length $m-1$ with initial vertices x_∞^i and x_∞^j and terminal vertices x_t^i and x_t^j, respectively, see Figure 5(b); similarly $A_1 \times ij = A_{1(1)}^{ij} \oplus A_{1(2)}^{ij}$, where the end vertices of $A_{1(1)}^{ij}$ (respectively, $A_{1(2)}^{ij}$) are x_∞^i (respectively, x_∞^j) and x_{t+1}^i (respectively, x_{t+1}^j), see Figure 5(d). Note that $V(K_m \times K_7) = V(H \times K_7)$. We construct three base cycles C', C'' and C''', each of length $4m$, in $H \times K_7$ as follows; see Figure 6.

Let $e_1 = 12$, $e_2 = 37$ and $e_3 = 46$ be the edges of F in K_7 and let

$$\begin{aligned}C' &= \big\{(H_0 \setminus \{x_t x_\infty\}) \times e_1\big\} \oplus \big\{(H_1 \setminus \{x_{t+1} x_\infty\}) \times e_2)\big\} \oplus x_\infty^1\, x_{t+1}^7 \oplus x_\infty^2\, x_{t+1}^3 \oplus \\ &\quad x_\infty^3\, x_t^1 \oplus x_\infty^7\, x_t^2 = A_{0(1)}^{12} \oplus x_t^1 x_\infty^3 \oplus A_{1(1)}^{37} \oplus x_{t+1}^3 x_\infty^2 \oplus A_{0(2)}^{12} \oplus x_t^2 x_\infty^7 \oplus A_{1(2)}^{37} \oplus x_{t+1}^7 x_\infty^1\end{aligned}$$

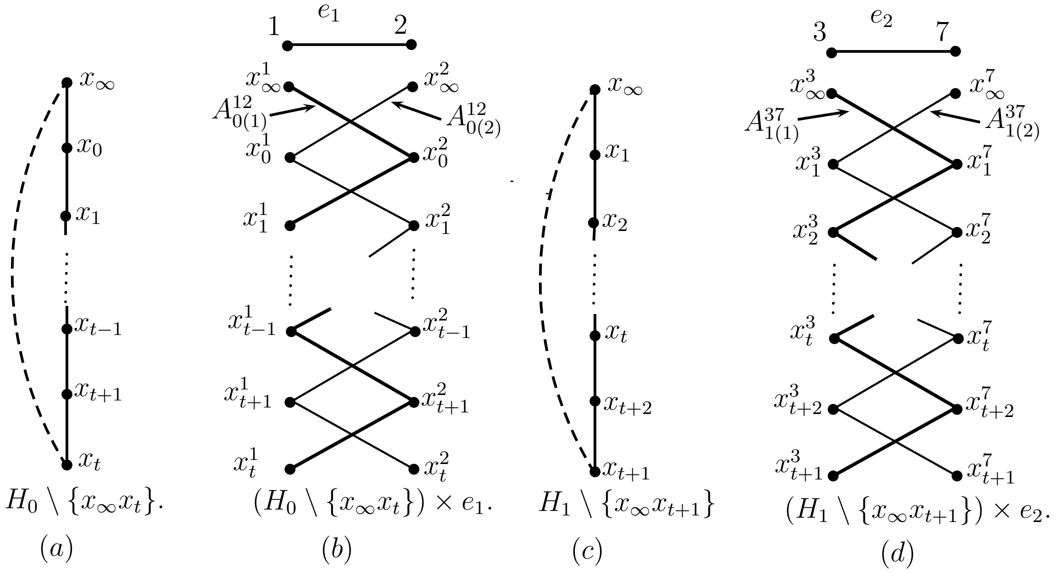

Figure 5: Broken edge in (a) (respectively, (c)) denotes the edge $x_\infty x_t$ (respectively, $x_\infty x_{t+1}$) which is removed from H_0 (respectively, H_1).

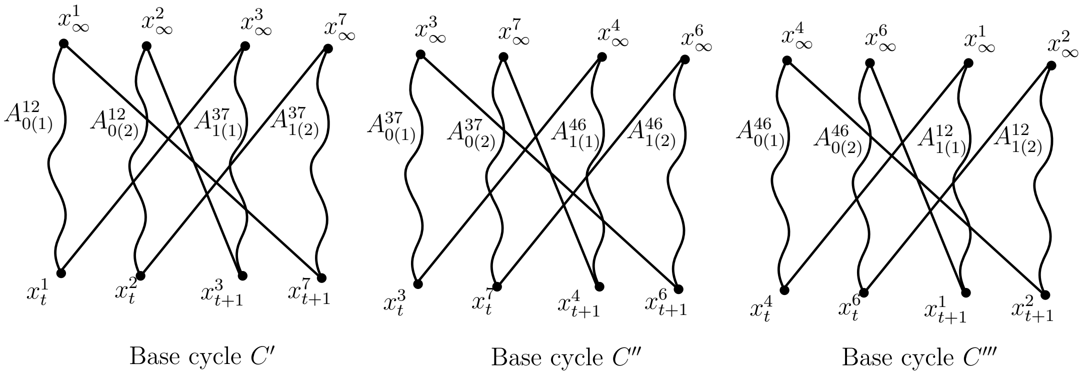

Figure 6: Base cycle C' of length $4m$ in $H \times K_7$ is constructed using the paths described in Figures 5(b) and 5(d). Similarly, the cycles C'' and C''' are shown using appropriate paths.

$$C'' = \left\{(H_0 \setminus \{x_t x_\infty\}) \times e_2\right\} \oplus \left\{(H_1 \setminus \{x_{t+1}\, x_\infty\}) \times e_3)\right\} \oplus x_\infty^3 x_{t+1}^6 \oplus x_\infty^7 x_{t+1}^4 \oplus x_\infty^4\, x_t^3 \oplus x_\infty^6\, x_t^7 = A_{0(1)}^{37} \oplus x_t^3 x_\infty^4 \oplus A_{1(1)}^{46} \oplus x_{t+1}^4 x_\infty^7 \oplus A_{0(2)}^{37} \oplus x_t^7 x_\infty^6 \oplus A_{1(2)}^{46} \oplus x_{t+1}^6 x_\infty^3$$

and

$$C''' = \left\{(H_0 \setminus \{x_t x_\infty\}) \times e_3\right\} \oplus \left\{(H_1 \setminus \{x_{t+1} x_\infty\}) \times e_1)\right\} \oplus x_\infty^4\, x_{t+1}^2 \oplus x_\infty^6\, x_{t+1}^1 \oplus x_\infty^1\, x_t^4 \oplus x_\infty^2\, x_t^6 = A_{0(1)}^{46} \oplus x_t^4 x_\infty^1 \oplus A_{1(1)}^{12} \oplus x_{t+1}^1 x_\infty^6 \oplus A_{0(2)}^{46} \oplus x_t^6 x_\infty^2 \oplus A_{1(2)}^{12} \oplus x_{t+1}^2 x_\infty^4.$$

If $\rho = (1\,2\,3\,4\,5\,6\,7)$ acts on the superscripts of the vertices of $H \times K_7$, then $\{C', \rho(C'), \ldots, \rho^6(C'), C'', \rho(C''), \ldots, \rho^6(C''), C''', \rho(C'''), \ldots, \rho^6(C''')\}$ is a C_{4m}-decomposition of $H \times K_7$, where $\rho(C') = A_{0(1)}^{\rho(1)\rho(2)} \oplus x_t^{\rho(1)} x_\infty^{\rho(3)} \oplus A_{1(1)}^{\rho(3)\rho(7)} \oplus x_{t+1}^{\rho(3)} x_\infty^{\rho(2)} \oplus A_{0(2)}^{\rho(1)\rho(2)} \oplus x_t^{\rho(2)} x_\infty^{\rho(7)} \oplus A_{1(2)}^{\rho(3)\rho(7)} \oplus x_{t+1}^{\rho(7)} x_\infty^{\rho(1)} = A_{0(1)}^{23} \oplus x_t^2 x_\infty^4 \oplus A_{1(1)}^{41} \oplus x_{t+1}^4 x_\infty^3 \oplus A_{0(2)}^{23} \oplus x_t^3 x_\infty^1 \oplus A_{1(2)}^{41} \oplus x_{t+1}^1 x_\infty^2$. □

Lemma 3.7. *If $n \geq 3$ and $n \equiv 2$ or $3 \pmod 4$, $m \equiv 1 \pmod 4$ and $m \equiv 0 \pmod p$, then $C_{4p} \,|\, K_m \times K_n$, where $p \geq 3$ is prime.*

Proof. Let $m = p\,s$; then $s \geq 1$ is odd as m is odd.
Case 1: $n \equiv 2 \pmod 4$.
Let $n = 4t + 2$, $t \geq 1$.
First we complete the proof for the case $s = 1$. If $t = 1$, the result follows by Lemma 3.5. For all $t \geq 2$, the graph

$$\begin{aligned}K_p \times K_n &= K_p \times K_{4t+2}\\ &= K_p \times (K_6 \oplus \underbrace{(K_6 - e) \oplus \cdots \oplus (K_6 - e)}_{(t-1)\,times} \oplus (K_t \circ \overline{K}_4))\\ &= K_p \times K_6 \oplus \underbrace{K_p \times (K_4 \oplus C_4 \oplus C_4) \oplus \cdots \oplus K_p \times (K_4 \oplus C_4 \oplus C_4)}_{(t-1)\,times}\\ &\quad \oplus K_p \times \underbrace{(K_{4,4} \oplus \cdots \oplus K_{4,4})}_{\binom{t}{2}\,copies}\\ &= (K_p \times K_6) \oplus ((K_p \times K_4) \oplus (K_p \times C_4) \oplus (K_p \times C_4)) \oplus \cdots \oplus\\ &\quad ((K_p \times K_4) \oplus (K_p \times C_4) \oplus (K_p \times C_4)) \oplus ((K_p \times K_{4,4}) \oplus \cdots \oplus (K_p \times K_{4,4})).\end{aligned}$$

The graphs $K_p \times K_6$ and $K_p \times K_4$ are C_{4p}-decomposable, by Lemma 3.5 and Theorem 2.6, respectively. Since $C_p \,|\, K_p$, $C_{4p} \,|\, K_p \times K_{4,4}$, by Lemma 3.3. Further, $K_p \times C_4 = C_p \times C_4 \oplus \cdots \oplus C_p \times C_4$ and $C_p \times C_4$ admits a C_{4p}-decomposition, by Theorem 2.7.
Next we consider the case $s \geq 3$.
Clearly, $K_m \times K_n = K_m \times K_2 \oplus \cdots \oplus K_m \times K_2$. Since $s \geq 3$, $2m > 4p$; also $4p \,|\, m(m-1)$ and hence the graph $K_m \times K_2 \cong K_{m,m} - I$, where I denotes a perfect matching, admits a C_{4p}-decomposition, by Theorem 2.4.

Case 2: $n \equiv 3 \pmod 4$.
Recall that s is odd. If $s \geq 3$, then $m > 2p + 1$; also $2p \,|\, \binom{m}{2}$ and hence $P_{2p+1} \,|\, K_m$, by Theorem 2.2. So $C_{4p} \,|\, K_m \times K_n$, by Lemma 2.12. Next we assume that $s = 1$. Let $n = 4t + 3$, $t \geq 1$. If $t = 1$, $C_{4p} \,|\, K_p \times K_7$, by Lemma 3.6. For $t \geq 2$,

$$\begin{aligned}K_p \times K_{4t+3} &= K_p \times (K_7 \oplus \underbrace{K_5 \oplus \cdots \oplus K_5}_{(t-1) times} \oplus K_{6,4,4,\ldots,4});\ \text{see Figure 7}\\ &= (K_p \times K_7) \oplus ((K_p \times K_5) \oplus \cdots \oplus (K_p \times K_5)) \oplus\\ &\quad \underbrace{(K_p \times K_{6,4} \oplus \cdots \oplus K_p \times K_{6,4})}_{(t-1)\,times} \oplus \underbrace{(K_p \times K_{4,4} \oplus \cdots \oplus K_p \times K_{4,4})}_{\binom{t-1}{2}\,times}\end{aligned}$$

The graphs $K_p \times K_7$ and $K_p \times K_5$ admit C_{4p}-decompositions, by Lemmas 3.6 and 3.4, respectively. Since $C_p \,|\, K_p$, $C_{4p} \,|\, K_p \times K_{4,4}$, by Lemma 3.3. By Theorem 2.3, $C_4 \,|\, K_{6,4}$ and hence $K_p \times K_{6,4} = C_p \times C_4 \oplus \cdots \oplus C_p \times C_4$. Now $C_{4p} \,|\, C_p \times C_4$, by Theorem 2.7. This completes the proof of the lemma. □

Lemma 3.8. *If $k \equiv 3 \pmod 4$, then $C_{4k} \,|\, K_{k+1} \times K_7$.*

Proof. Let $V(K_7) = \{1, 2, 3, 4, 5, 6, 7\}$ and $V(K_{k+1}) = \{x_0, x_1, \ldots, x_k\}$. Label the vertices of $K_{k+1} \times K_7$ as in Lemma 3.6. First we complete the proof for the case

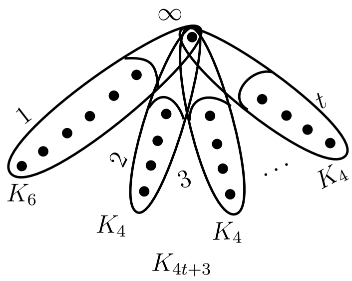

Figure 7: $K_{4t+3} = K_7 \oplus K_5 \oplus \cdots \oplus K_5 \oplus K_{6,4,4,\ldots,4}$. A copy of K_6 (respectively, K_4) together with ∞ induce a K_7 (respectively, K_5).

$k = 3$. Since $K_3 \,|\, K_7$, by Theorem 2.1, $K_4 \times K_7 = K_4 \times K_3 \oplus \cdots \oplus K_4 \times K_3$. The graph $K_4 \times K_3$ admits a C_{12}-decomposition, by Theorem 2.6. Thus, $C_{12} \,|\, K_4 \times K_7$.

Now we complete the proof for the case $k \geq 7$.
Let $k+1 = 2t$, for some even $t \geq 4$. A Hamilton path decomposition of K_{k+1} is $P_i = [x_i, x_{i+1}, x_{i-1}, x_{i+2}, x_{i-2}, \ldots, x_{i+t-2}, x_{i-t+2}, x_{i+t-1}, x_{i-t+1}, x_{i+t}]$, $0 \leq i \leq t-1$, where the addition in the subscripts is taken modulo $k+1$. Let $H_j = P_{2j} \oplus P_{2j+1}$, $0 \leq j \leq \frac{t}{2} - 1$, where P_{2j} and P_{2j+1} are two consecutive Hamilton paths of the above decomposition of K_{k+1}. As $K_{k+1} = H_0 \oplus H_1 \oplus \cdots \oplus H_{\frac{t}{2}-1}$, $K_{k+1} \times K_7 = H_0 \times K_7 \oplus \cdots \oplus H_{\frac{t}{2}-1} \times K_7$. Since $H_j \cong \rho^j(H_0)$, $1 \leq j \leq \frac{t}{2} - 1$, where $\rho = (x_0\, x_2\, \ldots\, x_{k-1})(x_1\, x_3\, \ldots\, x_k)$ is the permutation on the set $V(K_{k+1})$, to complete the proof of this lemma, it is enough to obtain a C_{4k}-decomposition of $H_0 \times K_7$.

First we describe three base cycles C_0', C_0'' and C_0''', each of length $4k$, in $H_0 \times K_7$ as follows:

Note that P_0 and P_1 have the same vertex set, but for our convenience we will view P_0 and P_1 to be on disjoint sets of vertices except for one particular vertex, x_{t-1}. Figure 8 shows this for $k = 11$, where P_0 and P_1 are Hamilton paths in K_{12}. In particular, Figure 8(c) shows the way in which we will view $H_0 = P_0 \oplus P_1$, so that each vertex of H_0, except the one vertex x_{t-1}, appears exactly twice. Each vertex x_i of H_0 gives rise to $X_i = x_i \times K_7 = \{(x_i, 1), (x_i, 2), \ldots, (x_i, 7)\}$ having seven vertices of $H_0 \times K_7$. This X_i also appears in both $P_0 \times K_7$ and $P_1 \times K_7$, except for X_{t-1} (see Figure 9). If we superimpose X_i of $P_0 \times K_7$ with X_i of $P_1 \times K_7$, $i \neq t-1$, we get $H_0 \times K_7$. If x_i and x_j are adjacent in H_0, then $\langle X_i \cup X_j \rangle$ is isomorphic to $K_{7,7} - F_0(X_i, X_j)$ and hence this subgraph $\langle X_i \cup X_j \rangle$ of $H_0 \times K_7$ has six 1-factors $F_1(X_i, X_j), F_2(X_i, X_j), \ldots, F_6(X_i, X_j)$. We construct three base cycles C_0', C_0'' and C_0''' of $H_0 \times K_7$, each of them having some of their sections in the graphs $P_0 \times K_7$ and $P_1 \times K_7$, in such a way that if C_0' (or C_0'' or C_0''') has a vertex of X_i in $P_0 \times K_7$, then the cycle does not have the vertex of X_i in $P_1 \times K_7$ (see Figure 9). So, when we superimpose X_i of $P_0 \times K_7$ with X_i of $P_1 \times K_7$, vertices of C_0' (or C_0'' or C_0''') are all distinct in $H_0 \times K_7$. The base cycles of $H_0 \times K_7$ are given below; see Figure 9.

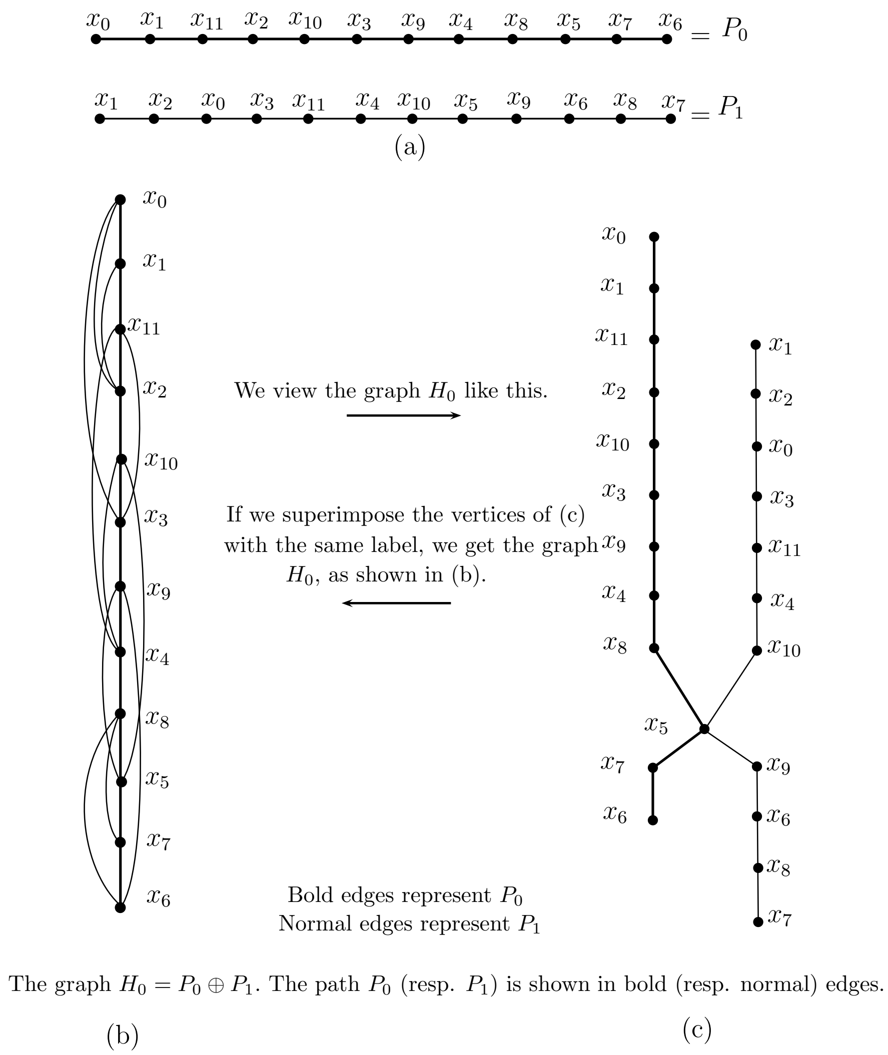

Figure 8: $H_0 = P_0 \oplus P_1$ is shown in (b), where P_0 and P_1 are the Hamilton paths of K_{12} as described in the text. In (b) P_0 and P_1 have the common vertex set whereas in (c) except for one vertex all other vertices are shown to be distinct.

$C_0' = (x_0^3, x_1^6, x_k^7, x_2^6, x_{k-1}^7, \ldots, x_{t-2}^6, x_{t+2}^7, x_{t-1}^4, x_{t+1}^7, x_t^3, x_{t+1}^6, x_{t-1}^2, x_{t+3}^1, x_t^2,$
$x_{t+2}^1, x_{t+1}^5, x_{t+2}^2, x_t^1, x_{t+3}^2, x_{t-1}^1, x_{t+4}^2, x_{t-2}^1, \ldots, x_3^1, x_0^2, x_2^1, x_1^5, x_2^2, x_0^1, x_3^2, \ldots,$
$x_{t-2}^2, x_{t+4}^1, x_{t-1}^5, x_{t+2}^6, x_{t-2}^7, x_{t+3}^6, x_{t-3}^7, \ldots, x_k^6, x_1^7),$

$C_0'' = (x_0^2, x_1^1, x_k^3, x_2^1, x_{k-1}^3, \ldots, x_{t-2}^1, x_{t+2}^3, x_{t-1}^4, x_{t+1}^3, x_t^2, x_{t+1}^1, x_{t-1}^7, x_{t+3}^5, x_t^7,$
$x_{t+2}^5, x_{t+1}^6, x_{t+2}^7, x_t^5, x_{t+3}^7, x_{t-1}^5, x_{t+4}^7, x_{t-2}^5, \ldots, x_3^5, x_0^7, x_2^5, x_1^6, x_2^7, x_0^5, x_3^7, \ldots,$
$x_{t-2}^7, x_{t+4}^5, x_{t-1}^6, x_{t+2}^1, x_{t-2}^3, x_{t+3}^1, x_{t-3}^3, \ldots, x_k^1, x_1^3)$ and

$C_0''' = (x_0^7, x_1^5, x_k^2, x_2^5, x_{k-1}^2, \ldots, x_{t-2}^5, x_{t+2}^2, x_{t-1}^4, x_{t+1}^2, x_t^7, x_{t+1}^5, x_{t-1}^3, x_{t+3}^6, x_t^3,$
$x_{t+2}^6, x_{t+1}^1, x_{t+2}^3, x_t^6, x_{t+3}^3, x_{t-1}^6, x_{t+4}^3, x_{t-2}^6, \ldots, x_3^6, x_0^3, x_2^6, x_1^1, x_2^3, x_0^6, x_3^3, \ldots,$
$x_{t-2}^3, x_{t+4}^6, x_{t-1}^1, x_{t+2}^5, x_{t-2}^2, x_{t+3}^5, x_{t-3}^2, \ldots, x_k^5, x_1^2),$

where the subscripts are taken modulo $k+1$.

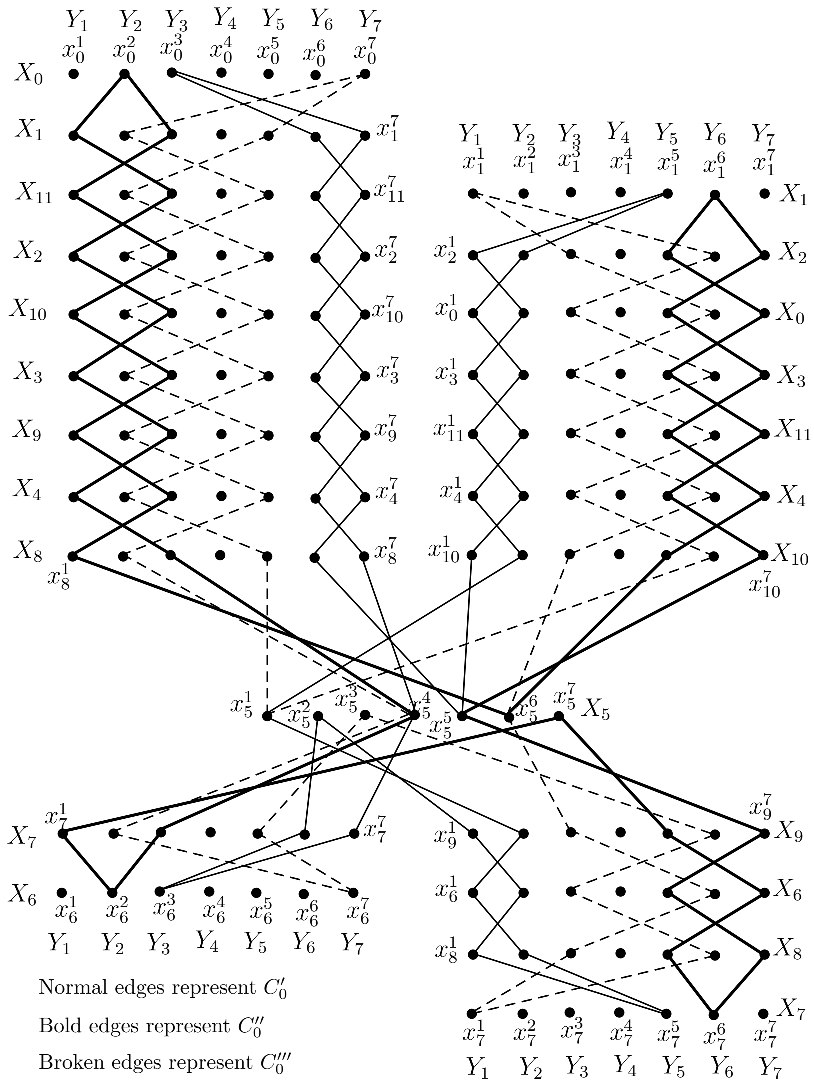

Figure 9: Three base cycles C_0', C_0'' and C_0''' of $H_0 \times K_7$ are given for a C_{44}-decomposition of $H_0 \times K_7$, where $H_0 = P_0 \oplus P_1$ and P_0 and P_1 are two Hamilton paths of K_{12} as obtained in the text. If we superimpose the X_i, except X_{t-1}, of $P_0 \times K_7$ with X_i of $P_1 \times K_7$, on the respective vertices, for all i we get three base cycles C_0', C_0'' and C_0''' of $H_0 \times K_7$. $Y_i's$ represent the columns of $H_0 \times K_7$. Note that $X_i's$ are not consecutive in the figure, but it appears as in the order of vertices of the Hamilton path of K_{12}.

If $\rho = (1\,2\,3\,4\,5\,6\,7)$ is the permutation acting on the superscripts of the vertices of $V(H_0 \times K_7)$, then $\{C_0', \rho(C_0'), \ldots, \rho^6(C_0'), C_0'', \rho(C_0''), \ldots, \rho^6(C_0''), C_0''', \rho(C_0'''), \ldots, \rho^6(C_0''')\}$ is a C_{4k}-decomposition of $H_0 \times K_7$. □

Lemma 3.9. *If $p \geq 3$ is prime, $m \equiv 3 \pmod 4$, $n \equiv 1 \pmod p$ and $n \equiv 0 \pmod 4$, then $C_{4p} \mid K_m \times K_n$.*

Proof. As $n \equiv 0 \pmod 4$ and $n \equiv 1 \pmod p$, $n = ps + 1$, $s \geq 1$ is odd.

First we deal with the case for $s \geq 3$ is odd.
By hypothesis, $2p \mid \binom{n}{2}$; also $n > 2p + 1$; then $P_{2p+1} \mid K_n$, by Theorem 2.2 and so $C_{4p} \mid K_m \times K_n$, by Lemma 2.12.

Now we consider the case for $s = 1$.
Clearly, $n = p + 1$ and $m = 4t + 3$ for some $t \geq 1$. If $t = 1$, then $C_{4p} \mid K_7 \times K_{p+1}$, by Lemma 3.8. So we assume that $t \geq 2$,

$$\begin{aligned} K_{4t+3} \times K_{p+1} &= (K_7 \oplus K_5 \oplus \cdots \oplus K_5 \oplus K_{6,4,4,\ldots,4}) \times K_{p+1} \\ &= (K_7 \times K_{p+1}) \oplus (K_5 \times K_{p+1} \oplus \cdots \oplus K_5 \times K_{p+1}) \oplus \\ &\quad (K_{6,4} \times K_{p+1} \oplus \cdots \oplus K_{6,4} \times K_{p+1}) \oplus (K_{4,4} \times K_{p+1} \oplus \cdots \oplus K_{4,4} \times K_{p+1}). \end{aligned}$$

$C_{4p} \mid K_7 \times K_{p+1}$, by Lemma 3.8. Since $P_{p+1} \mid K_{p+1}$, the graphs $K_5 \times K_{p+1}$ and $K_{4,4} \times K_{p+1}$ are C_{4p}-decomposable, by Lemmas 2.13 and 2.14, respectively. Clearly, $K_{6,4} \times K_{p+1} = C_4 \times K_{p+1} \oplus \cdots \oplus C_4 \times K_{p+1}$. A C_{4p}-decomposition of $C_4 \times K_{p+1}$ (isomorphic to $K_{p+1} \times C_4$) is described below:

Let $V(K_{p+1}) = \{x_0, x_1, \ldots, x_p\}$ and $C_4 = (1, 2, 3, 4)$. Then $V(K_{p+1} \times C_4) = \bigcup_{i=0}^{p} X_i$, where $X_i = x_i \times V(C_4) = \{(x_i, 1), (x_i, 2), (x_i, 3), (x_i, 4)\}$. The symmetric digraph K^*_{p+1} admits a $\vec{C}_p$-decomposition, by Theorem 2.9, say $\mathcal{C}$, where $\vec{C}_p$ denotes the directed cycle of length p. Based on each of the directed cycles in $\mathcal{C}$, we construct a cycle of length $4p$ in $K_{p+1} \times C_4$ as follows. Let $\vec{C}_p$ be in $\mathcal{C}$; corresponding to this $\vec{C}_p$ we consider in $K_{p+1} \times C_4$ the cycle $C'_p = \bigcup_{\overrightarrow{x_i x_j} \in A(\vec{C}_p)} F_1(X_i, X_j)$, where $F_1(X_i, X_j)$ is the 1-factor of jump 1 from X_i to X_j in $K_{p+1} \times C_4$ and $A(\vec{C}_p)$ denotes the arc set of $\vec{C}_p$. Clearly, C'_p is a cycle of length $4p$, in $K_{p+1} \times C_4$, as the sum of the jumps of the 1-factors occurring in $\bigcup_{\overrightarrow{x_i x_j} \in A(\vec{C}_p)} F_1(X_i, X_j)$ is p, which is relatively prime to 4. Thus to each $\vec{C}_p \in \mathcal{C}$ we obtain a C_{4p} in $K_{p+1} \times C_4$; as $\mathcal{C}$ is a directed p-cycle decomposition of K^*_{p+1}, we obtain a $4p$-cycle decomposition of $K_{p+1} \times C_4$. This completes the proof of the lemma. □

4 C_{4p}-decomposition of $K_m \times K_n$

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. We assume that $C_{4p} \mid K_m \times K_n$. As the cycle length cannot exceed the number of vertices of $K_m \times K_n$, $4p \leq mn$. As $C_{4p} \mid K_m \times K_n$, $K_m \times K_n$ is an even regular graph, that is, $(m-1)(n-1)$ is even and hence either m or n is odd. Further, $C_{4p} \mid K_m \times K_n$ implies $4p$ divides the number of edges of $K_m \times K_n$, that is, $4p \mid \binom{m}{2} n(n-1)$.

Next we prove the sufficiency. Since the tensor product is commutative, we assume that m is odd.

Case 1: $p \mid \binom{n}{2}$.
Subcase 1.1: $2 \mid \binom{m}{2}$.
Since m is odd and $2 \mid \binom{m}{2}$, $m \equiv 1 \pmod 4$; as $p \mid \binom{n}{2}$ and p is prime, $p \leq n$, we invoke Lemma 3.4 to complete the proof.

Subcase 1.2: $2 \nmid \binom{m}{2}$.
In this case, $m \equiv 3 \pmod 4$. Clearly, from the hypothesis of the theorem $2 \mid \binom{n}{2}$ and also from the hypothesis of the case $2p \mid \binom{n}{2}$. Now there are two possibilities, according to the parity of n.

(1) If n is even, then either $n \equiv 0 \pmod{4p}$ or, $n \equiv 0 \pmod 4$ and $n \equiv 1 \pmod p$. If $n \equiv 0 \pmod{4p}$, then $P_{2p+1} \mid K_n$, by Theorem 2.2; now apply Lemma 2.12. If $n \equiv 1 \pmod p$ with $4 \mid n$, then the proof follows by Lemma 3.9.

(2) If n is odd, then either $n \equiv 1 \pmod{4p}$ or, $n \equiv 1 \pmod 4$ and $n \equiv 0 \pmod p$. If $n \equiv 1 \pmod{4p}$, then the proof follows by Lemma 3.1; if $n \equiv 1 \pmod 4$ and $p \mid n$, then the proof follows by Lemma 3.7.

Case 2: $p \nmid \binom{n}{2}$.
Subcase 2.1: $2 \mid \binom{n}{2}$.
As $p \mid \binom{m}{2}$, $C_p \mid K_m$, by Theorem 2.1 and hence $K_m \times K_n = C_p \times K_n \oplus \cdots \oplus C_p \times K_n$. Since $2 \mid \binom{n}{2}$, either $n \equiv 0 \pmod 4$ or $n \equiv 1 \pmod 4$. If $n \equiv 0 \pmod 4$, then $C_{4p} \mid C_p \times K_n$, by Theorem 2.5; if $n \equiv 1 \pmod 4$, then $C_{4p} \mid K_m \times K_n$, by Lemma 3.4.
Subcase 2.2: $2 \nmid \binom{n}{2}$.
From the necessary conditions, $2p \mid \binom{m}{2}$; then either $m \equiv 0 \pmod p$ and $m \equiv 1 \pmod 4$ or, $m \equiv 1 \pmod{4p}$; recall that m is odd by assumption. If $m \equiv 1 \pmod{4p}$, then the proof follows by Lemma 3.1. Since $2 \nmid \binom{n}{2}$, $n \equiv 2$ or $3 \pmod 4$, and also $m \equiv 1 \pmod 4$ with $p \mid m$. The proof of this subcase now follows by Lemma 3.7. □

5 $\vec{C}_{4p}$-decomposition of $(K_m \circ \overline{K}_n)^*$

We quote the following two theorems which are used in the proof of Theorem 1.2.

Theorem 5.1. [13] *Let $\vec{G}$ be a directed closed trail of length m with maximum out degree Δ^+ and $\chi(\vec{G}) = s$. Then for all $n \geq \Delta^+$, $\vec{C}_m \mid \vec{G} \circ \overline{K}_n$ whenever at least $(s-2)$ mutually orthogonal latin squares of order n exist, where $\chi(\vec{G})$ denotes the chromatic number of $\vec{G}$.*

Theorem 5.2. [22] *$\vec{C}_k \mid \vec{C}_k \circ \overline{K}_n$ for all $k \geq 3$ and $n \geq 1$.*

Corollary 5.3. *If $\vec{C}_k \mid (K_m \circ \overline{K}_n)^*$, then $\vec{C}_k \mid (K_m \circ \overline{K}_{nt})^*$.*

Proof. Since $(K_m \circ \overline{K}_{nt})^* = (K_m \circ \overline{K}_n)^* \circ \overline{K}_t = \vec{C}_k \circ \overline{K}_t \oplus \vec{C}_k \circ \overline{K}_t \oplus \cdots \oplus \vec{C}_k \circ \overline{K}_t$, the proof is immediate from Theorem 5.2. □

Proof of Theorem 1.2. If $K_m \circ \overline{K}_n$ is an even regular graph, then the result is immediate by Theorem 2.11. So we assume that $K_m \circ \overline{K}_n$ is an odd regular graph and hence m is even and n is odd.

Case 1: $p \mid n^2$.
Clearly, $n \equiv 0 \pmod p$. Since m is even and n is odd, from the divisibility condition,

$m = 4t$, for some $t \geq 1$. By Corollary 5.3, it is enough to prove the case for $n = p$. Clearly,

$$\begin{aligned}(K_m \circ \overline{K}_p)^* &= ((tK_4 \oplus (K_t \circ \overline{K}_4)) \circ \overline{K}_p)^* \\ &= t(K_4 \circ \overline{K}_p)^* \oplus (K_{4,4} \circ \overline{K}_p \oplus \cdots \oplus K_{4,4} \circ \overline{K}_p)^*.\end{aligned}$$

Note that $\vec{C}_{4p} \mid (K_4 \circ \overline{K}_p)^*$, by Theorem 2.10. Also, $C_4 \mid K_{4,4}$, by Theorem 2.3, and $C_{4p} \mid C_4 \circ \overline{K}_p$, by Theorem 2.8; hence $\vec{C}_{4p} \mid (K_{4,4} \circ \overline{K}_p)^*$. This completes the proof of this case.

Case 2: $p \nmid n^2$.
From the necessary conditions, either $m \equiv 0 \pmod{4p}$ or $m \equiv 0 \pmod 4$ and $m \equiv 1 \pmod p$.

Subcase 2.1: $m \equiv 0 \pmod{4p}$.
Let $m = 4pt$, $t \geq 1$. Then

$$\begin{aligned}(K_m \circ \overline{K}_n)^* &= (K_{4pt} \circ \overline{K}_n)^* \\ &= nK^*_{4pt} \oplus (K_{4pt} \times K_n)^*,\end{aligned}$$

where the n copies of K^*_{4pt} are precisely the subdigraphs induced by the vertices of the n columns of $(K_{4pt} \circ \overline{K}_n)^*$ and the remaining subdigraph of $(K_{4pt} \circ \overline{K}_n)^*$ is isomorphic to $(K_{4pt} \times K_n)^*$. Since n is odd, $(K_{4pt} \times K_n)$ is an even regular graph. By Theorem 1.1, $C_{4p} \mid (K_{4pt} \times K_n)$ and hence $\vec{C}_{4p} \mid (K_{4pt} \times K_n)^*$. By Theorem 2.9, $\vec{C}_{4p} \mid K^*_{4pt}$. This completes the proof of this subcase.

Subcase 2.2: $m \equiv 0 \pmod 4$ and $m \equiv 1 \pmod p$.
Let $m = pt + 1$, $t \geq 1$ and odd.
First we consider the case $t = 1$. Clearly, $(K_{p+1} \circ \overline{K}_n)^* = K^*_{p+1} \circ \overline{K}_n$. Let $V(K_{p+1}) = \{a_1, a_2, \ldots, a_{p+1}\}$. For $1 \leq i \leq (\frac{p+1}{2})$, we define the Hamilton path $P_i = [a_i, a_{i+1}, a_{i-1}, \ldots, a_{i+(\frac{p+1}{2})-1}, a_{i+(\frac{p+1}{2})+1}, a_{i+(\frac{p+1}{2})}]$ in K_{p+1}, where the subscripts are taken modulo $p + 1$ with residues $1, 2, \ldots, p + 1$. Let $H_i = P_{2i-1} \oplus P_{2i}$, $1 \leq i \leq (\frac{p+1}{4})$; H_i has $2p$ edges and $\Delta(H_i) \leq 4$; note that $H_i = K_4$ when $p = 3$. As $\Delta(H_i) \leq 4$, $\chi(H_i) \leq 4$; see [8]. Since $H_i \mid K_{p+1}$, $H^*_i \mid K^*_{p+1}$; each H^*_i is a directed closed trail of length $4p$, $\Delta^+(H^*_i) \leq 4$ and $\chi(H^*_i) \leq 4$. Since n is odd and from the necessary conditions, $n \geq 5$. Consequently, at least two mutually orthogonal latin squares of order n exist (see [18]); then $\vec{C}_{4p} \mid H^*_i \circ \overline{K}_n$, by Theorem 5.1.

Next we assume that $t \geq 3$. Since $pt + 1 \equiv 0 \pmod 4$, $2p \mid \binom{pt+1}{2}$. Then $P^*_{2p+1} \mid K^*_{pt+1}$ as $P_{2p+1} \mid K_{pt+1}$ by Theorem 2.2. Clearly, each P^*_{2p+1} is a directed closed trail of length $4p$ with $\Delta^+(P^*_{2p+1}) = 2$ and $\chi(P^*_{2p+1}) = 2$ and hence $\vec{C}_{4p} \mid P^*_{2p+1} \circ \overline{K}_n$, by Theorem 5.1. This completes the proof. □

Acknowledgements

The first author would like to thank the Kalasalingam Academy of Research and Education, Tamil Nadu, India, for financial support through a University Research Fellowship. We are grateful to the reviewers for their careful reading and suggestions which improved the presentation of the paper.

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(Received 22 June 2019; revised 3 July 2020, 7 Dec 2020)