The expansion of polynomial invariants for 2-decompositions of generalized graphs

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Dedicated to Mahouton Norbert Hounkonnou on the occasion of his 60th birthday.

Abstract

We extend the notion of 2-decomposition to the class of half-edged ribbon graphs and then to rank D-weakly colored graphs, generalizing thereby the 2-sums and tensor products of these graphs. Using this extension for the 2-decompositions, we provide new expansion formulas for the Bollobás Riordan polynomial for half-edged ribbon graphs and also for the polynomial invariant for weakly colored stranded graphs.

1 Introduction

A graph-theoretic invariant called separability was recently introduced by Cicalese and Milanč [6]. In general, a graph is called k-separable if any two non-adjacent vertices can be separated by the removal of at most k vertices. The k-separability turns out to be an important property of a graph used to investigate the computational complexity of several optimization problems for graphs of bounded separability [6]. One of the main results in [6] is a decomposition theorem for the 2-separable graphs. This decomposition has been extended to ribbon graphs by taking into account the cyclic order of the vertices [10].

Defined as a neighborhood of a graph embedded in a surface, a ribbon graph [4, 7] can be decomposed into its 2-connected components according to the following description: assume that a ribbon graph $\widehat{\mathcal{G}}$ is 2-separable and then regard it as arising from a sequence of 2-sums of a collection of ribbon graphs $\{A_e\}_{e\in\mathcal{E}}$ with a ribbon graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. Along the edge $e \in \mathcal{E}$, we "glue" the distinguished ribbon graph A_e. Strictly speaking, a 2-sum of two graphs $\mathcal{G}$ and $\mathcal{F}$ with distinguished edges e

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and f respectively in $\mathcal{G}$ and $\mathcal{F}$ is defined by identifying e with f and deleting the identified edge [10]. The structure $(\mathcal{G}, \{H_e\}_{e \in \mathcal{E}})$, is called a 2-decomposition for $\widehat{\mathcal{G}}$ with $H_e = A_e - e$, for all $e \in \mathcal{E}$.

In [10], Huggett and Moffat find a connection between the Bollobás Riordan (BR) polynomial of $\widehat{\mathcal{G}}$ and those of $\mathcal{G}$ and H_e. This result generalizes Brylawski's result in [5] which expresses the Tutte polynomial of the tensor product $T(G \otimes A)$ in terms of those of its two factors G and A. The goal of the present work is to make one step further and to extend these series of results to new classes of generalized graphs appearing in quantum field theory and in theoretical physics [9, 3].

Graphs can be generalized as half-edged graphs (HEGs) [2]. A half-edge is defined as any edge incident to a unique vertex without forming a loop and a HEG is a graph together with an incidence relation which associates each half-edge with a unique vertex. HEGs are the natural class of graphs of quantum field theory [11], with half-edges representing field modes with much lower energy than internal processes represented by well-formed edges. Combining the definitions of HEGs and ribbon graphs, half-edged ribbon graphs (HERGs) arose as a class of graphs encompassing those. HERGs have both an underlying half-edged graph and ribbon graph structures. Ribbon graphs are also surfaces with boundary and each boundary component is called face of the ribbon. In the case of HERGs, the presence of the half-edges induces two kinds of faces: internal and external faces. The internal faces are components homeomorphic to the 2-dimensional disk D_2 and the external are the remaining ones which are homeomorphic to any open segment. Following the external faces, we obtain connected components called connected components of the boundary graph. As a new feature, the notion of 2-decomposition for HERGs that we introduce in the present work distinguishes the treatment of the internal faces and the connected components of the boundary graph during the 2-sum operation.

The Tutte and Bollobás Riordan polynomials have found an extension to HERGs by including extra variables: one for keeping track of the number of connected components of the boundary graph and another for the number of half-edges. This extension satisfies a similar recurrence to the contraction and deletion of edges and details have been given in [2]. Using the bijection between the states of $\widehat{\mathcal{G}}$ and those of $\mathcal{G}$ and H_e, we find the expression for the number of internal faces, number of connected components of the boundary graph and half-edges of a state of $\widehat{\mathcal{G}}$ in terms of those of the states of $\mathcal{G}$ and H_e. This task remains complex because, first, the definition by Hugget and Moffat must be modified to take into account the presence of the half-edges. Then, we must deal with the fact that the internal faces or connected components of the boundary graph of a state in $\widehat{\mathcal{G}}$ may be generated by different types of components in the states in $\mathcal{G}$ and the graphs H_e. To tackle this issue, we identify a matrix ϵ which captures the subtlety of the 2-decomposition of HERGs. This matrix called $\epsilon(f, x_e)$ is labelled by rows indexed by the internal faces of a state in $\mathcal{G}$ and columns indexed by the points x_e that define the meeting places of the graphs H_e on $\mathcal{G}$ in the construction of $\widehat{\mathcal{G}}$. The important quantity to master the decomposition of HERGs is the rank of ϵ. As a consequence, we relate the multivariate polynomial invariant of $\widehat{\mathcal{G}}$ and of those of $\mathcal{G}$ and of H_e.

The level of difficulty increases when we seek for such relations for the class of graphs called rank D weakly-colored (w-colored) stranded graphs [1]. Such graphs are called stranded graphs because they are made with stranded vertices which are chord diagrams and stranded edges which are collections of segments. Gurau introduced in [8] a coloring on them and proved that they are dual to simplicial pseudo-manifolds in any dimension D. This duality was the stimuli for particular quantum gravity models claiming that the geometry of spacetime at high energy is simplicial. In that sense, stranded graphs represent quantum (discrete) spaces. In a colored graph, the vertices are called 0-cells, lines or edges 1-cells and the faces 2-cells. A p-cell or p-bubble is defined as a connected subgraph made only of lines of p chosen colors. Once we impose upon the vertices a fixed coordination D, we obtain a specific stranded graph called colored tensor graph [1]. The coordination of the vertices in a colored tensor graph gets modified if we perform a contraction of an edge. Allowing such a contraction enlarges the class of graphs from the colored tensor graphs to what is called weakly-colored (w-colored) graphs. The BR polynomial invariant has a natural extension from ribbon graphs to rank D w-colored graphs [2, 1]. In the second part of this work, we introduce the 2-decomposition for rank D w-colored graphs and establish few properties of it. The expression of the generalized invariant on the graph $\widehat{\mathcal{G}}$ in terms of the invariants of the corresponding graphs $\mathcal{G}$ and H_e is much more involved.

In this paper, Section 2 reviews the ribbon graphs, half-edged ribbon graphs and the BR polynomials on such graphs. We show in Section 3, how to compute the BR polynomial of a graph $\widehat{\mathcal{G}} = (\mathcal{G}, \{H_e\}_{e \in \mathcal{E}})$, by introducing two matrices which capture the details generated by that 2-decomposition. Two particular cases are studied: the case where each graph H_e is embedded in the neighborhood of e in the embedded graph $\mathcal{G}$ and the general case. Theorems 3.11 and 3.19 establish the main results of this section. We investigate the 2-decomposition extended to the class of weakly colored graphs in Section 4 and report already a preliminary result of an expansion of the invariant of the weakly colored graphs. The complete expansion in terms of the invariants of $\mathcal{G}$ and of H_e is solved for a particular class of weakly colored graphs.

2 Preliminaries

In this section, we briefly review some essential concepts on ribbon graphs, HERGs and the Bollobás Riordan polynomial.

A ribbon graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a surface with boundary where the vertices are represented by a set of disks and the edges are represented by ribbons (rectangular disks) [4]. A spanning subgraph $s = (\mathcal{V}, \mathcal{E}')$ of $\mathcal{G}$, where $\mathcal{E}' \subseteq \mathcal{E}$, is called a state of $\mathcal{G}$ and we denote by $S(\mathcal{G})$ the set of states of $\mathcal{G}$. Let $v(s)$, $e(s)$, $k(s)$, $r(s)$, $n(s)$ and $\partial(s)$ be respectively the number of vertices, edges, connected components, rank, nullity and boundary components of s. Besides the parameters of this graph, there is $t(s)$, which records the orientability of an embedded graph s. By definition $t(s) = 0$ if s

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[1] Notice that this notion of tensor graph is technically different from the notion of tensor product of graphs defined as a particular 2-decomposition.

is orientable and $t(s) = 1$ otherwise.

There are three kinds of edges that we can identify in a ribbon graph $\mathcal{G}$. An edge e of $\mathcal{G}$ is called a bridge if its removal disconnects a component of $\mathcal{G}$. If the two ends of an edge e of $\mathcal{G}$ are incident to the same vertex v of $\mathcal{G}$ then e is a loop and e is a regular edge of $\mathcal{G}$ if it is neither a bridge nor a loop. A ribbon edge can be twisted as well (see Figure 1). A loop e at a vertex v of a ribbon graph $\mathcal{G}$ is twisted if $v \cup e$ forms a Möbius band as opposed to an annulus (an untwisted loop). We say that a loop e is trivial if there is no cycle in $\mathcal{G}$ which can be contracted to form a loop f interlaced with e.

Figure 1:   Untwisted (left) and twisted (right) edge notations

There are some graphs operations which we now describe

Definition 2.1 (Deletion and contraction [4])**.** Let $\mathcal{G}$ be a ribbon graph and e one of its edges.

• We call $\mathcal{G} - e$ the ribbon graph obtained from $\mathcal{G}$ by deleting e and keeping the end vertices as closed discs.

• If e is not a loop and is untwisted, consider its end vertices v_1 and v_2. The graph $\mathcal{G}/e$ obtained by contracting e is defined from $\mathcal{G}$ by replacing e, v_1 and v_2 by a single vertex disc $e \cup v_1 \cup v_2$. If e is a twisted non-loop, then untwist it (by flipping one of its incident vertex) and contract.

• If e is a trivial twisted loop, contraction is deletion: $\mathcal{G} - e = \mathcal{G}/e$. The contraction of a trivial untwisted loop e is the deletion of the loop and the addition of a new connected component vertex v_0 to the graph $\mathcal{G} - e$. We write $\mathcal{G}/e = (\mathcal{G} - e) \sqcup \{v_0\}$.

The Bollobás-Riordan polynomial $R(\mathcal{G}; x, y, z, w) \in Z[x, y, z, w]/w^2 - w$ for ribbon graphs is defined as the state sum:

$$R(\mathcal{G}; x, y, z, w) = \sum_{s \in S(\mathcal{G})} (x - 1)^{r(\mathcal{G}) - r(s)} y^{n(s)} z^{k(s) - \partial(s) + n(s)} w^{t(s)}. \tag{1}$$

In the following we set $w = 1$ and use an abuse of notation $e \in s$ rather than $e \in \mathcal{E}(s)$. Under these assumptions, the multivariate Bollobás-Riordan polynomial [13] is

$$Z(\mathcal{G}; a, b, c) = \sum_{s \in S(\mathcal{G})} a^{k(s)} \Big(\prod_{e \in s} b_e \Big) c^{\partial(s)}, \tag{2}$$

where a and c are indeterminates, and $b := \{b_e / e \in \mathcal{E}\}$ is a set of indeterminates indexed by $\mathcal{E}$. If we set all the variables $b_e = b$ in (2), then using equation (1), we obtain

$$R(G; x, y, z) = (x - 1)^{-k(\mathcal{G})} (yz)^{-v(\mathcal{G})} Z(\mathcal{G}; (x - 1)yz^2, yz, z^{-1}).$$

Let us now discuss a polynomial invariant for ribbon graphs with half-edges or half-ribbons.

Definition 2.2. A HERG $\mathcal{G}(V, E, \mathfrak{f}^0)$ (or simply $\mathcal{G}_{\mathfrak{f}^0}$ is a ribbon graph $\mathcal{G}(V, E)$ (or shortly $\mathcal{G}$) with a set $\mathfrak{f}^0$ of HR together with a relation which associates each HR with a unique vertex. The ribbon graph $\mathcal{G}$ is called the underlying ribbon graph of the HERG $\mathcal{G}_{\mathfrak{f}^0}$.

Starting with a HERG $\mathcal{G}_{\mathfrak{f}^0}$ we can find its underlying ribbon graph by deleting the half-ribbons or closing (pinching) the external points of each HR in $\mathfrak{f}^0$. The ribbon graph $\mathcal{G}$ is also denoted $\tilde{\mathcal{G}}$.

Note that the HERGs was originally studied in [11] where a half-ribbon edge (or simply half-ribbon, denoted henceforth HR) is a ribbon incident to a unique vertex by a unique segment and without forming loops. Half-ribbons allowed another graph operation called the cutting of an edge. Cutting an edge e in a ribbon graph $\mathcal{G}$, means that we remove e, and we let two HRs attached at the end vertices of e. If e is a loop, the two HRs are on the same vertex.

• A ribbon graph $\mathcal{G}$ with HRs or a HERG is defined as a ribbon graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ with a set $\mathfrak{f} = \mathfrak{f}^1 \cup \mathfrak{f}^0$, where $\mathfrak{f}^1$ is the set of HRs obtained from the cut of all edges of $\mathcal{G}$, and $\mathfrak{f}^0$ is the set of additional HRs together with a relation which associates with each additional HR a unique vertex. A ribbon graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ with the set $\mathfrak{f}^0$ of additional HRs is denoted $\mathcal{G}_{\mathfrak{f}^0}(\mathcal{V}, \mathcal{E})$. An illustration is given in Figure 2. The subgraphs of $\mathcal{G}_{\mathfrak{f}^0}(\mathcal{V}, \mathcal{E})$ are obtained by using the "cutting" operation to replace the usual operation of deletion.

• A c-subgraph $A_{\mathfrak{f}_A^0}$ of $\mathcal{G}_{\mathfrak{f}^0}(\mathcal{V}, \mathcal{E})$ is defined as a HERG $A_{\mathfrak{f}_A^0}(\mathcal{V}_A, \mathcal{E}_A)$ satisfying $\mathcal{V}_A \subseteq \mathcal{V}$ and $\mathcal{E}_A \subseteq \mathcal{E}$ such that the incidence relation between edges and vertices is respected. We now denote by $\mathcal{E}'_A$ the set of edges incident to the vertices of A and not contained in $\mathcal{E}_A$. The set of HRs of $A_{\mathfrak{f}_A^0}$ is $\mathfrak{f}_A^0 = \mathfrak{f}_A^{0;0} \cup \mathfrak{f}_A^{0;1}(\mathcal{E}_A)$ with $\mathfrak{f}_A^{0;0} \subseteq \mathfrak{f}^0$ and $\mathfrak{f}_A^{0;1}(\mathcal{E}_A) \subseteq \mathfrak{f}^1$, where $\mathfrak{f}_A^{0;1}(\mathcal{E}_A)$ is the set of HRs obtained by cutting all edges in $\mathcal{E}'_A$ and incident to the vertices of $A_{\mathfrak{f}_A^0}$. We denote $A_{\mathfrak{f}_A^0} \subseteq \mathcal{G}_{\mathfrak{f}^0}$. See an example of c-subgraph $A_{\mathfrak{f}_A^0}$ in Figure 2.

• A spanning c-subgraph $s_{\mathfrak{f}_s^0}$ or a state of $\mathcal{G}_{\mathfrak{f}^0}(\mathcal{V}, \mathcal{E})$ is defined as a c-subgraph $s_{\mathfrak{f}_s^0}(\mathcal{V}_s, \mathcal{E}_s)$ of $\mathcal{G}_{\mathfrak{f}^0}$ with all vertices and all additional HRs of $\mathcal{G}_{\mathfrak{f}^0}$. Then $\mathcal{E}_s \subseteq \mathcal{E}$ and $\mathcal{V}_s = \mathcal{V}$, $\mathfrak{f}_s^0 = \mathfrak{f}^0 \cup \mathfrak{f}_s^{0;1}(\mathcal{E}_s)$. We use the notation $s_{\mathfrak{f}_s^0} \Subset \mathcal{G}_{\mathfrak{f}^0}$ and denote by $S(\mathcal{G}_{\mathfrak{f}^0})$ the set of states of $\mathcal{G}_{\mathfrak{f}^0}$. (See $s_{\mathfrak{f}^0}$ in Figure 2.)
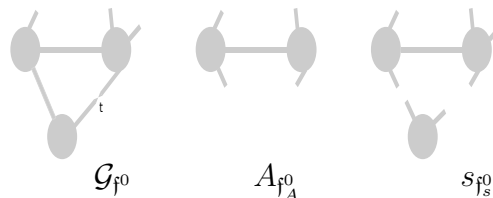

Figure 2:   A ribbon graph with HRs $\mathcal{G}_{\mathfrak{f}^0}$ together with a c-subgraph $A_{\mathfrak{f}_A^0}$ and a spanning c-subgraph $s_{\mathfrak{f}_s^0}$

The states or spanning c-subgraphs have crucial importance in this framework since they are involved in the state sum of the BR polynomial defined on HERGs.

There are two kinds of boundary on a HERG which deserve to be analyzed: the boundary faces following the contour of the HRs and the initial ones which follow

the boundary of well-formed edges. More precisely let us consider a ribbon graph with HRs $\mathcal{G}_{\mathfrak{f}^0}(\mathcal{V}, \mathcal{E})$.

• A closed or internal face is defined as a boundary component of a ribbon graph which never passes through any free segment of the additional HRs. We denote by $\mathcal{F}_{\text{int}}(\mathcal{G}_{\mathfrak{f}^0}(\mathcal{V}, \mathcal{E}))$ the set of closed faces of $\mathcal{G}_{\mathfrak{f}^0}(\mathcal{V}, \mathcal{E})$.

• A boundary component obtained by leaving an external point of some HR rejoining another external point is called an open or external face. We denote by $\mathcal{F}_{\text{ext}}(\mathcal{G}_{\mathfrak{f}^0}(\mathcal{V}, \mathcal{E}))$, the set of open faces of $\mathcal{G}_{\mathfrak{f}^0}(\mathcal{V}, \mathcal{E})$.

• The set of faces $\mathcal{F}(\mathcal{G}_{\mathfrak{f}^0}(\mathcal{V}, \mathcal{E}))$ of $\mathcal{G}_{\mathfrak{f}^0}(\mathcal{V}, \mathcal{E})$ is defined by

$$\mathcal{F}_{\text{int}}(\mathcal{G}_{\mathfrak{f}^0}(\mathcal{V}, \mathcal{E})) \cup \mathcal{F}_{\text{ext}}(\mathcal{G}_{\mathfrak{f}^0}(\mathcal{V}, \mathcal{E})).$$

If $\mathcal{F}_{\text{ext}}(\mathcal{G}_{\mathfrak{f}^0}(\mathcal{V}, \mathcal{E})) \neq \emptyset$, i.e. $\mathfrak{f}^0 \neq \emptyset$, then $\mathcal{G}_{\mathfrak{f}^0}(\mathcal{V}, \mathcal{E})$ is said to be open. Otherwise it is closed.

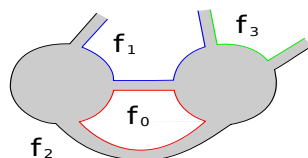An illustration for closed and open faces is given in Figure 3.

Figure 3:   A HERG $\mathcal{G}_{\mathfrak{f}^0}$ with $\mathcal{F}_{\text{int}}(\mathcal{G}_{\mathfrak{f}^0}) = \{f_0\}$, and $\mathcal{F}_{\text{ext}}(\mathcal{G}_{\mathfrak{f}^0}) = \{f_1, f_2, f_3\}$

The boundary $\partial \mathcal{G}_{\mathfrak{f}^0}$ of a ribbon graph $\mathcal{G}_{\mathfrak{f}^0}(\mathcal{V}, \mathcal{E})$ is a simple graph $\partial \mathcal{G}_{\mathfrak{f}^0}(\mathcal{V}_\partial, \mathcal{E}_\partial)$ such that $\mathcal{V}_\partial$ is one-to-one with $\mathfrak{f}^0$ and $\mathcal{E}_\partial$ is one-to-one with $\mathcal{F}_{\text{ext}}(\mathcal{G}_{\mathfrak{f}^0}(\mathcal{V}, \mathcal{E}))$. Then the boundary graph of a closed ribbon graph is empty. By construction, the boundary graph $\partial \mathcal{G}_{\mathfrak{f}^0}$, is obtained by inserting a vertex of valence or degree two at each HR, the edges of $\partial \mathcal{G}_{\mathfrak{f}^0}$ are nothing but the external faces of $\mathcal{G}_{\mathfrak{f}^0}$. An illustration is given in Figure 4. The operations of edge contraction and deletion for HERGs keep their meaning as in Definition 2.1.
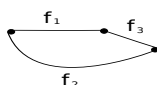

Figure 4:   The boundary graph associated to the HERG in Figure 3

Definition 2.3 (BR polynomial for HERGs)**.** Let $\mathcal{G}_{\mathfrak{f}^0}(\mathcal{V}, \mathcal{E})$ be a HERG. We define the ribbon graph polynomial of $\mathcal{G}_{\mathfrak{f}^0}$ to be

$$\mathcal{R}_{\mathcal{G}_{\mathfrak{f}^0}}(x, y, z, w, t) = \sum_{s \in S(\mathcal{G}_{\mathfrak{f}^0})} (x-1)^{\text{r}(\mathcal{G}_{\mathfrak{f}^0}) - \text{r}(s)} (y-1)^{n(s)} z^{k(s) - F_{\text{int}}(s) + n(s)} w^{C_\partial(s)} t^{f(s)}, \quad (3)$$

where $C_\partial(s) = |\mathcal{C}_\partial(s)|$ is the number of connected components of the boundary of s, $f(s)$, the number of half-edges and $F_{\text{int}}(s) = |\mathcal{F}_{\text{int}}(s)|$.

This definition gives a polynomial $\mathcal{R}$ (3) which is a generalization of the BR polynomial R (1) from ribbon graphs to HERGs. The multivariate Bollobás-Riordan polynomial for HERGs is

$$Z_{\mathcal{G}_{\mathfrak{f}^0}}(a, b, c, d, l) = \sum_{s \in S(\mathcal{G}_{\mathfrak{f}^0})} a^{k(s)} \Big(\prod_{e \in s} b_e \Big) c^{F_{\text{int}}(s)} d^{\partial(s)} l^{f(s)}, \tag{4}$$

where a, c and d are indeterminates, and $b := \{b_e / e \in \mathcal{E}\}$ is a set of indeterminates indexed by $\mathcal{E}$. If we set all the variables $b_e = b$ in (4), then using equation (3), we obtain

$$\mathcal{R}_{\mathcal{G}_{\mathfrak{f}^0}}(x, y, z, w, t) = (x-1)^{-k(\mathcal{G}_{\mathfrak{f}^0})} (yz)^{-v(\mathcal{G}_{\mathfrak{f}^0})} Z_{\mathcal{G}_{\mathfrak{f}^0}}((x-1)yz^2, yz, z^{-1}, w, t). \tag{5}$$

3   Expansion for the Bollobás-Riordan polynomial on half-edged ribbon graphs

The formation of the HERG $\widehat{\mathcal{G}}_{\mathfrak{f}^0}$ from its 2-decomposition $(\mathcal{G}_{\mathfrak{f}^0}, \{H_e\}_{e \in \mathcal{E}})$ is obtained by replacing each ribbon edge e by H_e. In fact from the 2-decomposition $(\mathcal{G}_{\mathfrak{f}^0}, \{H_e\}_{e \in \mathcal{E}})$ locally at $e = (u_e, w_e)$ of the template $\mathcal{G}_{\mathfrak{f}^0}$, the graph $\widehat{\mathcal{G}}_{\mathfrak{f}^0}$ is constructed by identifying the arcs m_e and n_e of the vertices u_e and w_e of H_e with the corresponding arcs m_e and n_e on the vertices u_e and w_e on the template $\mathcal{G}_{\mathfrak{f}^0} - e$. We use the same notations n_e and m_e according to the identification of the arcs. An illustration is given in Figure 5, where the arcs m_e and n_e are shown in red.
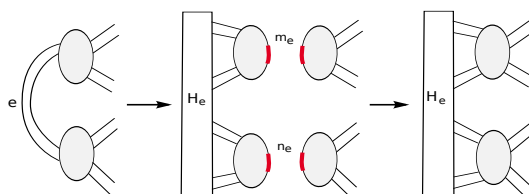

Figure 5:   Replace the edge e by H_e to obtain $\widehat{\mathcal{G}}_{\mathfrak{f}^0}$

3.1   The 2-decomposition of HERGs

We give in this subsection a natural extension of the 2-decomposition of graphs, from ribbon graphs to HERGs. Given a HERG $\mathcal{G}_{\mathfrak{f}^0} = (\mathcal{V}, \mathcal{E})$, we may wonder how to evaluate the number of its internal faces or connected components of the boundary graph using its 2-decomposition.

Definition 3.1. Let $\mathcal{G}_{\mathfrak{f}^0} = (\mathcal{V}, \mathcal{E})$ be a HERG and $\{A_e\}_{e \in \mathcal{E}}$ be a set of half-edged ribbon graphs each of which has a specific non-loop edge distinguished. For each $e \in \mathcal{E}$ take the 2-sum $\mathcal{G}_{\mathfrak{f}^0} \oplus_2 A_e$, along the edge e and the distinguished edge in A_e, to obtain the half-edged ribbon graph $\widehat{\mathcal{G}}_{\mathfrak{f}^0}$; $\mathfrak{f}^0 = f_0 \cup_e f^0(A_e)$ and $f^0(A_e)$ the set of half-ribbons of A_e. For each $e \in \mathcal{E}$ we define $H_e = A_e - \{e\}$. We will call the structure $(\mathcal{G}_{\mathfrak{f}^0}, \{H_e\}_{e \in \mathcal{E}})$ a 2-decomposition for $\widehat{\mathcal{G}}_{\mathfrak{f}^0}$.

We can construct the graphs A_e by performing the reverse operation. That is we look $\widehat{\mathcal{G}}_{\mathfrak{f}'^0}$ as a 2-decomposition $(\mathcal{G}_{\mathfrak{f}^0}, \{H_e\}_{e \in \mathcal{E}})$ where we identify two distinguished vertices u_e and w_e of each H_e with the corresponding end vertices of e in $\mathcal{G}_{\mathfrak{f}^0}$. From each H_e we can define a graph $A_e = H_e \cup \{e\}$.

The set of states $S(H_e)$ of each H_e is partitioned into two subsets: $S^1(H_e)$ consists of all states in $S(H_e)$ in which u_e and w_e lie in the same connected component, and $S^2(H_e)$ consists of all states in $S(H_e)$ in which u_e and w_e lie in different connected component. A state $\hat{s} \in \widehat{\mathcal{G}}_{\mathfrak{f}^0}$ is obtained by replacing the edges e in a state $s \in S(\mathcal{G}_{\mathfrak{f}^0})$ with elements of $S^1(H_e)$, and the edges f which are not in s by elements of $S^2(H_f)$.

Lemma 3.2. *If a state $\hat{s}$ of the embedded graph $\widehat{\mathcal{G}}_{\mathfrak{f}'^0}$ is decomposed into states $s \in S(\mathcal{G}_{\mathfrak{f}^0})$, $s_e \in S^1(H_e) \cup S^2(H_e)$, $e \in \mathcal{E}$ in the decomposition above, then*

$$k(\hat{s}) \;=\; \sum_{e \in \mathcal{E}} k(s_e) - |\{s_e \in S^1(H_e)\}| - 2|\{s_e \in S^2(H_e)\}| + k(s),$$

$$F_{\text{int}}(\hat{s}) + C_\partial(\hat{s}) \;=\; \sum_{e \in \mathcal{E}} F_{\text{int}}(s_e) - |\{s_e \in S^1(H_e)\}| - 2|\{s_e \in S^2(H_e)\}| + F_{\text{int}}(s)$$

$$+ \sum_{e \in \mathcal{E}} C_\partial(s_e) + C_\partial(s), \tag{6}$$

where $|\{s_e \in S^i(H_e)\}|$; $i = 1, 2$ is the number of states in $S^i(H_e)$, at fixed e.

Proof. The proof of this lemma will follow the one of Lemma 3 in [10] by considering the underling ribbon graphs $\tilde{\hat{s}}$ and $\tilde{s}_e$ associated respectively to the half edged ribbon graphs $\hat{s}$ and s_e and using the equalities

$$\partial(\tilde{\hat{s}}) = F_{\text{int}}(\hat{s}) + C_\partial(\hat{s}), \quad \partial(\tilde{s}_e) = F_{\text{int}}(s_e) + C_\partial(s_e).$$

$\square$

We are now interested in a separate formula relating $F_{\text{int}}(\hat{s})$, $F_{\text{int}}(s_e)$ and $F_{\text{int}}(s)$ and a formula relating $C_\partial(\hat{s})$, $C_\partial(s_e)$ and $C_\partial(s)$. This leads us to consider different cases. The case where a connected component of the boundary of s corresponds to a connected component of the boundary of $\hat{s}$ and the case where a closed face of s corresponds to a connected component of the boundary of $\hat{s}$. We take $H_e = A_e \vee \{e\}$ in the following subsection in order to overcome this issue.

3.2   The BR polynomial for HERGs embedding in a neighbourhood

In this subsection we assume that the graph $\widehat{\mathcal{G}}_{\mathfrak{f}^0} = (\mathcal{G}_{\mathfrak{f}^0}, \{H_e\}_{e \in \mathcal{E}})$ is embedded graph where $\mathcal{G}$ is embedded and each graph H_e is embedded in the neighborhood of e.

Let us remember the 2-decomposition in order to evaluate the BR polynomial on HERGs. Consider two HERGs $\mathcal{G}_{\mathfrak{f}^0}$ and $\mathcal{F}_{\mathfrak{f}'^0}$ with distinguished edges $e \in E(\mathcal{G}_{\mathfrak{f}^0})$ and $f \in E(\mathcal{F}_{\mathfrak{f}'^0})$. The 2-sum $\mathcal{G}_{\mathfrak{f}^0} \oplus_2 \mathcal{F}_{\mathfrak{f}'^0}$ is defined by identifying e with f and deleting the identified edge. We introduced here another way to perform this sum which will be generalized on tensor graphs. In the process of identification of e and f, assume that a vertex $u_e \in V(\mathcal{G}_{\mathfrak{f}^0})$ is identified with $u_f \in V(\mathcal{F}_{\mathfrak{f}'^0})$. We introduce an edge between u_e and u_f such that the end points of this edge coincide with the end points

of the identified edges. We contract this edge and obtain that the cyclic order around the new vertex will be $\{e_1, \cdots, e_n, f_1, \cdots, f_m\}$ if the cyclic order around u_e and u_f are $\{e, e_1, \cdots, e_n\}$ and $\{f, f_1, \cdots, f_m\}$ according to a choice of orientations on these ribbon graphs. An illustration is given in Figure 6.
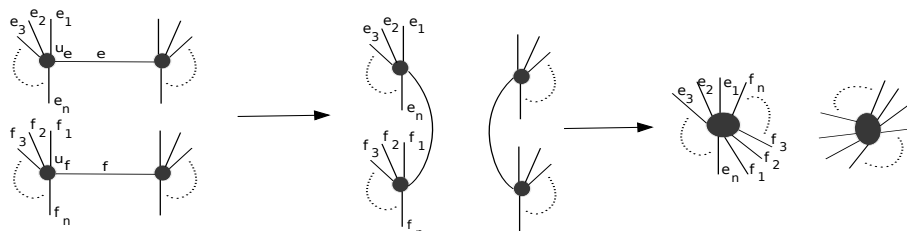

Figure 6:   The process of identification of e and f (on the left), through the introduction of an edge between u_e and u_f (midle) and the 2-sum obtained after contraction (on the right)

The definition 3.1, can be reformulated as follows:

Definition 3.3. Let $\mathcal{G}_{\mathfrak{f}^0} = (\mathcal{V}, \mathcal{E})$ be a HERG and $\{A_e\}_{e\in\mathcal{E}}$ be a set of half-edged ribbon graphs each of which has a specific non-loop edge distinguished. For each $e \in \mathcal{E}$ take the 2-sum $\mathcal{G}_{\mathfrak{f}^0} \oplus_2 A_e$, along the edge e and the distinguished edge in A_e as introduced above, to obtain the half-edge ribbon graph $\widehat{\mathcal{G}}_{\mathfrak{f}'^0}$; $\mathfrak{f}'^0 = f_0 \cup_e f^0(A_e)$ and $f^0(A_e)$ the set of half-ribbons of A_e. For each $e \in \mathcal{E}$ let us define $H_e = A_e \vee \{e\}$. We will call the structure $(\mathcal{G}_{\mathfrak{f}'^0}, \{H_e\}_{e\in\mathcal{E}})$ a 2-decomposition for $\widehat{\mathcal{G}}_{\mathfrak{f}'^0}$.

From Definition 3.3, each H_e has at least two half-edges attached to the points m_e and n_e. Definition 3.1 shows that $\widehat{\mathcal{G}}_{\mathfrak{f}'^0}$ is obtained by identifying the distinguished vertices u_e and w_e, for $e = (u_e, w_e)$, of each graph H_e with their correspondence in $\mathcal{G}_{\mathfrak{f}^0}$. Then we may not have some half-edges attached to m_e and n_e.

Let a_e and a'_e be the two endpoints of the arcs m_e and b_e and b'_e the endpoints of the arcs n_e. Some of these points may belong to the same internal face or connected component of the boundary graph. These points a_e, a'_e, b_e and b'_e induce points on the boundary of $\mathcal{G}_{\mathfrak{f}^0}$, $\widehat{\mathcal{G}}_{\mathfrak{f}'^0}$ and H_e, and then on the states $s \Subset \mathcal{G}_{\mathfrak{f}^0}$, $\hat{s} \Subset \widehat{\mathcal{G}}_{\mathfrak{f}'^0}$ and $s_e \Subset H_e$.

Consider one of such points, say, x in an internal face f of a state $s \in S(\mathcal{G})$. This point has a correspondent point x_e in each of the graphs s_e such that f will pass by e in $\hat{s}$. We can define a matrix ϵ which columns are indexed by the points x_e and rows indexed by the internal faces of s. The elements of this matrix are given by: $\epsilon(f, x_e) = 1$ if x_e corresponds to a point x in f and belongs to a connected component of the boundary graph in $s \cup e$ and $\epsilon(f, x_e) = 0$ otherwise. To the matrix ϵ we can associate a sub-matrix $\tilde{\epsilon}$ obtained by removing from ϵ, all the zero and collinear column vectors except one. It is clear that $\tilde{\epsilon}$ is a square matrix. We denote by $\{\tilde{x}_e\}_e$ the set of the remaining points indexing the columns of $\tilde{\epsilon}$ and τ the number of internal faces in s which become open in $\hat{s}$. The following result is straightforward.

Lemma 3.4. *Let us consider the 2-decomposition $\hat{s} = (s, \{s\}_e)$ where $\hat{s} \in S(\widehat{\mathcal{G}})$,*

$s \in S(\mathcal{G})$, $s_e \in S(H_e)$ and $\widehat{\mathcal{G}} = (\mathcal{G}, \{H\}_e)$ and ϵ as defined above. We have

$$\tau = \mathrm{rank}(\epsilon) = \mathrm{rank}(\tilde{\epsilon}).$$

We want to express τ as a sum over e in s. Consider another matrix σ whose rows are indexed by the points $\{x_e\}_e$ and columns indexed by the edges e. We say that $x_g \in e$ if and only if x_g has a correspondent point $x \in e$ and $x \in e = (m_e, n_e)$ means that x is one of the points a_e, a'_e, b_e and b'_e. Each matrix element is given by: $\sigma(x_g, e) = 1$ if $x_g \in e$ and x_g belongs to a connected component of the boundary graph in $s \cup e$ and $\sigma(x_e, e) = 0$ otherwise. We can deduce the matrix $\tilde{\sigma}$ indexed by the elements $\{\tilde{x}_e\}_e$ discussed above.
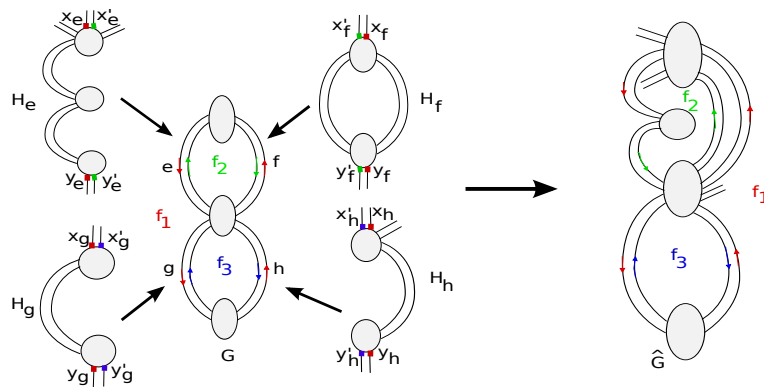

Figure 7:   The 2-decomposition of $\widehat{G} = (G, \{H_e, H_f, H_g, H_h\})$ with the meeting points colored according to the faces of G.

In Figure 7, we give the details of the 2-decomposition of $\hat{G} = (G, \{H_e, H_f, H_g, H_h\})$ in order to compute the matrices ϵ and σ. As discussed earlier, the set of the points is

$$\{x_e, x'_e, x_f, x'_f, x_g, x'_g, x_h, x'_h, y_e, y'_e, y_f, y'_f, y_g, y'_g, y_h, y'_h\},$$

the set of faces, $\{f_1, f_2, f_3\}$ and the edges set $\{e, f, g, h\}$. The columns vectors in ϵ indexed by x_e, x'_e, x_f, x'_f, x_g, x'_g, x_h and x'_h are respectively the same with the columns vectors indexed by y_e, y'_e, y_f, y'_f, y_g, y'_g, y_h and y'_h. Without lost of generality we can compute ϵ for the vectors indexed by x_e, x'_e, x_f, x'_f, x_g, x'_g, x_h, x'_h, say, in that order. Hence

$$\epsilon = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}.$$

We deduce the matrix $\tilde{\epsilon}$

$$\tilde{\epsilon} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{7}$$

From Figure 7 and equation (7), we have $\tau = 3 = \mathrm{rank}(\tilde{\epsilon})$.

The corresponding matrices σ and $\tilde{\sigma}$ are given by

$$\sigma = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

and

$$\tilde{\sigma} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Lemma 3.5. *The number τ introduced above is*

$$\tau = \sum_{e \in s} \sum_{f \in \mathcal{F}_{int}(s)} \sum_{\tilde{x}_g} \tilde{\epsilon}(f, \tilde{x}_g) \tilde{\sigma}(\tilde{x}_g, e).$$

Lemma 3.6. *Consider a state $\hat{s}$ of the embedded graph $\widehat{\mathcal{G}}_{\mathfrak{f}^0}$ decomposed into the states $s \in S(\mathcal{G}_{\mathfrak{f}^0})$ and $s_e \in S^1(H_e) \cup S^2(H_e)$, $e \in \mathcal{E}$ with the matrices $\tilde{\epsilon}$ and $\tilde{\sigma}$ as introduced above. Then*

$$C_\partial(\hat{s}) = \sum_{e \in \mathcal{E}} C_\partial(s_e) + C_\partial(s) - |\{s_e \in S^1(H_e)\}| - 2|\{s_e \in S^2(H_e)\}| +$$
$$\sum_{e \in s} \sum_{f \in \mathcal{F}_{int}(s)} \sum_{\tilde{x}_g} \tilde{\epsilon}(f, \tilde{x}_g) \tilde{\sigma}(\tilde{x}_g, e), \tag{8}$$

$$F_{int}(\hat{s}) = \sum_{e \in \mathcal{E}} F_{int}(s_e) + F_{int}(s) - \sum_{e \in s} \sum_{f \in \mathcal{F}_{int}(s)} \sum_{\tilde{x}_g} \tilde{\epsilon}(f, \tilde{x}_g) \tilde{\sigma}(\tilde{x}_g, e), \tag{9}$$

$$f(\hat{s}) = \sum_{e \in \mathcal{E}} f(s_e) + f(s) - 2|\{s_e \in S^1(H_e)\}| - 4|\{s_e \in S^2(H_e)\}|, \tag{10}$$

where $|\{s_e \in S^i(H_e)\}|$; $i = 1, 2$ is the number of states in $S^i(H_e)$, at fixed e.

Proof. From the proof of (8) we can deduce (9) by using the relation (6) in Lemma 3.2. Let $e = (m_e, n_e)$, a_e and a'_e be the end points of the arc m_e and b_e and b'_e those of n_e.

    - If a connected component of the boundary graph of the graph $s \in S(\mathcal{G}_{\mathfrak{f}^0})$ is such that it does not contain any of the points a_e, a'_e, b_e and b'_e, then there is a natural corresponding component in $\hat{s}$.

- If a connected component of the boundary graph of the graph of $s \in S(\mathcal{G}_{\mathfrak{f}^0})$ contains some of the points a_e, a'_e, b_e and b'_e, then there is a correspondence connected component of the boundary of the graph $\hat{s} \in S(\widehat{\mathcal{G}}_{\mathfrak{f}^0})$ containing the same set of points.

- Assume that an internal face of $s \in S(\mathcal{G}_{\mathfrak{f}^0})$ contains some of the points a_e, a'_e, b_e, b'_e. This face can correspond to a connected component of the boundary graph of the graph $\hat{s} \in S(\widehat{\mathcal{G}}_{\mathfrak{f}^0})$ containing the same set of points. The total number of the components for which some of the points a_e, a'_e, b_e and b'_e belong to connected component of the boundary graph in $\hat{s}$ is $\sum_{e \in s} \sum_{f \in \mathcal{F}_{int}(s)} \sum_{\tilde{x}_g} \tilde{\epsilon}(f, \tilde{x}_g) \tilde{\sigma}(\tilde{x}_g, e)$.

However, $\hat{s}$ has two kind of extra connected components: the unmarked components in the graphs s_e which do not contain any of the points a_e, a'_e, b_e and b'_e and the one containing some of these points belonging to connected components of the boundary graphs. The number of the unmarked components for each e for which $s_e \in S^1(H_e)$ or $s_e \in S^1(H_e)$ is $C_\partial(s_e) - 1$ and $C_\partial(s_e) - 2$ respectively. The number of connected components of the boundary graph of $\hat{s}$ is

$$
\begin{aligned}
C_\partial(\hat{s}) = \quad & C_\partial(s) + \sum_{s_e \in S^1(H_e)} \Big(C_\partial(s_e) - 1 \Big) + \sum_{s_e \in S^2(H_e)} \Big(C_\partial(s_e) - 2 \Big) + \\
& \sum_{e \in s} \sum_{f \in \mathcal{F}_{int}(s)} \sum_{\tilde{x}_g} \tilde{\epsilon}(f, \tilde{x}_g) \tilde{\sigma}(\tilde{x}_g, e).
\end{aligned}
$$

The proof of (10) is direct since by inserting $s_e \in S^1(H_e)$ in s, we keep all the half edges of s but two half edges of s_e are lost. Inserting $s_e \in S^2(H_e)$ in s we lose two half edges of s and two half-edges of s_e. This ends the proof. $\qquad\square$

In order to evaluate the connection between the BR polynomial of $\widehat{\mathcal{G}}_{\mathfrak{f}^0}$ and those of $\mathcal{G}_{\mathfrak{f}^0}$ and H_e let us come back to the relations in Lemma 3.6. It appears that the evaluation of the number of connected components of the boundary graph and internal faces of $\hat{s}$ in term of those in s_e and s becomes more complicated.

We consider the following state sums:

$$
\begin{aligned}
\eta_e^{(1)}(a,b,c,d,l) &:= \sum_{s_e \in S^1(H_e)} a^{k(s_e)-1} b^{e(s_e)} c^{F_{int}(s_e)} d^{C_\partial(s_e)-1} l^{f(s_e)-2}, \\
\eta_e^{(2)}(a,b,c,d,l) &:= \sum_{s_e \in S^2(H_e)} a^{k(s_e)-2} b^{e(s_e)} c^{F_{int}(s_e)} d^{C_\partial(s_e)-2} l^{f(s_e)-4}.
\end{aligned}
$$

Furthermore let $\mathfrak{F}$ be a map defined for $s \in S(\mathcal{G}_{f^0})$, by

$$
\mathfrak{F}\Big(s, \eta_e^{(1)}(a,b,c,d,l)\Big) = \sum_{s_e \in S^1(H_e)} a^{k(s_e)-1} b^{e(s_e)} c^{F_{int}(s_e) - \sum_{f \in \mathcal{F}_{int}(s)} \sum_{\tilde{x}_g} \tilde{\epsilon}(f, \tilde{x}_g) \tilde{\sigma}(\tilde{x}_g, e)} \times
$$
$$
d^{C_\partial(s_e) - 1 + \sum_{f \in \mathcal{F}_{int}(s)} \sum_{\tilde{x}_g} \tilde{\epsilon}(f, \tilde{x}_g) \tilde{\sigma}(\tilde{x}_g, e)} l^{f(s_e)-2}, \quad (11)
$$

where $\epsilon(.)$ and $\tilde{\epsilon}(.)$ are defined above.

We remark that if the graph s does not have any internal face i.e $\mathcal{F}_{int}(s) = \emptyset$ or if the product $\tilde{\epsilon}(.)\epsilon(.)$, is always vanishing then the relation (11) becomes

$$
\mathfrak{F}\Big(s, \eta_e^{(1)}(a,b,c,d,l)\Big) = \eta_e^{(1)}(a,b,c,d,l) \quad \forall s \in S(\mathcal{G}_{f_0}).
$$

As consequence, the relation (11) for $\eta_e^{(2)}(.)$ gives

$$\mathfrak{F}\left(s, \eta_e^{(2)}(a, b, c, d, l)\right) = \eta_e^{(2)}(a, b, c, d, l) \ \ \forall s \in S(\mathcal{G}_{f_0}).$$

Sometimes we will use the following notations:

$$\mathfrak{F}\left(s, \eta_e^{(1)}(a, b, c, d, l)\right) = \mathfrak{F}_e^s, \ \eta_e^{(1)}(a, b, c, d, l) = \eta_e^{(1)} \quad \text{and} \quad \eta_e^{(2)}(a, b, c, d, l) = \eta_e^{(2)}.$$

Lemma 3.7. *Let $(\mathcal{G}_{f^0}, \{H_e\}_{e \in \mathcal{E}})$ be a 2-decomposition of $\widehat{\mathcal{G}}_{f'^0}$ and $\mathfrak{F}_e^s$, $\eta_e^{(2)}$ two functions as introduced above. Then*

$$Z(\widehat{\mathcal{G}}_{f^0}; a, b, c, d, l) = \sum_{s \in S(\mathcal{G}_{f^0})} a^{k(s)} c^{F_{\text{int}}(s)} d^{C_\partial(s)} l^{f(s)} \times$$

$$\left(\prod_{e \in s} \mathfrak{F}_e^s\right)\left(\prod_{e \notin s} \eta_e^2\right).$$

Proof. We recall that any state $\hat{s}$ is decomposed into states $s \in S(\mathcal{G}_{f^0})$, $s_e \in S^1(H_e)$ and $t_h \in S^2(H_h)$. By Lemma 3.6, we have

$$a^{k(s)} c^{F_{\text{int}}(s)} d^{C_\partial(s)} l^{f(s)} \prod_{e \in s} a^{k(s_e)-1} b^{e(s_e)} c^{F_{\text{int}}(s_e) - \sum_{f \in \mathcal{F}_{int}(s)} \sum_{\tilde{x}_g} \tilde{\epsilon}(f, \tilde{x}_g) \tilde{\sigma}(\tilde{x}_g, e)} \times$$

$$d^{\partial(s_e)-1 + \sum_{f \in \mathcal{F}_{int}(s)} \sum_{\tilde{x}_g} \tilde{\epsilon}(f, \tilde{x}_g) \tilde{\sigma}(\tilde{x}_g, e)} l^{f(s_e)-2} \times \prod_{h \notin s} a^{k(t_h)-1} b^{e(t_h)} c^{F_{\text{int}}(t_h)} d^{\partial(t_h)-2} l^{f(t_h)-4}$$

$$= a^{k(s) + \sum(k(s_e)-1) + \sum(k(t_h)-2)} c^{\sum F_{\text{int}}(s_e) + \sum F_{\text{int}}(s_h) - \sum_{e \in s} \sum_{f \in \mathcal{F}_{int}(s)} \sum_{\tilde{x}_g} \tilde{\epsilon}(f, \tilde{x}_g) \tilde{\sigma}(\tilde{x}_g, e)} \times$$

$$d^{\sum(C_\partial(s_e)-1) + \sum(C_\partial(t_h)-2) + \sum_{e \in s} \sum_{f \in \mathcal{F}_{int}(s)} \sum_{\tilde{x}_g} \tilde{\epsilon}(f, \tilde{x}_g) \tilde{\sigma}(\tilde{x}_g, e)} l^{\sum(f(s_e)-2) + \sum(f(t_h)-4)} \times$$

$$b^{\sum e(s_e) + \sum e(t_h)}$$

$$= a^{k(\hat{s})} b^{e(\hat{s})} c^{F_{\text{int}}(\hat{s})} d^{C_\partial(\hat{s})} l^{f(\hat{s})}.$$

$\square$

Example 3.8. Let us compute $Z(.)$ for the 2-decomposition $(G, \{H_f, H_g\})$ of the graph $\hat{G}$ shown in Figure 8.
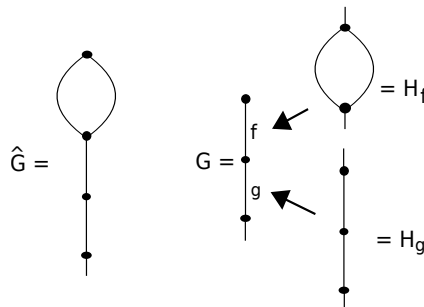

Figure 8:   A HERG $\widehat{G}$ on the left and its 2-decomposition $(G, \{H_f, H_g\})$ on the right

In this example, $\mathcal{F}_{\text{int}}(G) = \emptyset$ and

$$Z(\widehat{G}; a, b, c, d, l) = adl\eta_f^1 \eta_g^1 + a^2 d^2 l^3 \eta_f^1 \eta_g^2 + a^2 d^2 l^3 \eta_f^2 \eta_g^1 + a^3 d^3 l^5 \eta_f^2 \eta_g^2,$$

$$\eta_f^1(a,b,c,d,l) = b^2c + 2bl^2, \quad \eta_f^2(a,b,c,d,l) = l^2$$
$$\eta_g^1(a,b,c,d,l) = b^2, \quad \eta_g^2(a,b,c,d) = 2b + adl^2.$$

Hence

$$\begin{aligned}
Z(\widehat{G}; a,b,c,d,l) &= adl(b^2c + 2bl^2)b^2 + a^2d^2l^3(b^2c + 2bl^2)(2b + adl^2) + a^2b^2d^2l^2 \\
&\quad + a^3d^3l^5(2b + adl^2) \\
&= ab^4cdl + 2ab^3dl^3 + 2a^2b^3cd^2l^3 + a^3b^2cd^3l^5 + 4a^2b^2d^2l^5 \\
&\quad + 2a^3bd^3l^7 + a^2b^2d^2l^2 + 2a^3bd^3l^5 + a^4d^4l^7.
\end{aligned}$$

Lemma 3.9. *Let* $(\mathcal{G}_{\mathfrak{f}^0}, \{H_e\}_{e \in \mathcal{E}})$ *be a two decomposition of* $\widehat{\mathcal{G}}_{\mathfrak{f}^0}$. *We have*

$$Z(\widehat{\mathcal{G}}_{\mathfrak{f}^0}; a,b,c,d,l) = \Big(\prod_{e \in \mathcal{E}} \big(\eta_e^{(2)} \big) \Big) Z(\mathcal{G}_{\mathfrak{f}^0}; a, \{\mathfrak{F}_e^s / \eta_e^{(2)}\}_{e \in \mathcal{E}}, c, d, l). \tag{12}$$

Consider a graph A_e as introduced above with $H_e = A_e \vee e$ and $\widehat{\mathcal{G}}_{\mathfrak{f}^0} = (\mathcal{G}_{\mathfrak{f}^0}, \{H_e\})$. Let us study the effect of the insertion of the edge e in a state s of H_e. If $s \in S^2(H_e)$, then the insertion of e in s decreases the number of connected components and connected components of the boundary graph of s by one and the number of half edges by two. In other words, if the contribution of s is the term $a^k b^e c^{F_{\text{int}}} d^{C_\partial} l^f$ in the expression of $Z(.)$, then the state $s \cup e$ contributes the term $a^{k-1}(b^e x_e) c^{F_{\text{int}}} d^{C_\partial - 1} l^{f-2}$. Assume that $s \in S^1(H_e)$. Three possibilities occur. In all these possibilities, inserting e in $s \in S^1(H_e)$ will decrease by two the number of half edges.

• Inserting e in $s \in S^1(H_e)$ increases the number of internal faces by two and decreases by one the number of connected components of the boundary graph. If s contributes the term $a^k b^e c^{F_{\text{int}}} d^{C_\partial} l^f$ to the HERGs BR polynomial then the state $s \cup e$ obtained by inserting e contributes $a^k(b^e x_e) c^{F_{\text{int}}+2} d^{C_\partial - 1} l^{f-2}$.

• Inserting e in $s \in S^1(H_e)$ increases the number of internal faces by one such that if s contributes the term $a^k b^e c^{F_{\text{int}}} d^{C_\partial} l^f$ to the HERGs BR polynomial then the state $s \cup e$ obtained by inserting e contributes $a^k(b^e x_e) c^{F_{\text{int}}+1} d^{C_\partial} l^{f-2}$.

• Inserting e in $s \in S^1(H_e)$ increases the number of connected components of the boundary graph by one such that if s contributes the term $a^k b^e c^{F_{\text{int}}} d^{C_\partial} l^f$ to the HERGs BR polynomial then the state $s \cup e$ obtained by inserting e contributes $a^k(b^e x_e) c^{F_{\text{int}}} d^{C_\partial + 1} l^{f-2}$.

We can summarized all these possibilities by saying that if $s \in S^1(H_e)$ and s contributes the term $a^k b^e c^{F_{\text{int}}} d^{C_\partial} l^f$ to the HERGs BR polynomial then the state $s \cup e$ obtained by inserting e contributes $a^k(b^e x_e) c^{F_{\text{int}}+\theta(s)} d^{C_\partial + 1 - \theta(s)} l^{f-2}$; $\theta(s) \in \{0, 1, 2\}$.

Based on this we can write $\mathfrak{F}\big(s, \eta_e^{(1)} \big)$ as

$$\mathfrak{F}\big(s, \eta_e^{(1)} \big) = \mathfrak{F}^0\big(s, \eta_e^{(1)} \big) + \mathfrak{F}^1\big(s, \eta_e^{(1)} \big) + \mathfrak{F}^2\big(s, \eta_e^{(1)} \big), \tag{13}$$

with

$$\mathfrak{F}^0\big(s, \eta_e^{(1)} \big) = \sum_{s \in S^1(H_e) | \theta(s) = 0} a^{k(s)-1} b^{e(s)} c^{F_{\text{int}}(s) - \sum_{f \in \mathcal{F}_{int}(s)} \sum_{\tilde{x}_g} \tilde{\epsilon}(f, \tilde{x}_g) \tilde{\sigma}(\tilde{x}_g, e)} \times$$

$$d^{C_\partial(s)-1+\sum_{f\in\mathcal{F}_{int}(s)}\sum_{\tilde{x}_g}\tilde{\epsilon}(f,\tilde{x}_g)\tilde{\sigma}(\tilde{x}_g,e)}l^{f(s)-2},$$

$$\mathfrak{F}^1\left(s,\eta_e^{(1)}\right) = \sum_{s\in S^1(H_e)|\theta(s)=1} a^{k(s)-1}b^{e(s)}c^{F_{\text{int}}(s)-\sum_{f\in\mathcal{F}_{int}(s)}\sum_{\tilde{x}_g}\tilde{\epsilon}(f,\tilde{x}_g)\tilde{\sigma}(\tilde{x}_g,e)}\times$$
$$d^{C_\partial(s)-1+\sum_{f\in\mathcal{F}_{int}(s)}\sum_{\tilde{x}_g}\tilde{\epsilon}(f,\tilde{x}_g)\tilde{\sigma}(\tilde{x}_g,e)}l^{f(s)-2},$$

$$\mathfrak{F}^2\left(s,\eta_e^{(1)}\right) = \sum_{s\in S^1(H_e)|\theta(s)=2} a^{k(s)-1}b^{e(s)}c^{F_{\text{int}}(s)-\sum_{f\in\mathcal{F}_{int}(s)}\sum_{\tilde{x}_g}\tilde{\epsilon}(f,\tilde{x}_g)\tilde{\sigma}(\tilde{x}_g,e)}\times$$
$$d^{C_\partial(s)-1+\sum_{f\in\mathcal{F}_{int}(s)}\sum_{\tilde{x}_g}\tilde{\epsilon}(f,\tilde{x}_g)\tilde{\sigma}(\tilde{x}_g,e)}l^{f(s)-2}.$$

The map $\mathfrak{F}$ applied on $Z_{A_e}(a,b,c,d,l)$ and $s\in S(\mathcal{G}_{f^0})$, gives

$$\mathfrak{F}\left(s,Z_{A_e}(a,b,c,d,l)\right) = \sum_{s_e\in S(A_e)} a^{k(s_e)}b^{e(s_e)}c^{F_{\text{int}}(s_e)-\sum_{f\in\mathcal{F}_{int}(s)}\sum_{\tilde{x}_g}\tilde{\epsilon}(f,\tilde{x}_g)\tilde{\sigma}(\tilde{x}_g,e)}\times$$
$$d^{C_\partial(s_e)+\sum_{f\in\mathcal{F}_{int}(s)}\sum_{\tilde{x}_g}\tilde{\epsilon}(f,\tilde{x}_g)\tilde{\sigma}(\tilde{x}_g,e)}l^{f(s_e)}.$$

Then we can write the following decomposition

$$\mathfrak{F}\left(s,Z_{A_e}(a,b,c,d,l)\right) = \sum_{s\in S^1(H_e)} a^{k(s_e)}b^{e(s_e)}\times$$
$$c^{F_{\text{int}}(s_e)-\sum_{f\in\mathcal{F}_{int}(s)}\sum_{\tilde{x}_g}\tilde{\epsilon}(f,\tilde{x}_g)\tilde{\sigma}(\tilde{x}_g,e)}\times$$
$$d^{C_\partial(s_e)+\sum_{f\in\mathcal{F}_{int}(s)}\sum_{\tilde{x}_g}\tilde{\epsilon}(f,\tilde{x}_g)\tilde{\sigma}(\tilde{x}_g,e)}l^{f(s_e)}$$
$$+\ x_e \sum_{s\in S^1(H_e)|\theta(s_e)=0,1,2} a^{k(s_e)}b^{e(s_e)}\times$$
$$c^{F_{\text{int}}(s_e)-\sum_{f\in\mathcal{F}_{int}(s)}\sum_{\tilde{x}_g}\tilde{\epsilon}(f,\tilde{x}_g)\tilde{\sigma}(\tilde{x}_g,e)}\times$$
$$d^{C_\partial(s_e)+\sum_{f\in\mathcal{F}_{int}(s)}\sum_{\tilde{x}_g}\tilde{\epsilon}(f,\tilde{x}_g)\tilde{\sigma}(\tilde{x}_g,e)+1-\theta(s_e)}l^{f(s_e)}$$
$$+\ (1+a^{-1}x_ed^{-1}l^{-2}) \sum_{s\in S^2(H_e)} a^{k(s_e)}b^{e(s_e)}\times$$
$$c^{F_{\text{int}}(s_e)-\sum_{f\in\mathcal{F}_{int}(s)}\sum_{\tilde{x}_g}\tilde{\epsilon}(f,\tilde{x}_g)\tilde{\sigma}(\tilde{x}_g,e)}\times$$
$$d^{C_\partial(s_e)+\sum_{f\in\mathcal{F}_{int}(s)}\sum_{\tilde{x}_g}\tilde{\epsilon}(f,\tilde{x}_g)\tilde{\sigma}(\tilde{x}_g,e)}l^{f(s_e)},$$

or

$$\begin{aligned}
\mathfrak{F}\left(s,Z_{A_e}(a,b,c,d,l)\right) &= adl^2\mathfrak{F}\left(s,\eta_e^{(1)}\right)+x_el^2\Big(ad^2\mathfrak{F}^0\left(s,\eta_e^{(1)}\right)+\\
&\quad acd\mathfrak{F}^1\left(s,\eta_e^{(1)}\right)+\mathfrak{F}^2\left(s,\eta_e^{(1)}\right)\Big)+\\
&\quad (1+a^{-1}x_ed^{-1}l^{-2})a^2d^2l^4\eta_e^{(2)},\\
&= adl^2\Big(\mathfrak{F}\left(s,\eta_e^{(1)}\right)+adl^2\eta_e^{(2)}\Big)+ax_el^2\Big(d^2\mathfrak{F}^0\left(s,\eta_e^{(1)}\right)\\
&\quad +cd\mathfrak{F}^1\left(s,\eta_e^{(1)}\right)+\mathfrak{F}^2\left(s,\eta_e^{(1)}\right)+d\eta_e^{(2)}\Big). \quad (14)
\end{aligned}$$

Furthermore

$$Z_{A_e}(a,b,c,d,l) = Z_{H_e}(a,b,c,d,l)+x_eZ_{A_e/e}(a,b,c,d,l).$$

There is a one-to-one correspondence between each state $s' \in S(A_e)$ such that $e \in s'$, and a state $\bar{s'} := s'/e \in S(A_e/e)$. Let us consider the map

$$\mathfrak{F}'\Big(s, Z_{A_e/e}(a, b, c, d, l)\Big) = \sum_{s'/e \in S(A_e/e)} a^{k(s'/e)} b^{e(s'/e)} \times$$
$$c^{F_{\mathrm{int}}(s'/e) - \sum_{f \in \mathcal{F}_{int}(s')} \sum_{\tilde{x}_g} \tilde{\epsilon}(f, \tilde{x}_g) \tilde{\sigma}(\tilde{x}_g, e)} d^{C_\partial(s'/e) + \sum_{f \in \mathcal{F}_{int}(s')} \sum_{\tilde{x}_g} \tilde{\epsilon}(f, \tilde{x}_g) \tilde{\sigma}(\tilde{x}_g, e)} l^{f(s'/e)}, \quad (15)$$

where $\tilde{\epsilon}$ and $\tilde{\sigma}$ are defined for s'.

The following lemma is direct

Lemma 3.10. *Given a state $s \in S(\mathcal{G}_{f^0})$, we have*

$$\mathfrak{F}(s, Z_{A_e}(a, b, c, d, l)) = \mathfrak{F}(s, Z_{H_e}(a, b, c, d, l)) + x_e \mathfrak{F}'(s, Z_{A_e/e}(a, b, c, d, l)). \quad (16)$$

where $\mathfrak{F}'(s, Z_{A_e/e}(a, b, c, d, l))$ is defined as in (15).

An identification of (14) and (16) gives

$$adl^2\Big(\mathfrak{F}\big(s, \eta_e^{(1)}\big) + adl^2 \eta_e^{(2)}\Big) = \mathfrak{F}(s, Z_{H_e}(a, b, c, d, l))$$
$$al^2\Big(d^2 \mathfrak{F}^0\big(s, \eta_e^{(1)}\big) + cd\mathfrak{F}^1\big(s, \eta_e^{(1)}\big) + \mathfrak{F}^2\big(s, \eta_e^{(1)}\big) + d\eta_e^{(2)}\Big) =$$
$$\mathfrak{F}'(s, Z_{A_e/e}(a, b, c, d, l)). \quad (17)$$

A substitution of (17) in (12) together with (13) gives

Theorem 3.11. *Let $\widehat{\mathcal{G}}_{f^0}$ be a half edged embedded graph with 2-decomposition $(\mathcal{G}_{f^0}, \{H_e\}_{e \in \mathcal{E}})$, such that each graph H_e is embedded in a neighbourhood of the edge e of the embedded graph $\mathcal{G}_{f^0}$. Let A_e be a HERG as introduced above. Then*

$$Z_{\widehat{\mathcal{G}}_{f^0}}(a, b, c, d, l) = (adl^2)^{-e(\mathcal{G}_{f^0})} \Big(\prod_{e \in \mathcal{E}} g_e\Big) Z_{\mathcal{G}_{f^0}}(a, \{f_e/g_e\}, c, d, l),$$

where f_e and g_e are the solutions to

$$adl^2 g_e + f_e = \mathfrak{F}(s, Z_{H_e}(a, b, c, d, l))$$
$$adl^2(d - c)\mathfrak{F}^0(s, \eta_e^1) + acl^2(c - d)\mathfrak{F}^2(s, \eta_e^1) + cf_e + g_e =$$
$$\mathfrak{F}'(s, Z_{A_e/e}(a, b, c, d, l)).$$

This leads to

Corollary 3.12. *Let $\mathcal{G}_{f^0}$ be a HERG, A be a planar HERG [2] and $H = A \vee e$. Then*

$$\mathcal{R}_{\mathcal{G}_{f^0} \otimes A}(x, y, z, w, t) = h^{n(\mathcal{G}_{f^0})} h'^{r(\mathcal{G}_{f^0})} \mathcal{R}_{\mathcal{G}_{f^0}}\Big(\mathcal{G}_{f^0}; \frac{\mathcal{R}_H(x, y, z, w, t)}{h'}, \frac{yh'}{h}, z, w, t\Big),$$

where h and h' are the unique solution to

$$zwt^2\big(zwt^2(x - 1)h + h'\big) = \mathfrak{F}(s, \mathcal{R}_H(x, y, z, w, t)),$$
$$(x - 1)^{-k(H)+1} y^{-v(H)+2} z^{-v(H)+3} t^2 (w - z^{-1}) \times$$
$$\big(w\mathfrak{F}^0(s, \eta_e^1) - z^{-1}\mathfrak{F}^2(s, \eta_e^1)\big) + zwt^2(yh' + h) = \mathfrak{F}'(s, \mathcal{R}_{A/e}(x, y, z, w, t)).$$

---

[2] Planar here means that the underlying ribbon graph is planar.

Proof. From (5), we have

$$
\begin{aligned}
\mathcal{R}_{\mathcal{G}_{\mathfrak{f}^0} \otimes A}(x, y, z, w, t) &= (x-1)^{-k(\mathcal{G}_{\mathfrak{f}^0} \otimes A)} (yz)^{-v(\mathcal{G}_{\mathfrak{f}^0} \otimes A)} \times \\
& \qquad Z_{\mathcal{G}_{\mathfrak{f}^0} \otimes A}((x-1)yz^2, yz, z^{-1}, w, t).
\end{aligned}
$$

Applying Theorem 3.11, we obtain

$$
\begin{aligned}
\mathcal{R}_{\mathcal{G}_{\mathfrak{f}^0} \otimes A}(x, y, z, w, t) &= (x-1)^{-k(\mathcal{G}_{\mathfrak{f}^0} \otimes A)} (yz)^{-v(\mathcal{G}_{\mathfrak{f}^0} \otimes A)} g^{e(\mathcal{G}_{\mathfrak{f}^0})} \times \\
& \qquad Z_{\mathcal{G}_{\mathfrak{f}^0}}\Big((x-1)yz^2, \frac{f}{g}, z^{-1}, w, t\Big), \tag{18}
\end{aligned}
$$

where f and g are solutions to

$$
\begin{aligned}
(x-1)yz^2wt^2\big(f + (x-1)yz^2wt^2g\big) &= \mathfrak{F}(s, Z_H((x-1)yz^2, yz, z^{-1}, w, t)), \\
(x-1)yz^2t^2(w - z^{-1})\big(w\mathfrak{F}^0(s, \eta_e^1) - z^{-1}\mathfrak{F}^2(s, \eta_e^1)\big) + (x-1)yz^2wt^2\big(z^{-1}f + g\big) &= \\
\mathfrak{F}'(s, Z_{A/e}((x-1)yz^2, yz, z^{-1}, w, t)). \tag{19}
\end{aligned}
$$

Applying (5) in (18), we obtain

$$
\begin{aligned}
\mathcal{R}_{\mathcal{G}_{\mathfrak{f}^0} \otimes A}(x, y, z, w, t) &= (x-1)^{-k(\mathcal{G}_{\mathfrak{f}^0} \otimes A)} (yz)^{-v(\mathcal{G}_{\mathfrak{f}^0} \otimes A)} g^{e(\mathcal{G}_{\mathfrak{f}^0})} \Big(\frac{(x-1)yzg}{f}\Big)^{k(\mathcal{G}_{\mathfrak{f}^0})} \times \\
& \Big(\frac{f}{g}\Big)^{v(\mathcal{G}_{\mathfrak{f}^0})} \mathcal{R}_{\mathcal{G}_{\mathfrak{f}^0}}\Big(\frac{(x-1)yzg + f}{f}, \frac{f}{zg}, z, w, t\Big).
\end{aligned}
$$

Now we set

$$
h := (x-1)^{-k(H)+1}(yz)^{-v(H)+2}g, \quad h' := (x-1)^{-k(H)+1}(yz)^{-v(H)+1}f.
$$

Using the relations

$$
v(\mathcal{G}_{\mathfrak{f}^0} \otimes A) = (v(H) - 2)e(\mathcal{G}_{\mathfrak{f}^0}) + v(\mathcal{G}_{\mathfrak{f}^0}) \text{ and } k(\mathcal{G}_{\mathfrak{f}^0} \otimes A) = (k(H) - 1)e(\mathcal{G}_{\mathfrak{f}^0}) + k(\mathcal{G}_{\mathfrak{f}^0}),
$$

we have

$$
\mathcal{R}_{\mathcal{G}_{\mathfrak{f}^0} \otimes A}(x, y, z, w, t) = h^{n(\mathcal{G}_{\mathfrak{f}^0})} h'^{r(\mathcal{G}_{\mathfrak{f}^0})} \mathcal{R}_{\mathcal{G}_{\mathfrak{f}^0}}\Big(\frac{((x-1)h + h'}{h'}, \frac{yh'}{h}, z, w, t\Big).
$$

Again using (5) and the identities $k(H) = k(A/e)$ and $v(A/e) = v(H) - 1$, the system (19) becomes

$$
\begin{aligned}
zwt^2\big(zwt^2(x-1)h + h'\big) &= \mathfrak{F}(s, \mathcal{R}_H(x, y, z, w, t)), \\
(x-1)^{-k(H)+1}y^{-v(H)+2}z^{-v(H)+3}t^2(w - z^{-1}) \times & \\
\big(w\mathfrak{F}^0(s, \eta_e^1) - z^{-1}\mathfrak{F}^2(s, \eta_e^1)\big) + zwt^2(yh' + h) &= \mathfrak{F}'(s, \mathcal{R}_{A/e}(x, y, z, w, t)).
\end{aligned}
$$

$\square$

The following corollary gives a way to construct many pairs of distinct HERGs with the same BR polynomial.

Corollary 3.13. *Let $\mathcal{G}_{\mathfrak{f}^0} \otimes H$ and $\mathcal{G}'_{\mathfrak{f}^0} \otimes H$ be two embedded HERGs with the property that each copy of H is embedded in the neighborhood of an edge. Then if, $\mathcal{R}_{\mathcal{G}_{\mathfrak{f}^0}}(x, y, z, w, t) = \mathcal{R}_{\mathcal{G}'_{\mathfrak{f}^0}}(x, y, z, w, t)$, $\mathcal{R}_{\mathcal{G}_{\mathfrak{f}^0} \otimes H}(x, y, z, w, t) = \mathcal{R}_{\mathcal{G}'_{\mathfrak{f}^0} \otimes H}(x, y, z, w, t)$.*

Proof. The proof of this corollary is a direct consequence of Corollary 3.12. Let $z = 1$, $w = 1$ and $t = 1$. Then $T(\mathcal{G}_{\mathfrak{f}^0}) = T(\mathcal{G}'_{\mathfrak{f}^0})$ where T denotes the Tutte polynomial obtained by taking the summation over the spanning cutting subgraphs. Furthermore the rank and nullity can be recovered from this polynomial and $n(\mathcal{G}_{\mathfrak{f}^0}) = n(\mathcal{G}'_{\mathfrak{f}^0})$, $\mathrm{r}(\mathcal{G}_{\mathfrak{f}^0}) = \mathrm{r}(\mathcal{G}'_{\mathfrak{f}^0})$. □

Coming back to the definition of $\tilde{\sigma}(.)$ and $\tilde{\epsilon}(.)$, from now, in order to simplify our results we assume that the template has no internal face or the product $\tilde{\sigma}(.)\tilde{\epsilon}(.)$ is always equal to zero.

3.3 The BR polynomial for HERGs: the general case

Consider the construction of $\widehat{\mathcal{G}}_{\mathfrak{f}^0}$ from the 2-decomposition $(\mathcal{G}_{\mathfrak{f}^0}, \{H_e\}_{e \in \mathcal{E}})$ at an edge $e = (u_e, w_e)$ of the template $\mathcal{G}_{\mathfrak{f}^0}$. As discussed earlier, the graph $\widehat{\mathcal{G}}_{\mathfrak{f}^0}$ is obtained by the identification of the arcs m_e and n_e of the vertices u_e and w_e in H_e with their correspondence in $\mathcal{G}_{\mathfrak{f}^0}$. It is important to compare again in this subsection the two definitions of the 2-decomposition and discuss a partition of the set $S(H_e)$ in each case.

From Definition 3.1, the graph H_e is obtained from A_e by deleting the edge e, i.e $A_e = H_e - e$. The partition of $S(H_e)$ will be

• $\ddot{S}^2(H_e)$: the set of states of H_e in which m_e and n_e lie in different connected components and different internal faces or connected components of the boundary graph. We may need to split the set $\ddot{S}^2(H_e)$, in two subsets. The first containing the set of states of H_e in which m_e and n_e lie in different connected components and different internal faces. The second containing the set of states of H_e in which m_e and n_e lie in different connected components and different connected components of the boundary graph.

• $\bar{S}^1(H_e)$: the set of states of H_e in which m_e and n_e lie in the same connected component and the same internal face or the same connected component of the boundary graph. We may split in the same way as above the set $\bar{S}^1(H_e)$ in two subsets.

• $\ddot{S}^1(H_e)$: the set of states of H_e in which m_e and n_e lie in the same connected component and different connected components of the boundary graph or internal faces. In this case three different possibilities may occur. The points m_e and n_e lie in two different internal faces or two different connected components of the boundary graph. Else one of the points m_e and n_e lies in an internal face and the second in a connected component of the boundary graph.

Finally, Definition 3.1 may give use a partition of $S(H_e)$ in seven different subsets. Now from Definition 3.3, the partition of $S(H_e)$ will be

• $\ddot{S}^2(H_e)$: the set of states of H_e in which m_e and n_e lie in different connected components and different connected components of the boundary graph.

• $\bar{S}^1(H_e)$: the set of states of H_e in which m_e and n_e lie in the same connected component and the same connected components of the boundary graph.

• $\ddot{S}^1(H_e)$: the set of states of H_e in which m_e and n_e lie in the same connected component and different connected components of the boundary graph.

Then we obtain $S(H_e) = \ddot{S}^2(H_e) \cup \bar{S}^1(H_e) \cup \ddot{S}^1(H_e)$. We adopt Definition 3.3 and we have

Lemma 3.14. *The sets $\ddot{S}^2(H_e)$, $\bar{S}^1(H_e)$ and $\ddot{S}^1(H_e)$ partition the set $S(H_e)$, $e \in \mathcal{E}$. Moreover every state of $\widehat{\mathcal{G}}_{\mathfrak{f}'0}$ can be uniquely obtained by the replacement of an edge in a state of $\mathcal{G}_{\mathfrak{f}0}$ by an element of $\bar{S}^1(H_e)$ and a replacement of a non-edge in that state of $\mathcal{G}_{\mathfrak{f}0}$ in an element of $\bar{S}^1(H_e) \cup \ddot{S}^1(H_e)$.*
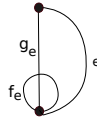
Figure 9:   A particular graph in the construction of the set of states.

If we want to construct the set of states of $S(\widehat{\mathcal{G}}_{\mathfrak{f}'0})$ by using the states of $\mathcal{G}_{\mathfrak{f}0}$, we run into a problem. We then introduce a class of states $\widetilde{\mathcal{G}}_{\mathfrak{f}0} = \mathcal{G}_{\mathfrak{f}0} \otimes T$ where T is given in Figure 9. We say that two states of $\widetilde{\mathcal{G}}_{\mathfrak{f}0}$ are equivalent if for some choice of e, one state contains neither of the edges g_e or f_e, the other state contains the edge f_e but not g_e, and the remaining edges contained in the two states are the same. This gives an equivalence relation $\sim$ and we say that two states are equivalent if they contain both of f_e and g_e or they contain both g_e but not f_e or one contains neither g_e or f_e and the second does not contain g_e but may or may not contain f_e.

Consider the set $S(\widetilde{\mathcal{G}}_{\mathfrak{f}0})/\sim$ and $[s] \in S(\widetilde{\mathcal{G}}_{\mathfrak{f}0})/\sim$ a class with s one of its representatives. A state of $\widehat{\mathcal{G}}_{\mathfrak{f}'0}$ is obtained by the following

• if the state s contains both of f_e and g_e, we remove both of them and replace in a state from $\ddot{S}^1(H_e)$.

• if the state s contains g_e but not f_e, we remove g_e and replace in a state from $\bar{S}^1(H_e)$.

• if the state s contains f_e but not g_e, we remove f_e and replace in a state from $\ddot{S}^2(H_e)$, or if s contains neither of the edges g_e or f_e, then we replace in a state from $\ddot{S}^2(H_e)$.

Lemma 3.15. *The sets $\ddot{S}^2(H_e)$, $\bar{S}^1(H_e)$ and $\ddot{S}^1(H_e)$ partition the set $S(H_e)$, $e \in \mathcal{E}$. Moreover every state of $\widehat{\mathcal{G}}_{\mathfrak{f}'0}$ can be uniquely obtained by replacing classes in $S(\widetilde{\mathcal{G}}_{\mathfrak{f}0})/\sim$ by an element of $\ddot{S}^2(H_e)$, $\bar{S}^1(H_e)$ and $\ddot{S}^1(H_e)$ in the manner described above.*

Let us consider the following sums:

$$\Phi_{\widetilde{\mathcal{G}}}(a, \{g_e, f_e\}_{e \in \mathcal{E}}, c, d) := \sum_{[s] \in S(\widetilde{\mathcal{G}}_{\mathfrak{f}0})/\sim} a^{k(s)} \big(\prod_{e \in s} x_e \big) c^{F_{\text{int}}(s)} d^{C_\partial(s)},$$

$$\ddot{\eta}_e^{(1)}(a, b, c, d) := \sum_{s \in \ddot{S}^1(H_e)} a^{k(s)-1} b^{e(s)} c^{F_{\text{int}}(s)} d^{C_\partial(s)-2},$$

$$\ddot{\eta}_e^{(2)}(a, b, c, d) := \sum_{s \in \ddot{S}^2(H_e)} a^{k(s)-2} b^{e(s)} c^{F_{\text{int}}(s)} d^{C_\partial(s)-2},$$

$$\bar{\eta}_e^{(1)}(a,b,c,d) := \sum_{s \in \bar{S}^1(H_e)} a^{k(s)-1} b^{e(s)} c^{F_{\text{int}}(s)} d^{C_{\partial}(s)-1},$$

where the product in $\Phi_{\widetilde{\mathcal{G}}_{\mathfrak{f}^0}}(.)$ is over a representative s of $[s]$. For simplicity we suppose that s has the fewest edges in its equivalence class and x_e stand for (f_e, g_e).

Lemma 3.16. *Suppose that a state $\hat{s}$ of $\widehat{\mathcal{G}}_{\mathfrak{f}^0}$ is obtained by the replacement from the states $[s] \in S(\widetilde{\mathcal{G}}_{\mathfrak{f}^0})$, $s_e \in \ddot{S}^1(H_e)$, $t_e \in \ddot{S}^1(H_e)$, and $u_e \in \bar{S}^1(H_e)$ using the decomposition above. Then*

$$k(\hat{s}) = \sum (k(s_e) - 1) + \sum (k(t_e) - 1) \sum (k(u_e) - 1) + k(s),$$
$$C_{\partial}(\hat{s}) = \sum (C_{\partial}(s_e) - 2) + \sum (C_{\partial}(t_e) - 2) + \sum (C_{\partial}(u_e) - 1) + C_{\partial}(s),$$
$$F_{\text{int}}(\hat{s}) = \sum F_{\text{int}}(s_e) + \sum F_{\text{int}}(t_e) + \sum F_{\text{int}}(u_e) + F_{\text{int}}(s),$$
$$e(\hat{s}) = \sum e(s_e) + \sum e(t_e) + \sum e(u_e). \tag{20}$$

where the representative $s \in [s]$ has a fewest number of edges in its class.

Proof. The proof of this lemma will follow the one given in Lemma 3.7. Consider a representative $s \in [s] \in S(\widetilde{\mathcal{G}}_{\mathfrak{f}^0})/ \sim$ which has the fewest number of edges. A connected component of s containing any of the points a_e, a'_e, b_e and b'_e corresponds to a connected component of the boundary of the graph $\hat{s}$ containing the same set of points. The extra connected components of the boundary graph of $\hat{s}$ arise from the states $s_e \in \ddot{S}^1(H_e)$, $t_e \in \ddot{S}^1(H_e)$, and $u_e \in \bar{S}^1(H_e)$. These components are precisely the unmarked components. For each e they are $(C_{\partial}(s_e) - 2)$, $(C_{\partial}(t_e) - 2)$ and $(C_{\partial}(u_e) - 1)$ extra components. This ends the proof of the second relation in (20). The remaining follows. $\square$

Let us consider the map $\mathcal{F} : \mathbb{Z}[\{f_e, g_e\}_{e \in \mathcal{E}}] \to \mathbb{Z}[\{\ddot{\eta}_e^{(1)}, \ddot{\eta}_e^{(1)}, \bar{\eta}_e^{(1)}\}_{e \in \mathcal{E}}]$ as the linear extension of

$$\mathcal{F} : \prod_{e \in \mathcal{E}} f_e^{\alpha_e} g_e^{\beta_e} \mapsto \prod_{e \in \mathcal{E}} (\ddot{\eta}_e^{(1)})^{\alpha_e \beta_e} (\ddot{\eta}_e^{(2)})^{1-\beta_e} (\bar{\eta}_e^{(1)})^{\beta_e - \alpha_e \beta_e}.$$

Lemma 3.17. *The multivariate version of BR polynomial on HERGs, Z, of the graph $\widehat{\mathcal{G}}_{\mathfrak{f}^0}$ introduced above is given by*

$$Z_{\widehat{\mathcal{G}}_{\mathfrak{f}^0}}(a,b,c,d) = \mathcal{F}(\Phi_{\widetilde{\mathcal{G}}_{\mathfrak{f}^0}}).$$

Proof. Each state $\hat{s}$ of $\widehat{\mathcal{G}}_{\mathfrak{f}^0}$ is obtained in the following way: we consider a representative $s \in [s] \in S(\widetilde{\mathcal{G}}_{\mathfrak{f}^0})/ \sim$ which has the fewest number of edges. We replace the three edges configurations $\{f_e, g_e\}$, g_e and $\emptyset$ of the pair of edges $\{f_e, g_e\}$ (as introduced above) by $s_e \in \ddot{S}^1(H_e)$, $\ddot{S}^1(H_e)$, and $\bar{S}^1(H_e)$ respectively. The contribution of $\hat{s}$ in $Z_{\widehat{\mathcal{G}}_{\mathfrak{f}^0}}(.)$ is

$$a^{k(s)} c^{F_{\text{int}}(s)} d^{C_{\partial}(s)} \prod_{e \in \mathcal{E}} \left(a^{k(s_e)-1} b^{e(s_e)} c^{F_{\text{int}}(s_e)} d^{C_{\partial}(s_e)-2} \right)^{\alpha_e \beta_e} \times$$

$$\left(a^{k(s_e)-2}b^{e(s_e)}c^{F_{\text{int}}(s_e)}d^{C_\partial(s_e)-2}\right)^{1-\beta_e}\left(a^{k(s_e)-1}b^{e(s_e)}c^{F_{\text{int}}(s_e)}d^{C_\partial(s_e)-1}\right)^{\beta_e-\alpha_e\beta_e} =$$
$$\mathcal{F}\left(a^{k(s)}\left(\prod_{e\in s}x_e\right)c^{F_{\text{int}}(s)}d^{C_\partial(s)}\right), \tag{21}$$

where $\alpha_e = \begin{cases} 1 & \text{if } f_e \in s, \\ 0 & \text{otherwise,} \end{cases}$ and $\beta_e = \begin{cases} 1 & \text{if } g_e \in s, \\ 0 & \text{otherwise} \end{cases}$. The expression (21) is also equal to

$$a^{k(s)+\sum(k(s_e)-1)+\sum(k(t_e)-1)+\sum(k(u_e)-1)}b^{\sum e(s_e)+\sum e(t_e)+\sum e(u_e)}\times$$
$$c^{\sum F_{\text{int}}(s_e)+\sum F_{\text{int}}(t_e)+\sum F_{\text{int}}(u_e)+F_{\text{int}}(s)}d^{\sum(C_\partial(s_e)-2)+\sum(C_\partial(t_e)-2)+\sum(C_\partial(u_e)-1)+C_\partial(s)}, \tag{22}$$

where $s_e \in \ddot{S}^1(H_e)$, $t_e \in \ddot{S}^1(H_e)$, and $u_e \in \bar{S}^1(H_e)$. Using Lemma 3.16, we deduce that the relation in (22) is equal to $a^{k(\hat{s})}b^{e(\hat{s})}c^{F_{\text{int}}(\hat{s})}d^{C_\partial(\hat{s})}$. Since the decomposition of $\hat{s}$ into s and s_e is unique, the result follows summing over the states. $\qquad\square$

We remark that the proof of this lemma follows the proof of Lemma 8 in [10]. Following the proof of Lemma 9 in the same paper, we can prove the following result.

Lemma 3.18. *Let $(\mathcal{G}_{\mathfrak{f}'0}, \{H_e\}_{e\in\mathcal{E}})$ be a two decomposition of $\widehat{\mathcal{G}}_{\mathfrak{f}'0}$, and $\mathcal{H}:\mathbb{Z}[\{f_e, g_e\}_{e\in\mathcal{E}}]$ $\to \mathbb{Z}[\{\ddot{\eta}_e^{(1)}, \ddot{\eta}_e^{(1)}, \bar{\eta}_e^{(1)}\}_{e\in\mathcal{E}}]$ be the linear extension of the map*

$$\mathcal{H} : \prod_{e\in\mathcal{E}} f_e^{\alpha_e}g_e^{\beta_e} \mapsto \prod_{e\in\mathcal{E}}(\ddot{\eta}_e^{(1)})^{\alpha_e\beta_e}(\ddot{\eta}_e^{(2)})^{1-\beta_e}(\bar{\eta}_e^{(1)})^{\beta_e-\alpha_e\beta_e}.$$

Then

$$Z(\widehat{\mathcal{G}}_{\mathfrak{f}'0}; a, b, c, d) = \mathcal{H}(Z(\widetilde{\mathcal{G}}_{\mathfrak{f}'0}; a, x, c, d)).$$

Let us study the effect of the insertion of the edge e in a state of H_e: $s \in \ddot{S}^1(H_e) \cup \ddot{S}^2(H_e) \cup \bar{S}^1(H_e)$. The effect of inserting s in $\bar{S}^1(H_e) \cup \ddot{S}^2(H_e)$ is similar to the one we discussed in the planar case. If $s \in \ddot{S}^2(H_e)$, then the insertion of e in s decreases the number of connected components and connected components of the boundary graph by one. This means, if s contributes the term $a^k b^e c^{F_{\text{int}}} d^{C_\partial}$, then the state $s \cup e$ contributes the term $a^{k-1}(b^e x_e)c^{F_{\text{int}}}d^{C_\partial-1}$. The case $s \in \bar{S}^1(H_e)$ is summarized in three possibilities: if s contributes the term $a^k b^e c^{F_{\text{int}}} d^{C_\partial}$ to the HERGs BR polynomial then the state $s \cup e$ obtained by inserting e contributes $a^k(b^e x_e)c^{F_{\text{int}}+\theta(s)}d^{C_\partial+1-\theta(s)}$; $\theta(s) \in \{0, 1, 2\}$. Assume that $s \in \ddot{S}^1(H_e)$. Two possibilities occur.

• Inserting e in $s \in \ddot{S}^1(H_e)$ increases the number of internal faces by one and decreases by two the number of connected components of the boundary graph. If s contributes the term $a^k b^e c^{F_{\text{int}}} d^{C_\partial}$ to the HERGs BR polynomial then the state $s \cup e$ obtained by inserting e contributes $a^k(b^e x_e)c^{F_{\text{int}}+1}d^{C_\partial+2}$.

• Inserting e in $s \in \ddot{S}^1(H_e)$ decreases the number of connected components of the boundary graph by one such that if s contributes the term $a^k b^e c^{F_{\text{int}}} d^{C_\partial}$ to the HERGs BR polynomial then the state $s \cup e$ obtained by inserting e contributes $a^k(b^e x_e)c^{F_{\text{int}}}d^{C_\partial-1}$.

We can summarized our discussion by saying that is $s \in \ddot{S}^1(H_e)$ contributes the term $a^k b^e c^{F_{\text{int}}} d^{C_\partial}$ to the HERGs BR polynomial, then the state $s \cup e$ obtained by inserting e in s contributes $a^k(b^e x_e)c^{F_{\text{int}}+\theta(s)}d^{C_\partial-1-\theta(s)}$; $\theta(s) \in \{0, 1\}$.

Based on the previous discussions, we can write $\bar{\eta}_e^{(1)}(a,b,c,d)$ as

$$
\begin{aligned}
\bar{\eta}_e^{(1)}(a,b,c,d) &= \bar{\eta}_e^{(1,0)}(a,b,c,d) + \bar{\eta}_e^{(1,1)}(a,b,c,d) + \bar{\eta}_e^{(1,2)}(a,b,c,d), \\
\ddot{\eta}_e^{(1)}(a,b,c,d) &= \ddot{\eta}_e^{(1,0)}(a,b,c,d) + \ddot{\eta}_e^{(1,1)}(a,b,c,d),
\end{aligned}
$$

with

$$
\bar{\eta}_e^{(1,0)}(a,b,c,d) = \sum_{s\in\bar{S}^1(H_e)|\theta(s)=0} a^{k(s)-1}b^{e(s)}c^{F_{\mathrm{int}}(s)}d^{C_\partial(s)-1},
$$

$$
\bar{\eta}_e^{(1,1)}(a,b,c,d) = \sum_{s\in\bar{S}^1(H_e)|\theta(s)=1} a^{k(s)-1}b^{e(s)}c^{F_{\mathrm{int}}(s)}d^{C_\partial(s)-1},
$$

$$
\bar{\eta}_e^{(1,2)}(a,b,c,d) = \sum_{s\in\bar{S}^1(H_e)|\theta(s)=2} a^{k(s)-1}b^{e(s)}c^{F_{\mathrm{int}}(s)}d^{C_\partial(s)-1},
$$

$$
\ddot{\eta}_e^{(1,0)}(a,b,c,d) = \sum_{s\in\ddot{S}^1(H_e)|\theta(s)=0} a^{k(s)-1}b^{e(s)}c^{F_{\mathrm{int}}(s)}d^{C_\partial(s)-2},
$$

$$
\ddot{\eta}_e^{(1,1)}(a,b,c,d) = \sum_{s\in\bar{S}^1(H_e)|\theta(s)=1} a^{k(s)-1}b^{e(s)}c^{F_{\mathrm{int}}(s)}d^{C_\partial(s)-2}.
$$

We also have

$$
\begin{aligned}
Z_{A_e}(a,b,c,d) &= \sum_{s\in\bar{S}^1(H_e)} a^{k(s)}b^{e(s)}c^{F_{\mathrm{int}}(s)}d^{C_\partial(s)} \\
&+ x_e \sum_{s\in\bar{S}^1(H_e)|\theta(s)=0,1,2} a^{k(s)}b^{e(s)}c^{F_{\mathrm{int}}(s)+\theta(s)}d^{C_\partial(s)+1-\theta(s)} \\
&+ \sum_{s\in\ddot{S}^1(H_e)} a^{k(s)}b^{e(s)}c^{F_{\mathrm{int}}(s)}d^{C_\partial(s)} \\
&+ x_e \sum_{s\in\ddot{S}^1(H_e)|\theta(s)=0,1} a^{k(s)}b^{e(s)}c^{F_{\mathrm{int}}(s)+\theta(s)}d^{C_\partial(s)-1-\theta(s)} \\
&+ (1+a^{-1}x_e d^{-1}) \sum_{s\in\ddot{S}^2(H_e)} a^{k(s)}b^{e(s)}c^{F_{\mathrm{int}}(s)}d^{C_\partial(s)}.
\end{aligned}
$$

$$
\begin{aligned}
Z_{A_e}(a,b,c,d) &= ad\bar{\eta}_e^{(1)} + x_e\big(ad^2\bar{\eta}_e^{(1,0)} + acd\bar{\eta}_e^{(1,1)} + ac^2\bar{\eta}_e^{(1,2)}\big) + ad^2\ddot{\eta}_e^{(1)} \\
&+ x_e\big(ad\ddot{\eta}_e^{(1,0)} + acd^2\ddot{\eta}_e^{(1,1)}\big) + (1+a^{-1}x_e d^{-1})a^2 d^2 \ddot{\eta}_e^{(2)} \\
&= ad(\bar{\eta}_e^{(1)} + d\ddot{\eta}_e^{(1)} + ad\ddot{\eta}_e^{(2)}) \\
&+ ax_e\big(d^2\bar{\eta}_e^{(1,0)} + cd\bar{\eta}_e^{(1,1)} + c^2\bar{\eta}_e^{(1,2)} + d\ddot{\eta}_e^{(1,0)} + cd^2\ddot{\eta}_e^{(1,1)} + d\ddot{\eta}_e^{(2)}\big).
\end{aligned}
$$

Furthermore

$$
Z_{A_e}(a,b,c,d) = Z_{H_e}(a,b,c,d) + x_e Z_{A_e/e}(a,b,c,d).
$$

Then

$$
ad(\bar{\eta}_e^{(1)} + d\ddot{\eta}_e^{(1)} + ad\ddot{\eta}_e^{(2)}) = Z_{H_e}(a,b,c,d),
$$

$$a\big(d^2\bar\eta_e^{(1,0)} + cd\bar\eta_e^{(1,1)} + c^2\bar\eta_e^{(1,2)} + d\ddot\eta_e^{(1,0)} + cd^2\ddot\eta_e^{(1,1)} + d\ddot\eta_e^{(2)}\big) = Z_{A_e/e}(a,b,c,d). \quad (23)$$

The second relation in (23) can also be written as

$$ad(d-c)\bar\eta_e^{(1,0)} + ac(c-d)\bar\eta_e^{(1,2)} + ad(1-cd)\ddot\eta_e^{(1,0)} + ad(c\bar\eta_e^1 + cd\ddot\eta_e^1 + \ddot\eta_e^2) =$$
$$Z_{A_e/e}(a,b,c,d).$$

Theorem 3.19. *Let $\widehat{\mathcal{G}}_{\mathfrak{f}'0}$ be a ribbon graph with the 2-decomposition $(\mathcal{G}_{\mathfrak{f}^0}, \{H_e\})$, and let A_e be the ribbon graph H_e with an additional ribbon e joining the vertices u_e and w_e. Then*

$$Z(\widehat{\mathcal{G}}_{\mathfrak{f}'0}; a, x, c, d) = \mathcal{H}(Z(\widetilde{\mathcal{G}}_{\mathfrak{f}'0}; a, x, c, d)),$$

where p_e, q_e and r_e are uniquely determine by the pair of equations

$$p_e + adq_e + dr_e = Z_{H_e}(a,b,c,d)$$
$$ad(d-c)\bar\eta_e^{(1,0)} + ac(c-d)\bar\eta_e^{(1,2)} + ad(1-cd)\ddot\eta_e^{(1,0)} + cp_e + q_e + cdr_e =$$
$$Z_{A_e/e}(a,b,c,d),$$

where the η are introduced above and $\mathcal{H}$ is deduced by

$$\mathcal{H} : \prod_{e\in\mathcal{E}} f_e^{\alpha_e} g_e^{\beta_e} \mapsto \prod_{e\in\mathcal{E}} \Big(\frac{r_e}{ac}\Big)^{\alpha_e\beta_e} \Big(\frac{q_e}{2ac}\Big)^{1-\beta_e} c^{\alpha_e\beta_e-\alpha_e} \Big(\frac{p_e}{ac}\Big)^{\beta_e-\alpha_e\beta_e}.$$

4   The rank D-weakly colored stranded graph 2-decomposition and polynomial invariant

4.1   Weakly colored stranded graph

In this subsection we assume that the reader is familiar with stranded graphs. We briefly review here the weakly-colored stranded graphs introduced in [1].

A graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ is stranded when its vertices and edges are stranded. A rank D stranded vertex is a trivial disc or a chord diagram with a collection of $2n$ points on the unit circle (called the vertex frontier) paired by n chords and drawn in a specific way. A rank D stranded edge is a collection of segments called strands respecting some conditions as introduced in [1].

A rank D stranded graph $\mathcal{G}$ is a graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ which admits: rank D stranded vertices, rank at most D stranded edges such that the intersections of vertices and edges are pairwise distinct. The graph $\mathcal{G}$ is a rank D tensor graph if the vertices of $\mathcal{G}$ have a fixed coordination $D+1$ and their pre-edges have a fixed cardinal D. The merged point graph is K_{D+1} and the edges of $\mathcal{G}$ are of rank D.

Collapsing the stranded vertices of $\mathcal{G}$ to points and edges to simple lines, the resulting object is a graph. The graph $\mathcal{G}$ is said to be connected if its corresponding collapsed graph is connected. From this point, stranded vertices and edges are always connected. If we assign a color from the set $\{0, \cdots, D\}$ to each edge of $\mathcal{G}$ such that no two adjacent edges share the same color, then the graph $\mathcal{G}$ is called a $(D+1)$ colored graph. It is a bipartite graph if the set $\mathcal{V}$ of vertices is split into two disjoint sets, i.e.

$\mathcal{V} = \mathcal{V}^+ \cup \mathcal{V}^-$ with $\mathcal{V}^+ \cap \mathcal{V}^- = \emptyset$, such that each edge connects a vertex $v^+ \in \mathcal{V}^+$ and a vertex $v^- \in \mathcal{V}^-$. The rank $D \geq 1$ tensor graph $\mathcal{G}$ is a rank $D \geq 1$-Colored tensor graph if it is a $(D+1)$-colored and bipartite graph.

The collapsed graph associated to a colored tensor graph is obtained by regarding the tensor graph as a simple bipartite colored graph and is called *compact* in the following. In such a colored graph, there are some informations that we address now.

Definition 4.1 (*p*-bubbles [9]). Let $\mathcal{G}$ be a rank D colored tensor graph.
- A 0-bubble is a vertex of $\mathcal{G}$.
- A 1-bubble is an edge of $\mathcal{G}$.
- For all $p \geq 2$, a p-bubble of $\mathcal{G}$ with colors $i_1 < i_2 < \cdots < i_p$, $p \leq D$, and $i_k \in \{0, \ldots, D\}$ is a connected rank $p-1$ colored tensor graph the compact form of which is a connected subgraph of the compact form of $\mathcal{G}$ made of edges of colors $\{i_1, \ldots, i_p\}$.

The concept of half-edge finds a natural extension on colored tensor graphs and allows one to recall the operation called "cut" of an edge.

In the following, a stranded graph having stranded half-edges is denoted by $\mathcal{G}_{\mathfrak{f}^0}(\mathcal{V}, \mathcal{E})$ or simply $\mathcal{G}_{\mathfrak{f}^0}$, where $\mathfrak{f}^0$ is the set of the half-edges.

Definition 4.2 (Cut of an edge [12]). Let $\mathcal{G}_{\mathfrak{f}^0}(\mathcal{V}, \mathcal{E})$ be a rank D stranded graph and e a rank d edge of $\mathcal{G}_{\mathfrak{f}^0}$, $1 \leq d \leq D$. The cut graph $\mathcal{G}_{\mathfrak{f}^0} \vee e$ or the graph obtained from $\mathcal{G}_{\mathfrak{f}^0}$ by cutting e is obtained by replacing the edge e by two rank d stranded half-edges at the end vertices of e and respecting the strand structure of e. (See Figure 10.) If e is a loop, the two stranded half-edges are on the same vertex.
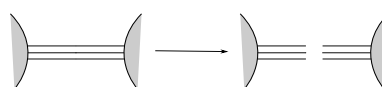

Figure 10:   The cut of a rank 3 stranded edge.

Using the operation of "cutting" of an edge, we obtain a c-subgraph $A_{\mathfrak{f}_A^0}(\mathcal{V}_A, \mathcal{E}_A)$ of a rank D stranded graph $\mathcal{G}_{\mathfrak{f}^0}(\mathcal{V}, \mathcal{E})$ by cutting a subset of edges of $\mathcal{G}_{\mathfrak{f}^0}$. A spanning c-subgraph called also state s of $\mathcal{G}_{\mathfrak{f}^0}$ is defined as a c-subgraph $s_{\mathfrak{f}_s^0}(\mathcal{V}_s, \mathcal{E}_s)$ of $\mathcal{G}_{\mathfrak{f}^0}$ with all vertices and all additional half-edges of $\mathcal{G}_{\mathfrak{f}^0}$. Then $\mathcal{E}_s \subseteq \mathcal{E}$ and $\mathcal{V}_s = \mathcal{V}$, $\mathfrak{f}_s^0 = \mathfrak{f}^0 \cup \mathfrak{f}_s^{0;1}(\mathcal{E}_s)$, where $\mathfrak{f}_s^{0;1}(\mathcal{E}_s)$ is the set of half-edges obtained by cutting all edges in $\mathcal{E}_s'$ (the set of edges incident to the vertices of s and not contained in $\mathcal{E}_s$) and incident to the vertices of s. We write $s \Subset \mathcal{G}_{\mathfrak{f}^0}$ to say that s is a spanning c-subgraph of $\mathcal{G}_{\mathfrak{f}^0}$.

The cutting of an edge modifies the strand structure of the graph. In fact as discussed earlier, the presence of half-edges immediately introduce another type of faces which pass through the external points of the half-edges. We then distinguish two kind of faces: open faces which are passing through the external points of the half-edges and the others called closed or internal faces. We denote by $\mathcal{F}_{\text{int}}$, the set of closed faces and $\mathcal{F}_{\text{ext}}$ the set of open faces. The set of faces $\mathcal{F}$ for a rank D

half-edged colored tensor graph is then the disjoint union $\mathcal{F}_{\text{int}} \cup \mathcal{F}_{\text{ext}}$. A bubble is called open or external if it contains open faces otherwise it is closed or internal. We denote respectively by $\mathcal{B}_{\text{int}}$ and $\mathcal{B}_{\text{ext}}$ the sets of closed and open bubbles for a rank D tensor graph.

Their is a graph directly associated to a color tensor graph called "boundary graph" which is obtained by setting a vertex to each half-edge [9]. The boundary graph $\partial \mathcal{G}(\mathcal{V}_\partial, \mathcal{E}_\partial)$ of a rank D half-edged colored tensor graph $\mathcal{G}_{\mathfrak{f}^0}(\mathcal{V}, \mathcal{E})$ is obtained by inserting a vertex with degree D at each additional stranded half-edge of $\mathcal{G}_{\mathfrak{f}^0}$ and taking the external faces of $\mathcal{G}_{\mathfrak{f}^0}$ as its edges. Thus, $|\mathcal{V}_\partial| = |\mathfrak{f}^0|$ and $\mathcal{E}_\partial = \mathcal{F}_{\text{ext}}$. If the rank D half-edged colored tensor graph is closed, then its boundary is empty.

Definition 4.3 (Equivalence class of half-edged stranded graph [1])**.** Let $D_{\mathcal{G}_{\mathfrak{f}^0}}$ be the subgraph in a rank D half-edge stranded graph $\mathcal{G}_{\mathfrak{f}^0}$ defined by all of its trivial disc vertices and $\mathcal{G}_{\mathfrak{f}^0} \setminus D_{\mathcal{G}_{\mathfrak{f}^0}}$ the rank D half-edges stranded graph obtained after removing $D_{\mathcal{G}_{\mathfrak{f}^0}}$ from $\mathcal{G}_{\mathfrak{f}^0}$.

Two rank D half-edged stranded graphs $\mathcal{G}_{1,\mathfrak{f}^0(\mathcal{G}_1)}$ and $\mathcal{G}_{2,\mathfrak{f}^0(\mathcal{G}_2)}$ are "equivalent up to trivial discs" if and only if $\mathcal{G}_{1,\mathfrak{f}^0(\mathcal{G}_1)} \setminus D_{\mathcal{G}_{1,\mathfrak{f}^0(\mathcal{G}_1)}} = \mathcal{G}_{2,\mathfrak{f}^0(\mathcal{G}_2)} \setminus D_{\mathcal{G}_{2,\mathfrak{f}^0(\mathcal{G}_2)}}$. We note $\mathcal{G}_{1,\mathfrak{f}^0(\mathcal{G}_1)} \sim \mathcal{G}_{2,\mathfrak{f}^0(\mathcal{G}_2)}$.

As a consequence of this definition, the contraction of all edges in arbitrary order of a half-edged tensor graph $\mathcal{G}_{\mathfrak{f}^0}$ yields a half-edged stranded graph $\mathcal{G}_{\mathfrak{f}^0}^0$ determined by the boundary $\partial(\mathcal{G}_{\mathfrak{f}^0})$ up to additional discs. Noting that contracting an edge in a rank D (colored) half-edged tensor graph does not change its boundary.

We can now address a precise definition of a rank D w-colored graph.

Definition 4.4 (Rank D w-colored graph [1])**.** A rank D weakly colored or w-colored graph is the equivalence class (up to trivial discs) of a rank D half-edged stranded graph obtained by successive edge contractions of some rank D half-edged colored tensor graph. An illustration is given in Figure 11 .
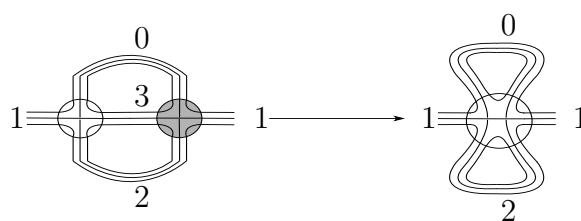

Figure 11:   Contraction of an edge in a rank 3 colored tensor graph (left) giving a rank 3 w-colored tensor graph (right)

4.2   Polynomial invariant expansion

Here again, some definitions and notations introduced in [1] deserve to be review as well.

Consider a representative $\mathcal{G}_{\mathfrak{f}^0}$ of any rank D w-colored graph. A d-bubble (closed or open) in $\mathcal{G}_{\mathfrak{f}^0}$ is denoted by $\mathbf{b}^d$, the set of d-bubbles by $\mathcal{B}^d$, and its cardinal B^d. The set of vertices and edges of $\mathbf{b}^d$ are denoted by $\mathcal{V}_{\mathbf{b}^d}$ and $\mathcal{E}_{\mathbf{b}^d}$ of cardinal $V_{\mathbf{b}^d}$ and

$E_{\mathbf{b}^d}$ respectively. We also denote by $\mathcal{F}_{\text{int};\mathbf{b}^d}$ and $\mathcal{B}^p_{\mathbf{b}^d}$, the sets of internal faces and p-bubbles ($p \le d$) of $\mathbf{b}^d$ of cardinal $F_{\text{int};\mathbf{b}^d}$ and $B^p_{\mathbf{b}^d}$ respectively.

Definition 4.5 (Topological invariant for rank n w-colored graph [1])**.** Let $\mathfrak{G}(\mathcal{V}, \mathcal{E}, \mathfrak{f}^0)$ be a rank n w-colored graph and $\boldsymbol{\alpha} = \{\alpha_k\}_{k=3,\cdots,n}$ some positive rational numbers. The generalized topological invariant associated with $\mathfrak{G}$ is given by the following function associated with any of its representatives $\mathcal{G}_{\mathfrak{f}^0}$.

$$\mathfrak{T}_{\mathfrak{G};\boldsymbol{\alpha}}(x,y,z,s,w,q,t) = \mathfrak{T}_{\mathcal{G}_{\mathfrak{f}^0};\boldsymbol{\alpha}}(x,y,z,s,w,q,t) = \tag{24}$$
$$\sum_{s \in \mathcal{G}_{\mathfrak{f}^0}} (x-1)^{\text{r}\,(\mathcal{G}_{\mathfrak{f}^0})-\text{r}\,(s)} y^{n(s)} z^{\frac{(n-1)(n+2)}{2}k(s)-\gamma_{n;\boldsymbol{\alpha}}(s)} s^{C_\partial(s)} w^{F_\partial(s)} q^{E_\partial(s)} t^{f(s)},$$

with

$$\gamma_{n;\boldsymbol{\alpha}}(s) = \frac{n(n-1)}{2}(V(s) - E(s)) + (n-1)F_{\text{int}}(s) - (2 + (n-2)\alpha_3)B^3(s)$$
$$+ \sum_{k=4}^{n} \Big[(k-1)\alpha_{k-1} - (n-k+1)\alpha_k\Big]B^k(s)$$

a negative integer.

Expanding the definitions of r (s) and $n(s)$ in (24) yields

$$\mathfrak{T}_{\mathfrak{G};\boldsymbol{\alpha}}(x,y,z,s,w,q,t) = \mathfrak{T}_{\mathcal{G}_{\mathfrak{f}^0};\boldsymbol{\alpha}}(x,y,z,s,w,q,t) =$$
$$(x-1)^{-k(\mathcal{G})}\Big(yz^{\frac{n(n-1)}{2}}\Big)^{-v(\mathcal{G})} \sum_{A \in \mathcal{G}_{\mathfrak{f}^0}} \Big((x-1)yz^{\frac{(n-1)(n+2)}{2}}\Big)^{k(s)} \Big(yz^{\frac{n(n-1)}{2}}\Big)^{e(s)} z^{(1-n)F_{\text{int}}(s)}$$
$$\times z^{(2+(n-2)\alpha_3)B^3(s)} \Big(\prod_{k=4,\cdots,n} z^{((1-k)\alpha_{k-1}+(n-k+1)\alpha_k)B^k(s)}\Big) s^{C_\partial(s)} w^{F_\partial(s)} q^{E_\partial(s)} t^{f(s)}.$$

Let us introduce the multivariate version of this polynomial

Definition 4.6 (Multivariate form)**.** The multivariate form associated with (24) is defined by:

$$\widetilde{\mathfrak{T}}_{\mathfrak{G};\boldsymbol{\alpha}}(x,\{\beta_e\},\{z_i\}_{i=1,2,3},s,w,q,t) = \widetilde{\mathfrak{T}}_{\mathcal{G}_{\mathfrak{f}^0}}(x,\{\beta_e\},\{z_i\}_{i=1,\cdots,n},z,s,w,q,t)$$
$$= \sum_{A \in \mathcal{G}} x^{\text{r}\,(s)} \Big(\prod_{e \in s} \beta_e\Big) \Big(\prod_{i=1,\cdots,n} z_i^{B^i}\Big) z^{F_{\text{int}}(s)} s^{C_\partial(s)} w^{F_\partial(s)} q^{E_\partial(s)} t^{f(s)},$$

for $\{\beta_e\}_{e \in \mathcal{E}}$ labelling the edges of the graph $\mathcal{G}$.

The operation of 2-sum introduced earlier can be generalized on the class of graph called weakly colored stranded graph. The only issue is the colors of the edges we want to identify. It is clear that for this operation to be possible, the edges must have the same color. This is illustrated in Figure 12.

The construction of a rank n weakly colored graph $\widehat{\mathcal{G}}_{\mathfrak{f}^0}$ from the 2-decomposition $(\mathcal{G}_{\mathfrak{f}^0}, \{H_e\}_{e \in \mathcal{E}})$ at an edge $e = (u_e, w_e)$ of the template $\mathcal{G}_{\mathfrak{f}^0}$ will be a direct extension
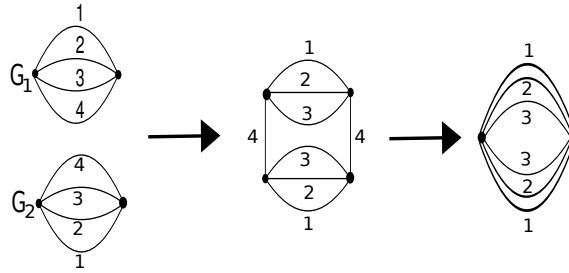
Figure 12:   The 2-sum of G_1 and G_2 by their edges of color 4

of the one introduced earlier. We can say that the graph $\widehat{\mathcal{G}}_{\mathfrak{f}'^0}$ is obtained by the identification of the $n-1$ ball (in topological point of view) m_e and n_e of the vertices u_e and w_e in H_e with their correspondence in $\mathcal{G}_{\mathfrak{f}^0} \vee e$. From this, we can give a partition of $S(H_e)$: the set of states for a given H_e.

Let $S^2(H_e)$ be the set of states of H_e in which m_e and n_e lie in different connected components and different connected components of the boundary graph. We set $S^1(H_e)$ as the set of states of H_e in which m_e and n_e lie in the same connected component and the same internal face or connected component of the boundary graph. We work under the condition that m_e and n_e lie in the same connected component if and only if they are in the same internal face or connected component of the boundary graph.

Lemma 4.7. *Let $\hat{s}$ be a state of the rank n weakly colored stranded graph $\widehat{\mathcal{G}}_{\mathfrak{f}'^0}$. $s_e \in S^1(H_e) \cup S^2(H_e)$, $e \in \mathcal{E}$ in the decomposition above. Then*

$$k(\hat{s}) = \sum_{e \in \mathcal{E}} k(s_e) + k(s) - |\{s_e \in S^1(H_e)\}| - 2|\{s_e \in S^2(H_e)\}|, \tag{25}$$

$$C_\partial(\hat{s}) = \sum_{e \in \mathcal{E}} C_\partial(s_e) + C_\partial(s) - |\{s_e \in S^1(H_e)\}| - 2|\{s_e \in S^2(H_e)\}|,$$

$$F_{\text{int}}(\hat{s}) = \sum_{e \in \mathcal{E}} F_{\text{int}}(s_e) + F_{\text{int}}(s), \tag{26}$$

$$f(\hat{s}) = \sum_{e \in \mathcal{E}} f(s_e) + f(s) - 2|\{s_e \in S^1(H_e)\}| - 4|\{s_e \in S^2(H_e)\}|, \tag{27}$$

$$E_\partial(\hat{s}) = \sum_{e \in \mathcal{E}} E_\partial(s_e) + E_\partial(s) - n|\{s_e \in S^1(H_e)\}| - 2 \times n|\{s_e \in S^2(H_e)\}|, \tag{28}$$

$$F_\partial(\hat{s}) \geq \sum_{e \in \mathcal{E}} F_\partial(s_e) + F_\partial(s) - n|\{s_e \in S^1(H_e)\}| - 2 \times n|\{s_e \in S^2(H_e)\}|. \tag{29}$$

Furthermore

$$B^p(\hat{s}) = \sum_{e \in \mathcal{E}} B^p(s_e) + B^p(s) - \mathbb{C}_n^{p-1}|\{s_e \in S^1(H_e)\}| - 2\mathbb{C}_n^{p-1}|\{s_e \in S^2(H_e)\}|. \tag{30}$$

With $|\{s_e \in S^i(H_e)\}|$; $i = 1, 2$ the number of states in $S^i(H_e)$, at fixed e.

Proof. The proof of this lemma can be performed on colored tensor graphs since the w-colored stranded graphs are obtained by a successive contraction of edges in a colored tensor graph. Furthermore this contraction does not modify the boundary or the number of bubbles in a colored tensor graph.

Equations (25), (26) and (27) are direct extension to the relations in Lemma 3.6.

Let us consider the graph $\partial(s)$. Each edge of $\partial(s)$ or open face of s corresponds to an edge in $\partial(\hat{s})$. However, $\partial(\hat{s})$ has extra edges which are the edges of $\partial(s_e)$; $s_e \in S^1(H_e) \cup S^2(H_e)$, $e \in \mathcal{E}(\mathcal{G}_1)$. Since s_e is inserted in s by n strands then there are $E(\partial(s_e)) - n$ such extra edges. This ends the proof of (28). The proof of (29) is similar to (28). In this case s_e is inserted in s by at most 3 faces of the boundary $\partial(s_e)$.

Let us assume that the graphs s_e are inserted in s by the edges of colors i. There is a one to one correspondence between the p-bubbles of s and s_e not containing any i and those of $\hat{s}$ not containing the same colors i. Furthermore each $s_e \in S^1(H_e)$ share $\mathbb{C}_n^{p-1}$ number of p-bubbles with s (these bubbles contain the color of e). If $s_e \in S^2(H_e)$, then the number is $2\mathbb{C}_n^{p-1}$. Hence the number of extra bubbles in $\hat{s}$ are $B^p(s_e) - \mathbb{C}_n^{p-1}$. This ends the proof of (30). □

We consider the following state sums:

$$\eta_e^{(1)}(a, b, c, d, f, \{g_p\}) := \sum_{s \in S^1(H_e)} a^{k(s)-1} b^{e(s)} c^{F_{\text{int}}(s)} d^{C_\partial(s)-1} f^{E_\partial(s)-n} \left(\prod_{p=3}^{p=n} g_p^{B^p(s)-\mathbb{C}_n^{p-1}} \right), \quad (31)$$

$$\eta_e^{(2)}(a, b, c, d, f, \{g_p\}) := \sum_{s \in S^2(H_e)} a^{k(s)-2} b^{e(s)} c^{F_{\text{int}}(s)} d^{C_\partial(s)-2} f^{E_\partial(s)-2n} \left(\prod_{p=3}^{p=n} g_p^{B^p(s)-2\mathbb{C}_n^{p-1}} \right).$$

$$(32)$$

We can observe in the expressions of $\eta_e^{(1)}$ and $\eta_e^{(2)}$ given in (31) and (32) respectively that there is no variable for the faces of the boundary graph. The reason of this choice comes from the inequality given by (29).

Proposition 4.8. *Let $(\mathcal{G}_{\mathfrak{f}^0}, \{H_e\}_{e \in \mathcal{E}})$ be a 2-decomposition of $\widehat{\mathcal{G}}_{\mathfrak{f}^0}$ and $\eta_e^{(1)}$, $\eta_e^{(2)}$ two functions as introduced above. Then*

$$Z(\widehat{\mathcal{G}}_{\mathfrak{f}^0}; a, b, c, d, f, \{g_p\}) = \sum_{s \in S(\mathcal{G})} a^{k(s)} c^{F_{\text{int}}(s)} d^{C_\partial(s)} f^{E_\partial(s)} \left(\prod_{p=3}^{p=n} g_p^{B^p(s)} \right) \times$$
$$\left(\prod_{e \in s} \eta_e^1 \right) \left(\prod_{e \notin s} \eta_e^2 \right).$$

Once again we can make a partition of the set $S(H_e)$ of states s of H_e where $e = (m_e, n_e)$. We can set $S(H_e) = S^1(H_e) \cup S^2(H_e)$; where $S^1(H_e)$ is the set of states having m_e and n_e in the same connected component and the same component of the boundary graph and $S^2(H_e)$ is the set of states having m_e and n_e in

different connected components and different components of the boundary graph. It is clear that the insertion of e in $s \in S^2(H_e)$ decreases the number of connected components of the boundary graph by one, the number of edges of the boundary by n and the number of p-bubbles by C_n^{p-1}. Then if s contributes with the term $a^{k(s)}c^{F_{\mathrm{int}}(s)}d^{C_\partial(s)}f^{E_\partial(s)}\left(\prod_{p=3}^{p=n}g_p^{B^p(s)}\right)$, the state $s \cup e$ will contribute with $a^{k(s)-1}c^{F_{\mathrm{int}}(s)}d^{C_\partial(s)-1}f^{E_\partial(s)-n}\left(\prod_{p=3}^{p=n}g_p^{B^p(s)-\mathsf{C}_n^{p-1}}\right)$.

The insertion of e in a state $s \in S^1(H_e)$ of H_e leads to different possible cases. For example this insertion can increase the number of internal faces from 0 to n. We also have multiple possibilities for the number of connected components of the boundary graph, the number of edges of the boundary graphs and the number of p-bubbles. A good analysis of the different possibilities may help to find a more explicit formula than the one given in Proposition 4.8. For doing this let us make a restriction to $n = 3$.

Consider $s \in S^1(H_e)$ and insert e in s. The number of 3-bubbles may decrease from 0 to 3; but the number of internal faces may increase from 0 to 3. Depending on the number of internal faces we add by the insertion of e, let us discuss the different cases for the number of connected components of the boundary graph and the number of edges of the boundary graph.

• Assume that the number of internal faces is stable after the insertion of e. The number of connected components of the boundary graph is also stable after the insertion of e or may increase by one. The number of edges of the boundary graph will decrease by 3. Considering the previous discussions about the evolving of the number of 3-bubbles under the insertion of e, if s contributes with the term

$$a^{k(s)}c^{F_{\mathrm{int}}(s)}d^{C_\partial(s)}f^{E_\partial(s)}\left(\prod_{p=3}^{p=n}g_p^{B^p(s)}\right),$$

in the polynomial, then the contribution to the polynomial of the state $s \cup e$ will be $a^{k(s)}c^{F_{\mathrm{int}}(s)}d^{C_\partial(s)+\beta(s)}f^{E_\partial(s)-3}g_3^{B^3(s)-\gamma(s)}$ for $\beta(s) = 0, 1$ and $\gamma(s) = 0, 1, 2, 3$.

• Assume that the number of internal faces increases by one or two after the insertion of e. Then number of edges of the boundary graph will decrease by two or one respectively. It is also clear that the number of connected components of the boundary graph is stable or may increase by one after the insertion of e. If s contributes with the term $a^{k(s)}c^{F_{\mathrm{int}}(s)}d^{C_\partial(s)}f^{E_\partial(s)}\left(\prod_{p=3}^{p=n}g_p^{B^p(s)}\right)$, in the polynomial, then the state $s \cup e$ will contribute with

$$a^{k(s)}c^{F_{\mathrm{int}}(s)+\alpha(s)}d^{C_\partial(s)+\beta(s)}f^{E_\partial(s)-3+\alpha(s)}g_3^{B^3(s)-\gamma(s)},$$

for $\alpha(s) = 1, 2$, $\beta(s) = 0, 1$ and $\gamma(s) = 0, 1, 2, 3$.

• Assume that the number of internal faces increases by three after the insertion of e. The number of connected components of the boundary graph is stable after the insertion of e or decreases by one. The number of edges of the boundary graphs is stable after the insertion. If s contributes with the term $a^{k(s)}c^{F_{\mathrm{int}}(s)}d^{C_\partial(s)}f^{E_\partial(s)}\left(\prod_{p=3}^{p=n}g_p^{B^p(s)}\right)$

in the polynomial, then the state $s \cup e$ will contribute with $a^{k(s)} c^{F_{\text{int}}(s)+3} d^{C_\partial(s)+\beta(s)}$ $f^{E_\partial(s)} g_3^{B^3(s)-\gamma(s)}$ for $\beta(s) = -1, 0$ and $\gamma(s) = 0, 1, 2, 3$.

We can summarized all the different cases in this way: if s contributes with the term

$$a^{k(s)} c^{F_{\text{int}}(s)} d^{C_\partial(s)} f^{E_\partial(s)} \Big(\prod_{p=3}^{p=n} g_p^{B^p(s)} \Big),$$

in the polynomial, then the state $s \cup e$ will contribute with

$$a^{k(s)} c^{F_{\text{int}}(s)+\alpha(s)} d^{C_\partial(s)+\beta(s)} f^{E_\partial(s)-3+\alpha(s)} g_3^{B^3(s)-\gamma(s)},$$

for $\alpha(s) = 0, 1, 2, 3$, $\beta(s) = -1, 0, 1$ and $\gamma(s) = 0, 1, 2, 3$.

Consider the following sum

$$Z_{A_e}(a, b, c, d, f, g_3) = \sum_{s \in S(A_e)} a^{k(s)} c^{F_{\text{int}}(s)} d^{C_\partial(s)} f^{E_\partial(s)} g_3^{B^3(s)},$$

which is also equal to

$$
\begin{aligned}
Z_{A_e}(a, b, c, d, f, g_3) = {}& \sum_{s \in S^1(H_e)} a^{k(s)} c^{F_{\text{int}}(s)} d^{C_\partial(s)} f^{E_\partial(s)} g_3^{B^3(s)} \\
& + x_e \sum_{\substack{s \in S^1(H_e) \\ \beta(s)=-1,0,1 \\ \gamma(s),\alpha(s)=0,1,2,3}} a^{k(s)} c^{F_{\text{int}}(s)+\alpha(s)} d^{C_\partial(s)+\beta(s)} f^{E_\partial(s)-3+\alpha(s)} g_3^{B^3(s)-\gamma(s)} \\
& + (1 + a^{-1} x_e d^{-1} f^{-3} g_3^{-3}) \sum_{s \in S^2(H_e)} a^{k(s)} c^{F_{\text{int}}(s)} d^{C_\partial(s)} f^{E_\partial(s)} g_3^{B^3(s)}. \quad (33)
\end{aligned}
$$

Furthermore

$$Z_{A_e}(a, b, c, d, l) = Z_{H_e}(a, b, c, d, l) + x_e Z_{A_e/e}(a, b, c, d, l). \quad (34)$$

A reformulation of the equation (33) in a great number of terms together with an identification with (34) will help us to find a theorem similar to Theorem 3.11. The case where the points m_e and n_e may belong to the same connected component but different boundary components is not studied in this work and remains an open question for future investigations.

Acknowledgements

References

[1] R. C. Avohou, Polynomial invariants for arbitrary rank D weakly-colored stranded graphs, *SIGMA* 12 (2016), 030, 23 pp.

[2] R. C. Avohou, J. Ben Geloun and M. N. Hounkonnou, A polynomial invariant for rank 3 weakly-colored stranded graphs, arXiv:1301.1987[math.CO].

[3] J. Ben Geloun and V. Rivasseau, Addendum to a renormalization 4-dimensional tensor field theory, *Commun. Math. Phys.* **322** (2013), 957–965.

[4] B. Bollobás and O. Riordan, A polynomial of graphs on surfaces, *Math. Ann.* **323** (2002), 81–96.

[5] T. H. Brylawsky, The Tutte polynomial: General Theory, in: *Matroid Theory and Its Applications (ed. A. Barlotti), Liguori, Naples, (1982).*

[6] F. Cicalese and M. Milanič, Graphs of separability at most 2, *Discrete Appl. Math.* **160** (2012), 685–696.

[7] J. A. Ellis-Monaghan and I. Moffat, *Graphs on Surfaces Dualities, Polynomials, and Knots*, Springer Briefs in Mathematics (Springer, NY, 2013).

[8] R. Gurau, Topological graph polynomials in colored group field theory, *Ann. Henri Poincare* **11** (2010), 565.

[9] R. Gurau, Colored group field theory, *Commun. Math. Phys.* **304** (2011), 69–93.

[10] S. Huggett and I. Moffatt, Expansions for the Bollobás-Riordan Polynomial of Separable Ribbon Graphs, *Ann. Comb.* **15** (2011), 675–706.

[11] T. Krajewski, V. Rivasseau and F. Vignes-Tourneret, Topological graph polynomials and quantum field theory, Part II: Mehler kernel theories, *Ann. Henri Poincare* **12** (2011), 483.

[12] T. Krajewski, V. Rivasseau, A. Tanasa and Z. Wang, Topological Graph Polynomials and Quantum Field Theory, Part I: Heat Kernel Theories, *J. Noncommutative Geom.* **4** (2010), 29.

[13] F. Vignes-Tourneret, The multivariate signed Bollobás-Riordan polynomial, *Discrete Math.* **309** (2009), 5968–5981.