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Some graph theoretical aspects of generalized truncations

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Abstract

A broader definition of generalized truncations of graphs is introduced followed by an exploration of some standard concepts and parameters with regard to generalized truncations.

1 Introduction

Truncations of Platonic and Archimedean solids were studied by the ancient Greeks. It is worth observing the use of the term "solid" when considering truncations. The act of slicing off a corner of a solid allows an immediate and intuitive understanding of what a truncation produces. The skeletons of these solids, that is, the graphs formed by the vertices and edges of these solids then inherit an obvious truncation. This suggests that a notion of truncation may be applied to arbitrary graphs. However, some care needs to be exerted when extending the notion of truncation to arbitrary graphs for the following reason. Upon truncating a vertex of a solid, a new face whose boundary is a cycle is formed in a straightforward manner. Thus, the temptation for an arbitrary graph would be to somehow join the ends of the dangling edges so that a cycle is formed. Indeed, this has been the case in some instances where truncation has been employed, but other graphs have been inserted as well. We now provide a brief discussion of some of the history in spite of delaying the precise definition of a generalized truncation.

Sachs [12] seems to be the first modern graph theorist to have used truncation to obtain graphs with specific properties. He did not restrict the replacement graphs to be cycles, but did use the same graph for each replacement, and used a Hamilton cycle in each to organize the edges between the replacement graphs. His work was then extended by Exoo and Jajcay in [9]. The gap between those two papers is essentially fifty years.

Perhaps the best known graph truncation is the cube-connected cycles graph introduced in [10]. It is obtained by replacing each vertex of the n-dimensional

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cube with an n-cycle. The resulting graph is trivalent and has cube-like properties. Closely related to this is the truncation that replaces each vertex of an arc-transitive graph with a cycle in such a way that a trivalent vertex-transitive graph is obtained. This is exploited nicely in [8, 9] and elsewhere. Another paper dealing with replacing vertices by cycles is [7].

When the replacement graphs are cycles, if the order of the vertices along the inserted cycles is not handled with some care, desirable properties of the original graph may be lost. This problem is addressed in [2] by using complete graphs for the replacements. Using complete graphs for the replacements also is used in [5] to produce connected Cayley graphs that do not have Hamilton decompositions. Generalized truncations also appear several times in [3]. They are used in articles about graph expanders under the name zig zag product (for example, see [11]).

The purpose of this paper is the introduction of a much broader definition of generalized truncations of graphs and an exploration of some standard graph parameters in this setting. We believe there is considerable scope for research in this topic and include eight research problems we encountered.

The terms *reflexive* and *multigraph* are used if loops and multiple edges, respectively, are allowed. (For our purposes, an edge of multiplicity k is viewed as k distinct edges having the same end vertices.) Thus, a graph has neither loops nor multiple edges. We use $V(X)$ to denote the set of vertices of a reflexive multigraph X and $E(X)$ to denote the set of edges. The *order* of X is $|V(X)|$ and the *size* of X is $|E(X)|$. Finally, the *valency* of a vertex u, denoted $\mathrm{val}(u)$, is the number of edges incident with u, where a loop contributes 2 to the valency.

Given a reflexive multigraph X, a *generalized truncation* of X is obtained as follows via a two-step operation. The first step is the *excision step.* Let M denote an auxiliary matching (no two edges have a vertex in common) of size $|E(X)|$. Let $F : E(X) \to M$ be a bijective function and for $uv \in E(X)$, label the ends of the edge $F(uv)$ with u and v. Let M_F denote the vertex-labelled matching thus obtained. So M_F represents the edges of X completely disassembled. Note that a loop at a vertex $v \in V(X)$ produces an edge in M_F with both end vertices labelled v.

The second step is the *assemblage step.* For each $v \in V(X)$, the set of vertices of M_F labelled with v is called the *cluster at* v and is denoted $\mathrm{cl}(v)$. Insert an arbitrary graph on $\mathrm{cl}(v)$. (We remind the reader that graphs have neither loops nor multiple edges.) The inserted graph on $\mathrm{cl}(v)$ is called the *constituent graph at* v and is denoted $\mathrm{con}(v)$. The resulting multigraph

$$M_F \cup_{v \in V(X)} \mathrm{con}(v)$$

is a *generalized truncation* of X. We usually think of the labels on the vertices of M_F as being removed following the assemblage stage, but there are many times when the labels are useful in the exposition. We use $\mathrm{TR}(X)$ to denote a generalized truncation of the reflexive multigraph X.

Truncations arise via action involving the edges incident with a vertex. Consequently, isolated vertices (an isolated vertex has no incident loops or edges) are useless and we make the important convention that the reflexive multigraphs from which we are forming generalized truncations do not have isolated vertices. This will not be mentioned in the subsequent material, but is required for the validity of a few statements. Note that we claim that a generalized truncation may be a multigraph. This issue is addressed in the next section.

A few words about "style" are in order. There are two styles we recognize: local theorems and global theorems. Some discussion and two examples should clarify the distinction we are trying to make.

A *local theorem* is a result that is achieved by considering only the constituent graphs. A *global theorem* is a result that requires accounting for the structure of the multigraph X in carrying out the construction producing a generalized truncation. This description is admittedly a little fuzzy so let's consider two examples arising later in the paper.

Theorem 4.2 is a local theorem even though the hypotheses require that X be eulerian. We consider it local because once we start with an eulerian multigraph the subsequent construction requires only that we build constituents so that every vertex has odd valency. The structure of X has nothing to do with constructing the constituents. On the other hand, Theorem 3.1 is global because the choices for edges for the constituents depend heavily on the structure of X.

2 Some Characterizations

According to the definition above, a generalized truncation of a reflexive multigraph X may be a multigraph. This follows because a loop in X generates an edge e in M_F with the same vertex label on each end vertex of e. Thus, another edge may be added in the assemblage stage between these two vertices with the same label resulting in an edge of multiplicity 2. Thus, loops do not arise in the assemblage stage and multiple edges may arise but only if X has loops.

A natural question to ask is which multigraphs are generalized truncations of a reflexive multigraph. One obvious fact is that a generalized truncation contains a perfect matching, but this is not sufficient as we shall see. Given a multigraph X and a set of edges $E' \subseteq E(X)$, we use $X \setminus E'$ to denote the submultigraph obtained from X by removing the edges in E'.

Theorem 2.1 *A multigraph Y is a generalized truncation of a reflexive multigraph if and only if Y contains a perfect matching M such that $Y \setminus M$ is a graph.*

PROOF. If Y is a generalized truncation of some reflexive multigraph X, then it contains the edges of M_F and this forms a perfect matching in Y. If we remove the edges of M_F from Y, the resulting submultigraph $Y \setminus M_F$ is a graph by definition because the constituents partition the vertex set of $Y \setminus M_F$.

For the other direction, let Y be a multigraph containing a perfect matching M such that $Y \setminus M$ is a graph. Let $A_1, A_2, \ldots, A_t$ be the components of $Y \setminus M$.

Perform a contraction on Y by contracting each component A_i, $i = 1, 2, \ldots, t$, to a single vertex. Then remove every loop corresponding to the edges of $E(Y) \setminus M$. The resulting reflexive multigraph X has Y as a generalized truncation. □

There are some facts we may derive from Theorem 2.1 and its proof. We state them as separate corollaries for clarity and as an algorithm. Note that a multiple edge appears in $\mathrm{TR}(X)$ only when there is an edge of M_F whose end vertices have the same label, that is, the edge of M_F arose from a loop in X. Moreover, because we insert graphs during the assemblage stage, no edge in $\mathrm{TR}(X)$ may have multiplicity 3 or more. From the theorem we see that the distinct edges of multiplicity 2 in $\mathrm{TR}(X)$ must not share any vertices. This proves the following corollary which actually is a reformulation of Theorem 2.1.

Corollary 2.2 *A multigraph Y is a generalized truncation of some reflexive multigraph X if and only if Y has no edges of multiplicity bigger than* 2*, and there is a perfect matching in Y whose edge set intersects every edge of multiplicity* 2.

Theorem 2.1 informs us when a multigraph is a generalized truncation of a reflexive multigraph but we now restrict ourselves to multigraphs for the following reason. If Y is a generalized truncation of a reflexive multigraph X of size m, then Y clearly is a generalized truncation of the reflexive multigraph with a single vertex and m loops. This follows because every vertex of the m-matching arising in the excision stage has the same label which enables use to insert any graph on the $2m$ vertices. Because of this we now exclude consideration of loops, that is, we consider only generalized truncations arising from multigraphs and graphs. Thus, the generalized truncations themselves always are graphs.

Definition 2.3 Given a graph Y, define the *source of* Y, denoted $\mathrm{src}(Y)$, by

$$\mathrm{src}(Y) = \{X : Y \text{ is a generalized truncation of } X \text{ and } X \text{ is a multigraph}\}.$$

Definition 2.4 A perfect matching M in a graph Y is called *isolating* if no edge of M has both end vertices in the same component of $Y \setminus M$.

The proof of one direction of Theorem 2.1 is algorithmic so that we list the steps for finding a multigraph in $\mathrm{src}(Y)$.

Step 1. Find an isolating perfect matching M in Y. If there is none, then Y is not a generalized truncation of a multigraph,

Step 2. If there is an isolating perfect matching M in Y, let $A_1, A_2, \ldots, A_t$ be the components of $Y \setminus M$.

Step 3. Contract each set A_i, $i = 1, 2, \ldots, t$, in Y to a single vertex and remove all the loops formed. The remaining multigraph X belongs to $\mathrm{src}(Y)$.

It is natural to wonder when src(Y) contains a graph. This does impose an additional restriction on the isolating perfect matching M. Namely, there cannot be two edges of M whose end vertices are in the same pair of distinct components A_i and A_j. This proves the following corollary.

Corollary 2.5 *The graph Y is a generalized truncation of a graph X if and only if Y contains an isolating perfect matching M such that there are no two edges of M having their end vertices in the same pair of distinct components of $Y \setminus M$.*

It is easy to see from the definition that in general a given reflexive multigraph has many generalized truncations. The other direction is more interesting and we state a general problem that is wide open.

Research Problem 1: What can we say about src(Y) for various families of graphs?

With regard to Research Problem 1, we can determine the graphs Y that have a unique source. As a first step we prove the following lemma.

Lemma 2.6 *If Y is a graph for which $|\mathrm{src}(Y)| = 1$, then Y is connected.*

PROOF. Let Y be a graph for which $\mathrm{src}(Y) \neq \emptyset$ and Y is not connected. We know that Y has an isolating perfect matching M so that M restricted to each component Γ of Y is an isolating perfect matching for Γ. We then obtain a multigraph X_Γ which is a source for Γ.

The disconnected multigraph X formed by the union of the X_Γs over the components of Y belongs to src(Y). If we now amalgamate two components of X at a single vertex, then this yields another element of src(Y) and the result follows. □

Let αK_n denote the complete multigraph for which every edge has multiplicity α. When $\alpha = 1$, simply write K_n. In general, a *complete multigraph* is a multigraph in which every pair of distinct vertices is joined by at least one edge and the multiplicities may vary over the various edges.

Lemma 2.7 *If Y is a graph that is a generalized truncation with a unique source X, then X is a complete multigraph.*

PROOF. Let Y' be a graph that is a generalized truncation and let $X' \in \mathrm{src}(Y')$. If there are two vertices $u, v \in V(X')$ not joined by an edge, then we may identify u and v to obtain another multigraph of smaller order in src(Y'). The result now follows. □

Theorem 2.8 *Let Y be a graph that is a generalized truncation. If Y has a unique isolating perfect matching M and there is at least one edge of M joining any two components of $Y \setminus M$, then $|\mathrm{src}(Y)| = 1$.*

Proof. Each component of $Y \setminus M$ corresponds to the same vertex label on the ends of the edges of M incident with vertices of the component. Because there is at least one edge of M between two distinct components and a source multigraph has no loops, the labels on the vertices of the different components are distinct. The result now follows. □

Using Theorem 2.8, we see that the unique source of the cartesian product of C_3 and K_2 is $3K_2$. On the other hand, the hypothesis that Y has a unique isolating perfect matching is not necessary because the 4-cycle C_4 has two isolating perfect matchings even though the unique source is $2K_2$. Finally, the cartesian product of P_4, the path of order 4, and K_2 has an isolating perfect matching that yields $4K_2$ as a source, and it has an isolating perfect matching that yields the multipath $2P_3$ as a source. From this it is seen that determining the isolating perfect matchings is a key towards progress on the research problem.

3 Connectivity

A fundamental question is when is a generalized truncation of a multigraph connected? Let $\mathcal{L}(X)$ denote the line graph of a multigraph X. Let Y be a generalized truncation of X. The *projection* of Y into $\mathcal{L}(X)$ is the subgraph of $\mathcal{L}(X)$ obtained by including an edge joining two vertices with labels $\{x, y\}$ and $\{z, w\}$ if and only if there is an edge of Y joining a vertex of the edge with labels x, y and a vertex of the edge with labels z, w.

Theorem 3.1 *The generalized truncation Y of a multigraph X is connected if and only if the projection of Y into $\mathcal{L}(X)$ is connected.*

Proof. The trivial proof is left to the reader. □

The line graph $\mathcal{L}(X)$ provides an obvious constructive method for producing a connected generalized truncation of X. Choose a spanning tree T of $\mathcal{L}(X)$. For each edge of T, insert one edge between the corresponding edges of M_F. It is easy to see that the result is a generalized truncation of X which is itself a tree.

We follow the convention of not specifying the noun "vertex" when discussing the vertex connectivity of a multigraph, whereas, we employ the word "edge" when discussing the edge connectivity. That is, we shall use the notations k-connected and k-edge-connected. Denote the connectivity and edge connectivity of a multigraph X by $\kappa(X)$ and $\kappa'(X)$, respectively.

The following material on connectivity applies the results of the classical theorems by Menger that tell us that the minimum number of vertices that must be deleted from a multigraph X in order to separate two vertices $u, v \in V(X)$ equals the maximum number of internally disjoint paths in X whose terminal vertices are u and v. The edge analogue replaces "number of vertices" with "number of edges," and "internally disjoint" with "mutually edge-disjoint." Thus, a multigraph X is

k-connected if and only if every pair of distinct vertices is joined by k internally disjoint paths, and is k-edge-connected if and only if every pair of distinct vertices is joined by k mutually edge-disjoint paths. That is why the following proofs talk about paths joining vertices.

Theorem 3.2 *If Y is a generalized truncation of a multigraph X, then $\kappa'(Y) \leq \kappa'(X)$.*

PROOF. It is clear that if X is disconnected, then every generalized truncation of X is disconnected. So assume X is connected and consider a minimum edge cut $\mathcal{E}$. The multigraph $X \setminus \mathcal{E}$ has two components. Let A be the vertices of one component and B be the vertices of the other component. It is clear that the only edges of any generalized truncation Y of X which may have a label from A and a label from B are the edges of the matching in M_F arising from $\mathcal{E}$. Thus, these edges separate Y into at least two components. The result now follows. □

The interesting problem that now arises is how we guarantee that a multigraph X and a generalized truncation of X have the same edge connectivity. The next lemma is useful for subsequent results but first we have a definition followed by a discussion of a method to be employed frequently.

Definition 3.3 A generalized truncation is said to be *cohesive* when every constituent is connected.

Given a path or a cycle in a graph X, we now discuss how to expand it to a path or cycle in $Y = \mathrm{TR}(X)$. Let uvw be three successive vertices in a path P in X. The edges uv and vw are in M_F and the two occurrences of v give rise to two distinct vertices $v(x)$ and $v(y)$ in $\mathrm{con}(v)$ which are the ends of the edges labelled with v. If there is a path in $\mathrm{con}(v)$ from $v(x)$ to $v(y)$, then we can add this path to the edges of M_F that arise from P. If we are able to do this for each constituent, we obtain a path in Y based on P. We call this *an expansion of* P to Y. It is obvious what we mean by an expansion of a cycle.

Lemma 3.4 *Let Y be a cohesive generalized truncation of a multigraph X. If $\mathcal{E}$ is an edge cut of Y using only edges from M_F, then the edges of X corresponding to the edges of $\mathcal{E}$ form an edge cut of X.*

PROOF. Let Y and $\mathcal{E}$ be as hypothesised. There is an edge of $\mathcal{E}$ whose end vertices x and y are in different components of $Y \setminus \mathcal{E}$ because $\mathcal{E}$ is an edge cut. Also, x and y belong to different constituents because the edges of $\mathcal{E}$ belong to M_F. Let $x \in \mathrm{con}(u)$ and $y \in \mathrm{con}(v)$, respectively.

Let $\mathcal{E}'$ be the edges in X corresponding to the edges of $\mathcal{E}$. Assume that $\mathcal{E}'$ is not an edge cut of X. Then there is a path P in $X \setminus \mathcal{E}'$ whose end vertices are u and v. The edges of P belong to $E(X) \setminus \mathcal{E}'$. Hence, the corresponding edges form a matching in $Y \setminus \mathcal{E}$ and successive edges share a label, say w. There is a path in con(w) joining the two vertices with the same label. This yields a path in $Y \setminus \mathcal{E}$ joining x and y which is a contradiction. Therefore, $\mathcal{E}'$ is an edge cut in X as claimed. □

Definition 3.5 A generalized truncation is called *complete* if every constituent graph is complete.

Theorem 3.6 *If X is a k-edge-connected multigraph, $k \geq 2$, then the complete generalized truncation Y of X is k-edge-connected.*

Proof. Let Y be a generalized truncation of X. Note that every constituent graph has order at least k because X is k-edge-connected.

First choose two vertices x and y in the same constituent con(v). If $|\text{con}(v)| > k$, then it is trivially the case that there are k mutually edge-disjoint paths whose terminal vertices are x and y. Hence, we assume that $|\text{con}(v)| = k$.

Because the latter subgraph is complete, we may choose the edge xy and the 2-paths xzy, as z runs through the remaining vertices of con(v), to obtain $k-1$ mutually edge-disjoint paths whose terminal vertices are x and y. If we find an additional path from x to y that is edge-disjoint from the other paths, then we shall have shown that an edge-separating set for the vertices x and y has cardinality at least k.

Let uv and wv be the two edges of X giving rise to the vertices x and y in con(v). There are two edge-disjoint paths joining u and w in X because X is 2-edge-connected. If one of the paths does not contain v, then the expansion of this path produces a path from x to y in Y that uses none of the edges of the initial $k-1$ paths.

On the other hand, if there is a path in X from u to w containing v and using neither of the edges uv nor vw , then the corresponding edges incident with vertices of con(v) are incident with two vertices distinct from x and y. Thus, we see there is an expansion of such a path which is edge-disjoint from the other paths.

If one of the paths from u to w contains the edge uv and the other contains wv, then there is a trail from u to w containing v but neither of the edges under discussion. This reduces to a preceding subcase because the trail contains a path from u to w.

When x and y lie in different constituents con(u) and con(v), respectively, the existence of k edge-disjoint paths joining them in Y is easy to establish. There are k edge-disjoint paths in X whose terminal vertices are u and v. Use expansion to obtain k edge-disjoint paths in Y from con(u) to con(v). We then may use edges in each of the constituents to make the terminal vertices of each path x and y because the constituents are complete graphs. This completes the proof. □

Corollary 3.7 *If X is a k-regular, k-edge-connected multigraph, $k \geq 2$, then a generalized truncation Y of X is k-edge-connected if and only if it is complete.*

Proof. If every constituent is complete, then Y is k-edge-connected by Theorem 3.6. If there is a constituent graph con(v) which is not complete, then there are two vertices x and y of con(v) which are not adjacent. This implies that val(x) $< k$ in Y. This, in turn, implies $\kappa'(Y) \leq k-1$. By the contrapositive, if Y is k-edge-connected, then each constituent graph is complete. □

Theorem 3.8 *If X is a k-connected multigraph, $k \geq 2$, then a complete generalized truncation Y of X is k-connected.*

Proof. Let Y be a complete generalized truncation of the k-connected multigraph X. First consider two vertices x and y of Y which belong to constituents con(u) and con(v), $u \neq v$, respectively. There are k internally disjoint paths from u to v in X. Expanding the paths gives us k mutually vertex-disjoint paths from vertices of con(u) to vertices of con(v) in Y. We then use edges of each of the constituents to obtain k internally disjoint paths from x to y and may do so because each constituent is complete.

Suppose now that x and y belong to the same constituent con(v). If val(v) $> k$ in X, then there are trivially at least k internally disjoint paths joining x and y in con(v) because it is a complete graph. So we may assume that $|\mathrm{con}(v)| = k$.

There are $k-1$ internally disjoint paths joining x and y in con(v) and we need to find one more path that is internally disjoint from the $k-1$ paths. Let $u'x$ and $w'y$ be the edges of Y incident with x and y such that $u' \in \mathrm{con}(u)$ and $w' \in \mathrm{con}(w)$, where u, v and w are distinct. Because X is 2-connected, there is a path in X missing the vertex v. Extending this path gives a path Q in Y from a vertex of con(u) to a vertex of con(w) not containing any vertex of con(v). We may then use edges in the two constituents, if necessary, to obtain a path in Y from u' to w'. Then adding the edges $u'x$ and $w'y$ gives the desired path in Y completing the proof. □

The proof of the following corollary is easy and shall not be given.

Corollary 3.9 *If X is a k-connected k-regular graph, then a generalized truncation of X is k-connected if and only if every constituent graph is complete.*

Research Problem 2. Determine conditions on the original multigraph X and the constituent graphs of a generalized truncation TR(X) that determine the connectivity and/or the edge-connectivity of TR(X).

4 Eulerian Truncations

Recall that an *Euler tour* in a multigraph is a closed trail that covers each edge precisely once. A multigraph X is *eulerian* if it possesses an Euler tour. Also recall the following well-known theorem of Euler.

Theorem 4.1 *A connected multigraph X is eulerian if and only if every vertex has even valency.*

As we shall soon see, determining when a generalized truncation Y is eulerian is straightforward.

Theorem 4.2 *Let X be a connected multigraph. Every component of a generalized truncation Y of X is eulerian if and only if X is eulerian and every constituent has only vertices of odd valency.*

Proof. Let X be an eulerian multigraph so that every vertex of X has even valency. This implies that every constituent $\mathrm{con}(u)$ has even order. The valency of a vertex $x \in \mathrm{con}(u)$ in Y is one plus its valency in $\mathrm{con}(u)$. Thus, if every vertex in every constituent has odd valency in the constituent, then every vertex has even valency in Y. Hence, every component of Y is eulerian.

On the other hand, if every component of Y is eulerian, then every vertex has even valency in Y. This implies that every vertex has odd valency in its constituent. This implies that every constituent has even order which implies that X is eulerian because it is given that X is connected. □

The preceding is a local theorem because we need only consider each constituent in order to achieve the conclusion. However, considering the constituents individually does not guarantee that the generalized truncation Y itself is eulerian because it may not be connected. So we need to consider the structure of X in order to obtain a generalized truncation Y that is eulerian.

5 Hamiltonicity

There are three hamiltonicity problems we consider in this section. The first deals with the hamiltonian problem, that is, does a graph contain a Hamilton cycle. The hamiltonian problem is one of the earliest problems arising in graph theory, and is one that has been widely studied in many contexts.

It is apparent that there will be no easy answers regarding the existence of Hamilton cycles in generalized truncations. We may safely say this because there are many ways for a Hamilton cycle to contain all the vertices of a constituent graph. For example, it might enter a constituent once and pass through all the vertices of the constituent before exiting. On the other hand, it might enter and exit multiple times. In any case, a Hamilton cycle in a generalized truncation partitions a constituent into a collection of vertex-disjoint paths covering the vertices of the constituent.

The following theorem appears in [2].

Theorem 5.1 *If $\mathrm{TR}(X)$ is a complete generalized truncation of a connected multigraph X, then $\mathrm{TR}(X)$ is hamiltonian if and only if X contains a spanning eulerian subgraph.*

The preceding theorem is special because each of the constituent graphs is complete suggesting the following question.

Research Problem 3. Determine conditions on the source multigraph and constituents that imply a generalized truncation is hamiltonian.

The second hamiltonicity problem we consider is Hamilton connectivity. Recall that a multigraph X is *Hamilton-connected* if for every pair of vertices u and v in X there is a Hamilton path in X whose terminal vertices are u and v. Similarly, a bipartite multigraph X with parts of the same cardinality is *Hamilton-laceable* if for any two vertices in opposite parts there is a Hamilton path in X from one to the other.

The only Hamilton-connected graph with a vertex of valency 1 is K_2. The generalized truncation of K_2 is K_2 itself so that every generalized truncation of a Hamilton-connected graph with a vertex of valency 1 is Hamilton-connected.

The Hamilton-connected multigraphs with a vertex of valency 2 are $2K_2$ (an edge of multiplicity 2) and the multigraphs obtained from K_3 by replacing one of its edges with arbitrarily many edges, that is, it has an arbitrary multiplicity. The complete generalized truncations of these multigraphs are not Hamilton-connected so that there are no Hamilton-connected generalized truncations of any Hamilton-connected multigraph with a vertex of valency 2.

From the preceding comments we may assume that the multigraphs under consideration have minimum valency at least 3. Note that once a multigraph has a vertex of valency 3 or more, then its complete generalized truncation is not bipartite. Hence, the complete generalized truncation of a bipartite graph may not be bipartite. Therefore, bipartiteness may not be a barrier to generalized truncations being Hamilton-connected. For example, the complete bipartite $K_{3,3}$ graph is easily seen to be Hamilton-laceable. It turns out that its complete generalized truncation is Hamilton-connected.

Theorem 5.2 *A generalized truncation of the complete graph K_n, $n > 3$, is Hamilton-connected if every constituent graph is Hamilton-connected.*

PROOF. When $n = 4$, it is straightforward to verify that the complete truncation of K_4 is Hamilton-connected. Hence, we assume that $n \geq 5$ for the rest of the proof.

Let Y be a generalized truncation of K_n, $n > 4$, in which every constituent graph is Hamilton-connected. Let x and y be vertices of Y in different constituent graphs $\text{con}(u)$ and $\text{con}(v)$, respectively. Let $[u, w]$ be the edge of K_n such that x is the vertex of $\text{con}(u)$ corresponding to u and, similarly, let $[z, v]$ be the edge of K_n such that y is the vertex of $\text{con}(v)$ corresponding to v. We can find a Hamilton path P in K_n from u to v such that w is not the vertex following u on P, and z is not the vertex preceding v on P because $n \geq 5$.

It is now easy to find a Hamilton path from x to y by extending P. The vertex of $\text{con}(u)$ corresponding to the vertex u in P is $x' \neq x$. Because $\text{con}(u)$ is Hamilton-connected, there is a path from x to x' spanning all the vertices of $\text{con}(u)$. It is easy to use all of the vertices of the constituent graphs as we work along P because they are Hamilton-connected and the entering and departing vertices are distinct. The completion of the Hamilton path from x to y in Y in con(v) is done in the same way as the path was started in con(u).

Now let x and y both belong to con(u). Because con(u) is Hamilton-connected, there is a path Q from x to y in con(u) spanning the vertices of con(u). Let z be the predecessor of y on Q. Let $[u, w]$ and $[u, w']$ be edges of K_n corresponding to the vertices y and z in con(u).

There is a Hamilton cycle in K_n containing the edges $[u, w]$ and $[u, w']$. We then perform cycle expansion in the obvious way to obtain a Hamilton path in Y whose terminal vertices are x and y. □

The conditions for Theorem 5.2 are special and suggest two further problems.

Research Problem 4. If X is a Hamilton-connected or Hamilton-laceable multigraph with minimum valency at least 3, is the complete generalized truncation of X Hamilton-connected?

Research Problem 5. What conditions on the source multigraph X and the constituents of a generalized truncation Y of X guarantee that Y is Hamilton-connected?

The final problem we consider deals with Hamilton decompositions. A regular graph is *Hamilton-decomposable* if its edge set can be partitioned into Hamilton cycles when the valency is even, and into Hamilton cycles and a single perfect matching when the valency is odd. For the next result we require two facts that we encapsulate as a lemma. These facts are based on the Walecki decompositions given in [1]. The first fact is presented directly in [1]. The second fact is obtained by removing the diameter edge from each Hamilton cycle in the decomposition of a complete graph of odd order into Hamilton cycles which also is given in [1].

Lemma 5.3 *Let X be a complete graph of order n.*

(i) *If n is even, then X has a decomposition into $n/2$ Hamilton paths.*

(ii) *If n is odd, then X has a decomposition into $(n-1)/2$ Hamilton paths and a matching with $(n-1)/2$ edges.*

Theorem 5.4 *If X is a Hamilton-decomposable graph, then the complete generalized truncation of X also is Hamilton-decomposable.*

Proof. Let X have a decomposition into Hamilton cycles $H_1, H_2, \ldots, H_n$. Let Y denote the complete generalized truncation of X. Note that each constituent of Y has order $2n$. The edges of H_i in Y intersect each constituent in two vertices. Hence, the $2n$ vertices of a given constituent are partitioned into n 2-sets. By Lemma 5.3(i), we may decompose a constituent into n spanning paths such that the end vertices of each path belong to the same 2-set. It now is obvious that we may expand each Hamilton cycle H_i of X into a Hamilton cycle in Y using the spanning paths of the constituents. This decomposes Y into n Hamilton cycles.

When X has a decomposition into n Hamilton cycles and a single perfect matching, we slightly modify the preceding construction. Each constituent now has odd order so we use Lemma 5.3(ii) to decompose the constituent into n spanning paths and an n-matching. The n-matching misses precisely one vertex of the constituent and we make sure that the missing vertex is the vertex which is incident with the matching edge of X that is incident with a vertex of the constituent. It is now easy to see how to complete the Hamilton decomposition of Y. □

There are other ways to obtain a Hamilton decomposition of a generalized truncation. For example, if we start with a spanning eulerian subgraph of valency 4 in X, we may use that to obtain a Hamilton cycle in Y. This suggests the following problem.

Research Problem 6. Find conditions on the source graph and the constituents that produce a Hamilton-decomposable generalized truncation.

6 Planarity

Planarity is another basic topic that has been studied extensively in graph theory. It is natural to consider which generalized truncations are planar. After the excision stage before any edges have been added to the constituents, the generalized truncation certainly is planar which suggests two questions. First, what can we say about planarity in terms of the number of edges we introduce in the constituents. Second, what can we say about planarity if we insist that the generalized truncation is cohesive. We now investigate the second question.

Lemma 6.1 *If X is a non-planar graph, then every cohesive generalized truncation of X is non-planar.*

Proof. Because X is non-planar, it has either $K_{3,3}$ or K_5 as a minor. If Y is a cohesive generalized truncation of X, then X is a minor of Y by contracting each constituent of Y to a single vertex and removing the loops. Thus, Y has either a $K_{3,3}$-minor or a K_5-minor as the minor relation is transitive. □

Because of Lemma 6.1, we now consider cohesive generalized truncations of planar graphs and describe a process that produces a planar cohesive generalized truncation. Let X be a plane graph, that is, it is given embedded in the plane with no edges crossing. Draw a small closed disc around each vertex of X so that none of the discs overlap. Remove the intersection of each edge with the interior of the discs surrounding its end vertices, and let the intersections of the edges with the boundaries of the discs be the end vertices of the fragments of the original edges.

After performing the preceding operations, we have the perfect matching $F(M)$ embedded in the plane. Recall that a graph is *outerplanar* if it has an embedding in the plane so that every vertex belongs to the boundary of the infinite face. If we now insert an outerplanar graph for each constituent, it is clear that the resulting generalized truncation is planar. However, we shall now see that there are planar generalized truncations for which there are constituents that are not outerplanar. To get a handle on this we use the following result from [6].

Theorem 6.2 *A graph is outerplanar if and only if it contains no subgraph homeomorphic to $K_{2,3}$ or K_4.*

Consider Figure 1. Suppose that the vertices labelled 1 through 4 are the vertices of con(u) for a vertex u of valency 4 in a planar graph X and the graph depicted in the figure is a subgraph of a generalized truncation of X. These four vertices have been joined to form a constituent that is K_4. The crucial vertex here is 4 because the edge of M_F incident with 4 cannot pass through the edges of the 3-cycle formed by 1, 2 and 3. Hence, this edge must be the edge from 4 to the subgraph indicated by A. The edges of X incident with u corresponding to 1 and 3 may or may not be incident with vertices in A. Figure 1 has been drawn so that the edge corresponding to 1 also joins a vertex in A. This figure indicates how a planar generalized truncation of a planar graph may possess a constituent graph which is not outerplanar.

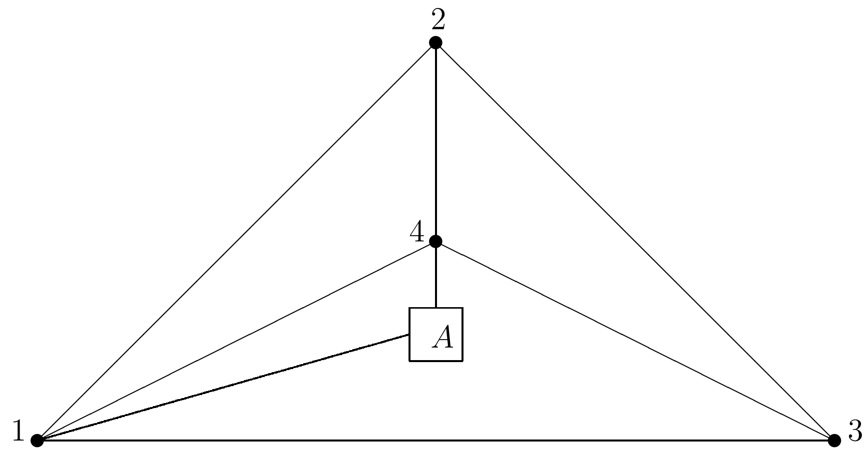

FIGURE 1

Let Y be a planar cohesive generalized truncation of a planar graph X. If we have a connected constituent con(u) that contains a subdivision of either K_4 or $K_{2,3}$, then there is some vertex labelled u such that the edge of M_F incident with it has its other end vertex, labelled $v \neq u$, in some face $\mathcal{F}$ of the constituent con(u). All vertices of con(v) must lie in the face $\mathcal{F}$ because every vertex of the boundary of $\mathcal{F}$ is labelled u and con(v) is connected.

Theorem 6.3 *If Y is a cohesive generalized truncation Y of a 2-connected planar graph X,then every constituent of Y is outerplanar.*

Proof. Let Y be a cohesive generalized truncation of a 2-connected planar graph X. Suppose that con(u) contains a a subdivision of either K_4 or $K_{2,3}$ for some vertex u in X. Note that this implies that the order of X is at least 5.

From the discussion preceding the statement of the theorem, there is a face $\mathcal{F}$ of Y containing all the vertices of con(v) for some $v \neq u$. There is a vertex y labelled u that lies in the exterior of the face $\mathcal{F}$. This implies that the other end vertex, say w, of the M_F edge incident with y lies in a face $\mathcal{F}'$ distinct from $\mathcal{F}$. Then con(w) must lie in $\mathcal{F}'$. Therefore, every path from v to w in X passes through u. This contradicts the fact that X is 2-connected and the conclusion follows. □

7 Colorings

We now consider vertex and edge colorings of generalized truncations. Recall that a *proper coloring* of a multigraph X is a coloring of the vertices so that adjacent vertices do not have the same color. Similarly, a *proper edge coloring* is a coloring of the edges so that adjacent edges do not have the same color. The *chromatic number* of X, denoted $\chi(X)$, is the fewest number of colors for which a proper coloring exists, and the *chromatic index*, denoted $\chi'(X)$, is the fewest number of colors for which a proper edge coloring exists.

A small hint of the kind of behavior that may occur is exemplified by the following. The graph K_3 has both chromatic number and chromatic index 3. The complete generalized truncation is the 6-cycle which has chromatic number and chromatic index 2. On the other hand, if X is a bipartite graph, then we need at least k colors to color the vertices of a complete truncation, where X has a vertex of valency k. So we may need to introduce many colors when we move from a graph with chromatic number 2 to a generalized truncation.

Vizing's well-known theorem tells us that the chromatic index of a graph equals the maximum valency or the maximum valency plus one. This, in turn, leads to a classification of graphs as follows. A graphs is *class* I if its chromatic index is equal to its maximum valency and is *class* II otherwise.

Theorem 7.1 *If X is a class* I *graph, then its complete generalized truncation also is class* I. *If X is a class* II *graph and its maximum valency is even, then its complete generalized truncation is class* I.

Proof. Let X be a graph whose maximum valency d is even and let its complete generalized truncation be Y. Every constituent of Y of order d admits a proper edge coloring with $d-1$ colors because d is even. Any constituent of order less than d admits a proper edge coloring with at most $d-1$ colors. The edges of Y with end vertices in different constituents form a perfect matching in Y. Color all of these edges with a single new color, thereby obtaining a proper edge coloring of Y with d colors. The maximum valency of Y is d so that Y is class I.

The preceding argument takes care of the case that the maximum valency of X is even leaving us with the case that the maximum valency d is odd, but from the hypotheses we know that X is then class I. Because X is class I, it has a proper edge coloring using d colors. In forming Y, retain the colors on the edges of M_F between the constituents. We now describe decompositions of the constituents into matchings which may be used to color the edges of the constituents so that we obtain a proper edge coloring of Y without introducing any new colors, thereby establishing that Y is class I.

The edges of M_F that are incident with the vertices of a given constituent $\text{con}(u)$ all have different colors because they arose from a proper edge coloring of X. It is well known that the edges of a complete graph of odd order m can be properly edge-colored with m colors so that each vertex misses a distinct color. If a constituent has odd order m, then properly edge color it with m colors so that each vertex misses the color of the edge from M_F incident with it. This properly colors the edges of $\text{con}(u)$ without introducing new colors.

If a constituent $\text{con}(v)$ has even order m, add an artificial vertex and color the edges of the new complete graph of order $m+1$ with $m+1$ colors as in the preceding case. Removing the artificial vertex leaves a proper edge coloring using $m+1$ colors and nothing is violated because $m+1 \leq d$. We now have a proper edge coloring of Y with d colors so that Y is class I. □

There is a notable missing possibility in Theorem 7.1, namely, X is class II and its maximum valency is odd. As is typical for a situation such as this, we look at the Petersen graph. It is not difficult to see that the complete generalized truncation Y of the Petersen graph is class II. Suppose this was not the case. Then Y would have a 1-factorization and the union of two of the 1-factors would form a 2-factor of Y whose components would be cycles of even length. Because a 2-factor must contain every vertex of Y, each cycle of the 2-factor must use all three vertices of a constituent when it passes through a constituent. Thus, the 2-factor of Y corresponds to a 2-factor of the Petersen graph. However, all the 2-factors in the Petersen graph consist of two 5-cycles which implies the only 2-factors in Y consist of two 15-cycles. A 15-cycle cannot be a cycle in the union of two 1-factors.

Research Problem 7. Characterize the class II generalized truncations of multigraphs.

Corollary 7.2 *Let X be a regular graph of valency d. If d is even or X is class* I*, then the complete generalized truncation Y of X admits a 1-factorization.*

Proof. The complete generalized truncation of X is regular of valency d and is class I by Theorem 7.1. This implies that each color class of edges must be a 1-factor. The result now follows. □

Research Problem 8. Determine conditions on the source multigraph and constituents so that a generalized truncation has a 1-factorization.

The spectrum problem for chromatic indices of generalized truncations of a given graph X is straightforward because the minimum generalized truncation is a perfect matching of size $|E(X)|$ for which the chromatic index is 1. The maximum value occurs for the chromatic index of the complete generalized truncation of X which is either the maximum valency of X or the maximum valency plus one. By adding one edge at a time and realizing the chromatic index stays the same or increases by one, it is easy to see that there are generalized truncations of X realizing all possible values between one and the upper bound. However, the problem takes on more interest if we restrict ourselves to cohesive generalized truncations.

Given a graph X, what is the minimum chromatic index of a cohesive generalized truncation of X? Once that is known, it is easy to see that all values from that point to the maximum possible value are achieved by a cohesive generalized truncation.

Theorem 7.3 *Let X be a multigraph with maximum valency $d > 2$. If the chromatic index of the complete generalized truncation of X is D, then for every k satisfying $3 \leq k \leq D$, there is a cohesive generalized truncation of X with chromatic index k.*

PROOF. The idea is to make the generalized truncation cohesive using as few edges as possible. The way to do this is to insert a spanning path on the vertices with the same label in $F(M)$ so that each constituent is a path. We then color the edges of the constituents with one or two colors and note that two colors are required because one of the constituents has order at least three. We then color the edges whose ends lie in different constituents with a third color giving us a cohesive generalized truncation with chromatic index 3.

We then add one edge at a time until reaching the complete generalized truncation. It is clear that we achieve a cohesive generalized truncation with chromatic index k for all k satisfying $3 \leq k \leq D$. □

Recall that Brooks' Theorem [4] states that the chromatic number of a graph X is bounded above by its maximum valency unless X is complete or an odd length cycle. This gives us a quick proof of the next result.

Theorem 7.4 *If X is a multigraph with maximum valency $d > 1$, then its complete generalized truncation Y satisfies $\chi(Y) = d$.*

PROOF. Let X be a multigraph satisfying the hypotheses and let Y be its complete generalized truncation. Then Y contains a clique of order d from which it follows that $\chi(Y) \geq d$. The result follows from Brooks' Theorem if we show that Y is neither an odd length cycle nor a complete graph. The order of Y is even so that it cannot be an odd length cycle. The order of X is at least two so that Y contains at least two constituents and there is at least one constituent $\text{con}(u)$ of order bigger than one. The edges between constituents form a perfect matching so that Y is not complete. □

Consider the spectrum problem for the chromatic numbers of generalized truncations for a fixed graph X. Theorem 7.4 provides an upper bound so that we want to determine the minimum chromatic number for a cohesive generalized truncation of

X. If we again use a spanning path for each constituent, then the maximum valency for Y is three except for a few exceptions. So Brooks' Theorem tells us the chromatic number for such a generalized truncation is at most 3 for the unexceptional graphs.

The exceptions arise if the maximum valency of X is 2. The complete generalized truncation of $2K_2$ is a 4-cycle, the complete generalized truncation of an n-cycle is a $2n$-cycle, and the complete generalized truncation of a path of length n is a path of length $2n-1$. All of these graphs have chromatic number 2. The following result follows from these comments and Theorem 7.4.

Theorem 7.5 *Let X be a multigraph with maximum valency $d > 1$. If $d = 2$, then every cohesive generalized truncation of X also has chromatic number 2. If $d > 2$, then for every k satisfying $3 \leq k \leq d$, there is a cohesive generalized truncation of X with chromatic number k.*

8 Conclusion

The topic of generalized truncations of reflexive multigraphs may be viewed as very old in the sense that a special version of it was studied by the ancient Greeks. A more general version has been introduced, studied somewhat and even then only in special circumstances. The general version presented in this paper is a further extension in what we see as a natural way to proceed.

The purpose of this paper is to encourage people to study the many possible directions in which the topic may proceed. We have only scratched the surface. If others pursue this topic, the authors will be pleased.

Finally, many of the results presented in this paper are contained in the honours thesis submitted by the second author in June 2020 to the University of Newcastle.

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