ONE-POINT EXTENSIONS OF FINITE INVERSIVE PLANES

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Dedicated to the memory of Alan Rahilly, 1947 – 1992

Abstract

A one-point extension of an egglike inversive plane of order n exists if, and only if, n is 2 or 3. Hence there are no 4–(18, 6, 1) designs and no 4–(66, 10, 1) designs.

An **inversive plane** of order n is a one-point extension of an affine plane of order n, that is, a $3\text{–}(n^2+1, n+1, 1)$ design.

An **ovoid** of $PG(3,q)$, $q > 2$, is a set of q^2+1 points, no three collinear. An ovoid of $PG(3,2)$ is a set of 5 points, no four coplanar. Every plane of $PG(3,q)$ meets an ovoid in either 1 or $q+1$ points. The incidence structure whose points are those of an ovoid in $PG(3,q)$, whose blocks are the plane sections of size $q+1$ of the ovoid, and whose incidence is given by set membership, is an inversive plane of order q. An inversive plane of this form is called **egglike**.

Theorem 1 [3, 6.2.14] An inversive plane of even order is egglike. In consequence, its order is a power of two.

A **one-point extension of an inversive plane** of order n is a $4\text{–}(n^2+2, n+2, 1)$ design.

Lemma If a $4\text{–}(n^2+2, n+2, 1)$ design exists, then $n = 2, 3, 4, 8$ or 13.

Proof: By the standard divisibility conditions [5,p.7] for designs, $n+2$ divides $n(n^2+1)(n^2+2)$. Hence $n+2$ divides 60. It follows that n is $2, 3, 4, 8, 10, 13, 18, 28$ or 58. By Theorem 1, n is $2, 3, 4, 8$ or 13. ■

Remarks

1. A 4–(6, 4, 1) design exists and is unique; it is complete. It is a one-point extension of the (egglike) inversive plane of order 2.

2. A 4–(11, 5, 1) design exists and is unique [5, §4.4]; it has automorphism group the Mathieu group M_{11} of degree 11. It is a one-point extension of the (egglike) inversive plane of order 3.

Theorem 2 If a 4–$(n^2+2, n+2, 1)$ design exists, and the residual structure of some point P is egglike, then n is 2 or 3.

Proof: A block B of the 4–$(n^2+2, n+2, 1)$ design not on P consists of $n+2$ points of the egglike inversive plane of order n, no 4 concircular. It follows that B is a set of $n+2$ points of $PG(3, q)$, no 4 coplanar. By [4, Theorems 21.2.4 and 21.3.8], it follows that $n+2 \leq \text{Max}(5, n)$, so that n is 2 or 3. ■

Corollary 1 [6] There are no 4–(18, 6, 1) designs.

Proof: Suppose there is a 4–(18, 6, 1) design. The residual structure at any point is an inversive plane of order 4 must be egglike, by Theorem 1. But this contradicts Theorem 2. ■

Corollary 2 [2] There are no 4–(66, 10, 1) designs.

Proof: An argument similar to that which proved Corollary 1 suffices. ■

It follows from the above that the only order for which the existence of an inversive plane is undecided is 13, and that, if a 4–(171, 15, 1) design exists, then the residual structure at any point is an inversive plane of order 13 which is not egglike.

Finally, note that in [1, 2.4] it was shown in another way that, if an inversive plane has a one-point extension, then it has order 2, 3 or 13.

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