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About paths with three blocks

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Abstract

We show that every oriented path of order $n \geqslant 4$ with three blocks, in which two consecutive of them are of length 1, is contained in every $(n+1)$-chromatic digraph.

1 Introduction

Digraphs considered here are finite having no loops, multiple edges or circuits of length 2. Let x and y be two vertices in a digraph D. The arc directed from x to y will be denoted by (x, y). We say that $xy \in E(G[D])$, where $G[D]$ is the underlying graph of D, if (x, y) or (y, x) is an arc in D. We denote by $N^+_D(x)$ (respectively, $N^-_D(x)$), the set of out-neighbors of x in D (respectively, the set of in-neighbors of x in D). The out-degree of x will be denoted by $d^+_D(x)$ and its in-degree by $d^-_D(x)$. A block of an oriented path P is a maximal directed sub-path of P.

In this paper, we are dealing with the following problem: which oriented path of order n is contained in any n-chromatic digraph. Havet and Thomassé [8] proved that every tournament of order n contains any oriented path of length $n-1$ except in three cases: the directed 3-cycle; the regular tournament on 5 vertices; and the Paley tournament on 7 vertices. In these cases it contains no antidirected path of length $n-1$.

In the general case, the situation is radically different. Gallai, Hasse, Roy and Vitaver [6, 7, 9, 10] proved that an n-directed path is contained in any n-chromatic digraph. Addario-Berry et al. [3] proved the same for any n-path with exactly two

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blocks. Due to a result of Burr we may show that any n-path is contained in any $(n-1)^2$-chromatic digraph [2]. Beyond paths with two blocks, no linear bound is established. As a first step in this direction, we prove in this paper that any $(n+1)$-chromatic digraph contains any n-path with three blocks in which two consecutive of them are of length 1. We denote by $P(k,l,r)$ an oriented path formed by k forward arcs followed by l backward arcs followed by r forward arcs. By considering a digraph D and its complement (the digraph obtained from D by reversing the orientation of all its arcs), it will suffice to prove that any $(n+1)$-chromatic digraph contains a path of type $P(n-3,1,1)$.

2 Maximal and final forest

An out-branching (respectively, in-branching) B is a digraph containing a vertex of in-degree (respectively, out-degree) 0, which is called the source (respectively, the sink) of B, and the other vertices are of in-degree (respectively, out-degree) 1.

The level of a vertex v in an out-branching B, denoted by $l_B(v)$, is the order of the unique directed path starting from the source of B and ending at v.

An out-forest F is a digraph in which each connected component is an out-branching. The level of a vertex v in an out-forest F, denoted by $l_F(v)$, is its level in the out-branching containing it. For $i \geqslant 1$, set $L_i(F) = \{v \in V(F) : l_F(v) = i\}$. We denote by $\ell(F)$ the maximum integer i such that $L_i(F) \neq \phi$. For all $v \in V(F)$, denote by P_v the unique directed path in F, starting from the source of the out-branching containing it and reaching v, and by $T_v(F)$ the sub-out-branching of F of source v.

Note that any digraph contains a spanning out-forest. Let F be a spanning out-forest of a digraph D. An arc $(u,v) \in E(D)$ is said to be a forward arc with respect to F if $l_F(u) < l_F(v)$; otherwise it is called a backward arc with respect to F. Addario-Berry et al. called a final forest of a digraph D each spanning out-forest F of D such that for any backward arc (u,v) with respect to F, the forest F contains a vu-directed path. A spanning out-forest of a digraph D is said to be maximal if $\sum_{v\in V(D)} l_F(v)$ is maximal. After introducing the concept of maximal forest, El Sahili and Kouider [4] proved that a maximal forest is a final forest.

It can be easily seen that if F is a final forest of a digraph D then $L_i(F)$ is stable in D for all $i \geqslant 1$, and consequently the number of levels in F should be at least $\chi(D)$.

Note that if D contains a Hamiltonian path, then this path is maximal and so is a final forest of D. Moreover, if F is a final forest of D, then the sub-forest F' of F induced by the vertices of levels at least k (respectively, at most k), $k \geqslant 1$, is a final forest of D', the sub-digraph of D induced by the vertices of F'. Also, if any leaf is removed from F, the remaining forest is final in the remaining digraph. But, in general, these properties may be not true for a maximal forest. In the following, we will need more characteristics for final and maximal forests that will be introduced in a sequence of lemmas.

Lemma 2.1. *Let D be a digraph with $d^-(v) \leqslant 2$ for all $v \in D$, and suppose that D contains a final forest F with no backward arcs with respect to F. Then there exists a proper 3-coloring of D such that all the vertices of $L_1(F)$ are of the same color.*

Proof. We establish the proof by induction on $v(D)$. It is trivial for $v(D) = 1$ and whenever $\ell(F) = 1$ and $v(D) \geqslant 1$. Now suppose that $v(D) \geqslant 2$ and let $v \in L_l(F)$ where $l = \ell(F) = \max\{i \in \mathbb{N}^* | L_i(F) \neq \phi\}$. Then all vertices of $D - v$ are of in-degree at most 2, and $F - v$ is a final forest of $D - v$ which contains no backward arc with respect to it. By induction, there exists a proper 3-coloring c' of $D - v$ such that $|c'(L_1(F - v))| = 1$. Since $v \in L_l(F)$ and D has no backward arc with respect to F, it follows that $d(v) = d^-(v) \leqslant 2$; hence c' can be extended to a proper 3-coloring c such that $|c(L_1(F))| = 1$. □

Lemma 2.2. *Let F be a maximal forest and let x be a leaf of F such that $(x, y) \in E(D)$ with $y \in L_1(F)$. Then $T_y(F) = P_x$.*

Proof. If the theorem is false, then $T_y(F) \setminus P_x \neq \phi$. Set $P_x = v_1 \dots v_s$, where $v_s = x$. Since F is a maximal forest and (x, y) is a backward arci, we have $v_1 = y$. Consider the spanning out-forest of D, $F' = F + (x, y) - (w, x)$, where w is the in-neighbor of x in F. Then $l_{F'}(x) = l_F(x) - (s - 1)$, $l_{F'}(z) = l_F(z) + 1$ for all $z \in T_y(F) \setminus \{x\}$, and $l_{F'}(z) = l_F(z)$ for all $z \notin T_y(F)$. Thus,

$$\begin{aligned}
\sum_{z \in D} l_{F'}(z) &= l_{F'}(x) + \sum_{z \in P_x \setminus x} l_{F'}(z) + \sum_{z \in T_y(F) \setminus P_x} l_{F'}(z) + \sum_{z \notin T_y(F)} l_{F'}(z) \\
&= l_F(x) - (s-1) + \sum_{z \in P_x \setminus x} (l_F(z) + 1) + \sum_{z \in T_y(F) \setminus P_x} (l_F(z) + 1) \\
&\quad + \sum_{z \notin T_y(F)} l_F(z) \\
&= l_F(x) - (s-1) + \sum_{z \in P_x \setminus x} l_F(z) + (s-1) + \sum_{z \in T_y(F) \setminus P_x} l_F(z) \\
&\quad + |T_y(F) \setminus P_x| + \sum_{z \notin T_y(F)} l_F(z) \\
&\geqslant \sum_{z \in D} l_F(z) + 1,
\end{aligned}$$

a contradiction. □

Note that the above proof indicates that F' is a maximal forest.

Lemma 2.3. *Let F be a maximal forest of a digraph D, and let x be a leaf in F and $y \in L_1(F)$. If $(x, y) \in E(D)$, then $uv \notin E(G[D])$ for all $u \in V(C)$ and $v \notin V(C)$ with $l_F(v) \leqslant l_F(x)$, where C is the circuit formed by P_x and (x, y).*

Proof. If the theorem is false, let $u \in C$ and $v \notin C$ such that $uv \in E(G[D])$ with $l_F(v) \leqslant l_F(x)$. Note that the maximal forest $F' = F + (x, y) - (w, x)$, where w is the in-neighbor of x in F, can be viewed as the forest obtained from F by rotating the circuit C exactly once. The same argument proves that the forest obtained from F after rotating C any number of times is maximal. So we rotate C until we reach the maximal forest F' in which u and v are in the same level, a contradiction. □

Note that if the digraph D considered in the above lemma is connected and $l_F(x) = \ell(F)$, then D is Hamiltonian due to the fact that $uv \notin E(G[D])$ for all $u \in C$ and $v \notin C$.

3 The main result

Theorem 3.1. *Let D be an $(n+1)$-chromatic digraph, $n \geqslant 4$. Then D contains a path $P(n-3, 1, 1)$.*

Proof. For the case $n = 4$, dealing with the existence of the antidirected path $P(1, 1, 1)$ in a 5-chromatic digraph, we may prove even more: the existence of such a path in a 4-chromatic digraph based on [5]. In fact, consider a 4-chromatic digraph D and let D_3 be the sub-digraph of D induced by the vertices of degree at least 3. Suppose to the contrary that D contains no $P(1, 1, 1)$; then D_3 contains no acyclic triangle. Indeed, if x, y, z is an acyclic triangle in D_3 such that (x, y), (x, z) and $(y, z) \in E(D_3)$, then let $w \notin \{x, y, z\}$ such that $yw \in E(G[D])$. Note that w exists since $d(y) \geqslant 3$. Thus, either $xzyw$ or $wyxz$ is a $P(1, 1, 1)$, a contradiction. Then by Beineke [1], D_3 is a line digraph. El Sahili in [5] proved that such a digraph is of chromatic number 3, a contradiction.

In what follows, we may suppose that $n \geqslant 5$. Let D be a digraph with chromatic number $\chi(D) = n+1$. We are going to establish our proof by contradiction. We may suppose that D is connected. Suppose that D contains no path $P(n-3, 1, 1)$ and let F be a maximal forest of D. Set $l = \ell(F)$; then $l \geqslant n+1$. Let H be the sub-digraph of D induced by the vertices of level at least $n-2$. Set $L = \{x \in H \mid x \text{ is a leaf in } F\}$ and let $H' = D[L]$.

Claim 1. *H' is an in-forest.*

Proof. H' contains no backward arc with respect to F, since each backward arc (x, y) generates a yx-directed path and so $d^+_F(y) \geqslant 1$, a contradiction. Thus it contains no circuit. Now H' contains no vertex x such that $d^+_{H'}(x) \geqslant 2$, since otherwise let x be such a vertex and let $\{y, z\} \subseteq N^+_{H'}(x)$. Then $P_y \cup (x, y) \cup (x, z)$ contains a path $P(n-3, 1, 1)$, a contradiction. Now one can easily prove that H' contains no cycle, since any non-directed cycle contains a vertex of out-degree at least 2. Thus H' is a forest. But the out-degree of all vertices is at most 1, so H' is an in-forest. □

Consequently H' is a bipartite digraph. Set $V(H') = S_1 \cup S_2$ such that S_i is stable for $i = 1, 2$ and S_1 contains all the sinks of H'. Note that any $x \in S_2$ has an out-neighbor in S_1. Let $M = H - S_1$ and let F' be the sub-forest of F induced by $V(M)$.

Clearly, F' is a final forest of M. Set $S_1^{n-2} = S_1 \cap L_{n-2}(F)$, $S_1^{n-1} = S_1 \cap L_{n-1}(F)$ and $S_1^n = S_1 - (S_1^{n-2} \cup S_1^{n-1})$.

Claim 2. *M has no backward arcs with respect to F'.*

Proof. If not, let (x, y) be a backward arc of M with respect to F'. Note that (x, y) is also a backward arc with respect to F. If $x \in S_2$ then x has out-neighbor in S_1, and if $x \notin S_2$ then x is not a leaf in F and so $d_F^+(x) \geqslant 1$. In both cases there exists $x' \in N^+(x) - V(P_y)$, and so $P_y \cup (x, y) \cup (x, x')$ contains a path $P(n-3, 1, 1)$, a contradiction. □

Claim 3. *M has no vertex x such that $d_M^-(x) \geqslant 3$.*

Proof. If not, let $x \in V(M)$ such that $d_M^-(x) \geqslant 3$. Let $\{x', y, z\} \subseteq N_M^-(x)$ such that $\{x'\} = N_F^-(x)$. Clearly, (y, x) and (z, x) are two forward arcs with respect to F. Without loss of generality, we can suppose $l_F(y) \leqslant l_F(z)$. So if $z \in S_2$, then z has an out-neighbor in S_1, and if $z \notin S_2$ then $d_F^+(z) \geqslant 1$, and since $x' \neq z$ then $x \notin N_F^+(z)$. In both cases there exists $z' \in N_D^+(z) - (V(P_y) \cup \{x\})$ such that $P_y \cup (y, x) \cup (z, x) \cup (z, z')$ contains a path $P(n-3, 1, 1)$, a contradiction. □

Claim 4. $\chi(D[V(M) \cup S_1^{n-2} \cup S_1^{n-1}]) \leqslant 3$

Proof. We proved that M contains no backward arc with respect to F' and all its vertices are of in-degree at most 2. Then by Lemma 2.1, we can color the vertices of M by a proper 3-coloring c that uses the colors $\{1, 2, 3\}$ such that $|c(L_1(F'))| = 1$.

Let $x \in S_1^{n-2}$. Then one can easily prove that x has no in-neighbor in $V(M)$. Thus all neighbors of x in M are out-neighbors. Moreover, x has at most one out-neighbor in M; otherwise, let y and z be two out-neighbors of x in M where $l_F(y) \leqslant l_F(z)$. Then $P_y \cup (x, y) \cup (x, z)$ contains a path $P(n-3, 1, 1)$, a contradiction. Thus $|d_M^+(x)| \leqslant 1$, and so we can give x an appropriate color from the set $\{1, 2, 3\}$.

Let $x \in S_1^{n-1}$. Clearly $L_1(F') = L_{n-2}(F) - S_1$. Using the same reasoning as above, we may show that x has at most one neighbor with level at least n, and all its neighbors in $L_{n-2}(F) - S_1$ have the same color. Thus $|c(N_M(x))| \leqslant 2$, and we may give x an appropriate color from the set $\{1, 2, 3\}$. □

Claim 5. *D is not Hamiltonian.*

Proof. If not, let $C = v_1 v_2 \dots v_s$ be a Hamiltonian circuit in D.

If $n \in \{5, 6\}$, then $s \geqslant 2(n-3) + 1$, since otherwise we have $l(C) < \chi(D)$, which is impossible. In both cases, $\chi(D) \geqslant 6$ and then D contains a vertex x such that $d^-(x) \geqslant 3$. Otherwise, $d^-(v) \leqslant 2$ for every $v \in D$ and this easily gives $\chi(D) < 5$, a contradiction. Suppose that $d^-(v_1) \geqslant 3$. If there exists $v_i \in N^-(v_1) \cap \{v_2, \dots, v_{n-3}\}$, then $v_{n-1} v_n \dots v_s v_1 \cup (v_i, v_1) \cup (v_i, v_{i+1})$ contains a path $P(n-3, 1, 1)$, a contradiction. Otherwise, $N^-(v_1) \subseteq \{v_{n-2}, \dots, v_s\}$. Let $\{v_i, v_j\} \subseteq N^-(v_1) - \{v_s\}$ where $i < j$; then $v_2 v_3 \dots v_i \cup (v_i, v_1) \cup (v_j, v_1) \cup (v_j, v_{j+1})$ contains a path $P(n-3, 1, 1)$, a contradiction.

For $n \geqslant 7$, $\chi(D) \geqslant 8$. We will consider two cases:

i) $l(C) \geqslant 2(n-3)$. As above, D contains a vertex of indegree at least 4, say v_1. If there exists a vertex $v_i \in N^-(v_1) \cap \{v_2, \ldots, v_{n-4}\}$, then $v_{n-2}v_n \ldots v_s \cup (v_i, v_1) \cup (v_i, v_{i+1})$ contains a path $P(n-3, 1, 1)$. Otherwise, let $\{v_i, v_j, v_k\} \subseteq N^-(v_1) - \{v_s\}$ where $n-3 \leqslant i < j < k$. Then $v_2v_3 \ldots v_i \ldots v_j \cup (v_j, v_1) \cup (v_k, v_1) \cup (v_k, v_{k+1})$ contains a path $P(n-3, 1, 1)$, a contradiction.

ii) $l(C) < 2(n-3)$. Let $x, y \in V(D)$; then either $l(C_{[x,y]}) \leqslant (n-4)$ or $l(C_{[y,x]}) \leqslant (n-4)$. Without loss of generality we can suppose that $l(C_{[x,y]}) \leqslant (n-4)$ and $v_1 = x$. Clearly, $v_1v_2 \ldots v_s$ is a maximal forest of D. Then, by Claim 4, we have $\chi(D[\{v_{n-2}, \ldots, v_{s-1}\}]) \leqslant 3$, and thus $\chi(D[\{v_{n-2}, \ldots, v_s\}]) \leqslant 4$. So, $\chi(D[\{v_1, \ldots, v_{n-3}\}]) \geqslant \chi(D) - \chi(D[\{v_{n-2}, \ldots, v_s\}]) \geqslant n+1-4 = n-3$, but $|\{v_1, \ldots, v_{n-3}\}| = n-3$, so then $D[\{v_1, .., v_{n-3}\}]$ is a tournament. Since $l(C_{[v_1,y]}) \leqslant (n-4)$, we have $y \in \{v_2, \ldots, v_{n-3}\}$, and so $xy \in E(G[D])$. Therefore D is a tournament of order $n+1$ containing a path $P(n-3, 1, 1)$ [8], a contradiction.

□

Claim 6. *D has no backward arc (x, y) where $x \in L$ and $y \in L_1(F)$.*

Proof. If not, let $C = P_x \cup (x, y)$ as noted in Lemma 2.2, $T_y(F) = P_x$, and by Lemma 2.3, $uv \notin E(G[D])$ for all $u \in P_x$, $v \notin P_x$ and $l_F(v) \leqslant l_F(x)$. If there exists $uv \in E(G[D])$ such that $u \in C$ and $v \notin C$, then $l_F(v) > l_F(x) \geqslant n-2$, and so uv represents a forward arc with respect to F, since otherwise D contains a uv-directed path and so $C \subsetneq T_y(F)$, contradiction. Therefore $(u, v) \in E(D)$, and so $P_v \cup (u, v) \cup (u, u')$ contains $P(n-3, 1, 1)$ where u' is the successor of u on C, a contradiction. Consequently $uv \notin E(G[D])$ for all $u \in C$ and $v \notin C$, and so D is Hamiltonian containing a $P(n-3, 1, 1)$, contradiction. □

Let $N_1(S_1^n) = N(S_1^n) \cap L_1(F)$ and $N_1^-(S_1^n) = N^-(S_1^n) \cap L_1(F)$. Then by Claim 6 we have $N_1(S_1^n) = N_1^-(S_1^n)$. Let $L_2' = L_2(F) \cup N_1(S_1^n)$. Then L_2' is a stable set because, if not, there exists $u_1 \in L_1(F)$ with at least two out-neighbors, u_2 in $L_2(F)$ and u_n in S_1^n. i Since $u_n \in S_1^n$, we have $l_F(u_n) \geqslant n$ and so $l(P_{u_n}) \geqslant n-1$. Thus $P_{u_n} \cup (u_1, u_n) \cup (u_1, u_2)$ contains a $P(n-3, 1, 1)$, a contradiction.

Let $L_1' = (L_1(F) - N_1(S_1^n)) \cup S_1^n$. Then L_1', L_2', $L_3(F), \ldots, L_{n-3}(F)$ are $n-3$ stable sets covering $D - (V(M) \cup S_1^{n-2} \cup S_1^{n-1})$, and $\chi(D[V(M) \cup S_1^{n-2} \cup S_1^{n-1}]) \leqslant 3$ by Claim 4. Then $\chi(D) \leqslant n$, a contradiction. This completes the proof of Theorem 3.1. □

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