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Reconstruction of 2-connected parity graphs

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Abstract

A graph G is *reconstructible* if it is determined up to isomorphism from the collection of all one-vertex deleted unlabeled subgraphs of G. A graph G is a *parity graph* if for every pair of vertices (u, v) of G, the lengths of all induced paths joining u and v have the same parity. A *domino* is a cycle on six vertices with only one chord joining a pair of vertices at distance 3. It is shown that all 2-connected graphs G, with $\mathrm{diam}(G) = 2$ or $\mathrm{diam}(G) = \mathrm{diam}(\overline{G}) = 3$, such that G is a domino-free parity graph or a triangle-free graph of girth 5, are reconstructible.

1 Introduction

All graphs considered in this paper have finite orders and have no loops or multiple edges. The terms not defined here are taken as in Harary [8]. The *distance* $d(u, v)$ between two vertices u and v in G is the minimum length of a path joining them and the *distance* $d(u, S)$ between a vertex u and $S \subseteq V(G)$ is $\min\{d(u, s) : s \in S\}$, where $V(G)$ is the vertex set of G. The *eccentricity* of a vertex v, denoted $e(v)$, is $\max\{d(u, v) : u \in V\}$. The *radius* $\mathrm{rad}(G)$ is the minimum eccentricity of the vertices and the *diameter* $\mathrm{diam}(G)$ is the maximum eccentricity. If W is a nonempty subset of $V(G)$, then the *subgraph induced by* W (denoted by $G[W]$) is the subgraph of G whose vertex set is W and whose edge set consists of those edges of G incident with two elements of W. If W is a nonempty proper subset of $V(G)$, then $G - W$ denotes the induced subgraph $G[V(G) - W]$. A *vertex-cut* of G is a subset S of $V(G)$ such that $G - S$ is disconnected. A k*-vertex-cut* is a vertex-cut of k elements. The *connectivity* $\kappa(G)$ of a graph G is the minimum number of vertices whose removal results in a disconnected or trivial graph. A *vertex-deleted subgraph* (or *card*) $G - v$

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of a graph G is the unlabeled subgraph obtained from G by deleting the vertex v and all edges incident with v. The collection of all cards of G is the *deck* of G. A graph H is a *reconstruction* of G if H has the same deck as G. A graph is *reconstructible* if it is isomorphic to all its reconstructions. A family $\mathscr{F}$ of graphs is *recognizable* if, for each $G \in \mathscr{F}$, every reconstruction of G is also in $\mathscr{F}$, and *weakly reconstructible* if, for each graph $G \in \mathscr{F}$, all reconstructions of G that are in $\mathscr{F}$ are isomorphic to G. A family $\mathscr{F}$ of graphs is *reconstructible* if, for any graph $G \in \mathscr{F}$, G is reconstructible (i.e. if $\mathscr{F}$ is both recognizable and weakly reconstructible). A property (parameter) Q defined on a class $\mathscr{C}$ of graphs is a *recognizable property* (*reconstructible parameter*) if $Q(G) = Q(H)$ whenever $G \in \mathscr{C}$ and H is a reconstruction of G.

The Reconstruction Conjecture (RC), proposed by Kelly and Ulam [3] in 1941, asserts that every graph with at least three vertices is reconstructible. This conjecture has proved notoriously difficult, and has motivated a large amount of work in graph theory. Several classes of graphs have already been proved to be reconstructible in the hope that one day all graphs would be proved reconstructible by including enough classes of graphs. The manuscripts [2, 3, 5, 9, 10] are surveys of work done on this problem.

The class of graphs G with $\text{diam}(G) = 2$ or $\text{diam}(G) = \text{diam}(\overline{G}) = 3$ is denoted by $\mathscr{DR}$-class. Yongzhi [17] settled Problem 3 listed in the survey [2] when he proved that the RC is true if and only if every 2-connected graph is reconstructible. Gupta et al. [7] have proved that the RC is true if and only if all connected graphs G in the $\mathscr{DR}$-class are reconstructible. Ramachandran and Monikandan [14] have combined the above two reductions of the RC and proved that the RC is true if and only if all 2-connected graphs G in the $\mathscr{DR}$-class are reconstructible. Recently, the authors [6] have shown that all distance hereditary 2-connected graphs G in the $\mathscr{DR}$-class are reconstructible.

A *chord* of a cycle of length at least 4 is an edge joining two non adjacent vertices in the cycle. Two chords x_1y_1 and x_2y_2 of a cycle C are *crossing chords* if the ends of the edges come in the order x_1, x_2, y_1, y_2 around C. A *domino* is a cycle on six vertices with only one chord joining a pair of vertices at distance 3. A graph G is H-free if no induced subgraph of G is isomorphic to H.

A graph G is a *parity graph* if for every pair of vertices (u, v), the lengths of all induced paths joining u and v have the same parity (that is, they are both odd or both even); this was introduced by Burlet and Uhry in [4]. Many classes of graphs such as bipartite graphs and distance-hereditary graphs are subclasses of parity graphs, and parity graphs are perfect. In this paper, we show that all 2-connected domino-free parity graphs and graphs with no induced C_4 and no C_3 in the $\mathscr{DR}$-class are reconstructible.

2 Recognition of Parity Graphs

Theorems 2.1, 2.2 and 2.3 will be used while proving the main result.

Theorem 2.1. (Tutte [16]) *The number of non separable spanning subgraphs of G with a given number of edges is reconstructible.*

Theorem 2.2. (Gupta et al. [7]) *Graphs G with* $\mathrm{diam}(G) = 2$ *and graphs H with* $\mathrm{diam}(H) = \mathrm{diam}(\overline{H}) = 3$ *are recognizable.*

Theorem 2.3. (Burlet and Uhry [4]) *A graph G is a parity graph if and only if every cycle of odd length at least* 5 *has at least two crossing chords.*

As Bondy stated in his paper [3], "The RC has no direct algorithmic implications. It is concerned not with the process of reconstruction but with the endproduct, and asserts that this endproduct, the reconstructed graph is unique up to isomorphism". The expressions "Hence G is reconstructible", "G can be obtained uniquely" and "G is uniquely determined" are used in this paper in the above sense.

Lemma 2.4. *Parity graphs G are recognizable.*

Proof. If G itself is a cycle, then G is reconstructible. Otherwise, let $\mathscr{C} = \{H : |V(H)| < |V(G)|$ and H is an odd cycle with no crossing chords$\}$. Now, for every graph H in $\mathscr{C}$, we can determine whether G contains H as an induced subgraph or not by using Kelly's Lemma [2]. If no graph $H \in \mathscr{C}$ is an induced subgraph of G, then G is a parity graph by Theorem 2.3. □

Notation. Let V_1 and V_2 be two disjoint subsets of $V(G)$ of a graph G. By $V_1 \sim V_2$, we mean that every vertex in V_1 is adjacent to every vertex in V_2. When $v_1 \notin V_2$, by $v_1 \sim V_2$, we mean that v_1 is adjacent to every vertex in V_2. Similarly, by $v_1 \sim v_2$, we mean that v_1 is adjacent to v_2; otherwise we write $v_1 \not\sim v_2$. By $v_1 \not\sim V_2$, we mean that v_1 is not adjacent to at least one vertex in V_2; and by $v_1 \not\sim\not\sim V_2$, we mean that v_1 is not adjacent to any vertex in V_2. For $u \in V(G)$, let $N_i(u) = \{v \in V(G) : d(u, v) = i\}$.

3 Reconstruction of Parity graphs

Here we focus on the reconstruction of a parity graph G that is 2-connected and of diameter 2. Since the connectivity is reconstructible, these parity graphs are recognizable by Theorem 2.2 and Lemma 2.4. Suppose there was an induced cycle C of length at least 5 in G. Then C would contain two nonadjacent vertices connected by an induced path of length 3, giving a contradiction to G being a parity graph of diameter 2. Hence G contains no cycle of length at least 5 as an induced subgraph. Since G is a parity graph, it contains no induced $C_5 + e$ or $C_5 + \{e_1, e_2\}$, where e_1 and e_2 are non-crossing chords (by Theorem 2.3). Thus G is distance-hereditary and it is proved to be reconstructible in [6]. So we have the following theorem.

Theorem 3.1. *All* 2*-connected parity graphs* G *with* $\mathrm{diam}(G) = 2$ *are reconstructible.*

We now focus on the reconstruction of a domino-free parity graph G that is 2-connected and with $\mathrm{diam}(G) = \mathrm{diam}(\overline{G}) = 3$. Although the next lemma is well-known, we give the proof for the sake of clarity.

Lemma 3.2. *If* G *is a graph with* $\mathrm{diam}(G) = \mathrm{diam}(\overline{G}) = 3$*, then* $\mathrm{rad}(G) = 2$.

Proof. Clearly $\mathrm{rad}(G)$ cannot be 1 as G is connected. Since $\mathrm{diam}(\overline{G})$ is 3, there is a vertex u with $e(u) = 3$ in $\overline{G}$. Now since the vertices in $N_2(u) \cup N_3(u)$ are nonadjacent to u in $\overline{G}$, they are adjacent to u in G. Also, in $\overline{G}$ no vertex in $N_1(u)$ is adjacent to a vertex in $N_3(u)$, and hence $N_1(u) \sim N_3(u)$ in G. Therefore $e(u)$ is 3 in G. □

From now until the end of this section, we assume that G is a 2-connected domino-free parity graph with $\mathrm{diam}(G) = \mathrm{diam}(\overline{G}) = 3$. Then, by Lemma 3.2, there is a vertex u with $e(u) = 2$. Let $Z = N_2(u)$, $X = \{v \in V(G) : d(v, z) \geq 2,$ for any $z \in N_2(u)\}$ and $Y = N_1(u) - X$ (Figure 3.1). We use here the notation u, X, Y and Z in the sense of this meaning; and the connectivity of G is taken to be k, $k \geq 2$.

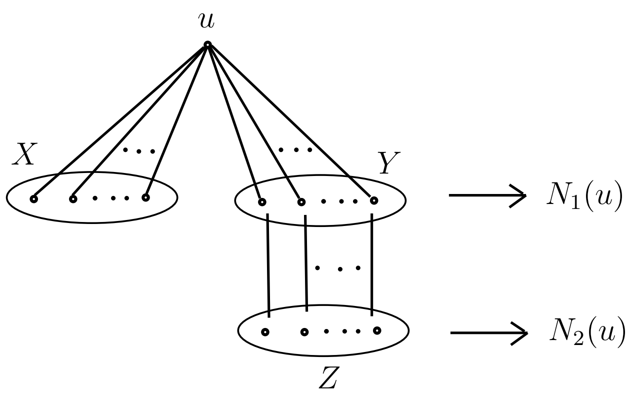

Figure 3.1. General structure of graphs with radius 2

Lemma 3.3. *If* W *is a* k*-vertex-cut of* G*, then* W *is not a subset of* Z*. Moreover, if* $W \cap X \neq \emptyset$*, then* W *contains the vertex* u*.*

Proof. Since $u \sim X \cup Y$ and every vertex in Z is adjacent to a vertex in Y, it follows that W is not a subset of Z. If W intersects X, then, by the minimality of W, separation is possible only among the vertices in $N_1(u)$ and hence W must contain the vertex u. □

Remark 3.4. For any k-vertex-cut W of G, Lemma 3.3 eliminates the following cases.

(i) $W \subseteq X$.
(ii) $W \cap X \neq \emptyset, W \cap Y \neq \emptyset$ and $W \cap [G - (X \cup Y)] = \emptyset$.
(iii) $W \cap X \neq \emptyset, W \cap Z \neq \emptyset$ and $W \cap [G - (X \cup Z)] = \emptyset$.
(iv) $W \cap X \neq \emptyset, W \cap Y \neq \emptyset, W \cap Z \neq \emptyset$ and $u \notin Z$.

Lemma 3.5. *If W is a k-vertex-cut of G such that $W \cap Y \neq \emptyset$ and $W \cap Z \neq \emptyset$, then $W \cap [V(G) - (Y \cup Z)] \neq \emptyset$.*

Proof. Suppose, to the contrary, that $W \cap [V(G) - (Y \cup Z)]$ was empty. Then, in $G - W$ there would be a component C such that $V(C) \subseteq Z$. Since $G - (W - \{z\})$, where $z \in Z \cap W$, is connected, we would get $Y - W \neq \emptyset$. If $z \not\sim\not\sim Y - W$, then z is nonadjacent to any vertex of $Z - [W \cup V(C)]$ lying in the same component as $Y - W$, because $e(u) = 2$. Thus $G - (W - \{z\})$ would be disconnected, giving a contradiction. Otherwise, that is, if z is adjacent to a vertex in $Y - W$, then again we will get a contradiction, but to $e(u) = 2$ (because z is adjacent to a vertex in C). □

Lemma 3.6. *If W is a k-vertex-cut of G such that $u \in W$ and $W \cap Y \neq \emptyset$, then $W \cap Z \neq \emptyset$.*

Proof. Suppose, to the contrary, that $W \cap Z = \emptyset$. It is clear that not all the vertices in $N_1(u) - W$ are confined to a single component of $G - W$. If a vertex y in $W \cap Y$ is adjacent to a vertex in $N_1(u)$, then y must be adjacent to all those vertices in $N_1(u) - W$, since $d(a, b) \leq 2$ for all $a, b \in N_1(u)$. Moreover, since the vertex u is in W, the graph $G - W$ has no component with vertices from Z alone. We now proceed with two cases as below.

Case 1. Vertices in $Y - W$ lie in different components of $G - W$.

Suppose that there was a vertex y in $W \cap Y$ such that $y \not\sim\not\sim N_1(u) - W$. Since $G - (W - \{y\})$ is connected, the vertex y is adjacent to at least two vertices, say $z_1, z_2 \in Z$, where z_1 and z_2 lie in different components; let y_1 and y_2 be their respective neighbours in $Y - W$. Now, the six vertices u, y, y_1, y_2, z_1 and z_2 together form a domino as induced subgraph of G, a contradiction. So we can assume that $W \cap Y \sim N_1(u) - W$. By the definition of Y, there exists a vertex $z \in Z$ such that $z \sim y$. Therefore the five vertices u, y, z, any vertex from $N_{Y-W}(z)$ and a vertex $y\prime \in N_{Y-W}(u)$ lying in the component not containing z, together form $C_5 + \{e_1, e_2\}$ as induced subgraph of G, where e_1 and e_2 are non-crossing edges, giving a contradiction.

Case 2. Vertices in $Y - W$ lie in the same component of $G - W$.

Now $X - W$ is nonempty and $(W \cap Y) \sim (N_1(u) - W)$. Since every vertex in $W \cap Y$ is adjacent to a vertex in Z, we would obtain a similar contradiction by proceeding as in Case 1 with a vertex from $X - W$ instead of y'. □

Remark 3.7. For any k-vertex-cut W of G, Lemma 3.6 eliminates the following cases.

(i) $u \in W, W \cap Y \neq \emptyset$ and $W \cap (G - (u \cup Y)) = \emptyset$.

(ii) $u \in W, W \cap X \neq \emptyset, W \cap Y \neq \emptyset$ and $W \cap Z = \emptyset$.

Theorem 3.8. *If W is a k-vertex-cut of G, then $N_{G-W}(w_i) = N_{G-W}(w_j)$ for any $w_i, w_j \in W$, where $1 \leq i, j \leq k$.*

Proof. First, we consider the case that $W \cap X \neq \emptyset$ and $W \cap ((G-u)-X) = \emptyset$. Now W must contain u by Lemma 3.6. Since not all the vertices in $N_1(u) - W$ are confined to the same component of $G-W$, it follows that $W \cap X \sim N_1(u) - W$ (because $d(a,b) = 2$ for any two nonadjacent vertices $a, b \in N_1(u)$). Hence all the vertices in W have the same set of neighbours in $G-W$ and $u \sim (W - \{u\})$.

Now, let us discuss the cases which are not eliminated in Remarks 1 and 2 and Lemma 3.5. That is, we consider the case when the k-vertex-cut W contains u and a vertex from Z. Now $G-W$ has at least two components with vertices from $Y-W$ and it has no component i with vertices from Z alone.

Case 1. The set $Z-W$ is empty.

Since $G-(W-z)$, where $z \in Z \cap W$, is connected and $d(a,b) = 2$ for any two nonadjacent vertices a, b in $N_1(u)$, we get $Z \sim Y - W$, which implies $X - W = \emptyset$ and $W \cap (X \cup Y) \sim Y - W$. That is, all the vertices in W have the same set of neighbours in $G-W$.

Case 2. The set $Z-W$ is nonempty.

Suppose there was a vertex z, where $z \in Z \cap W$, nonadjacent to every vertex y in $Y-W$. Then, since $G-(W-z)$ is connected, the vertex z would be adjacent to a vertex in $Z-W$, giving a contradiction to u being a vertex of eccentricity 2. Thus $z \sim y_1$, where $y_1 \in Y-W$, and it is nonadjacent to any vertex $z' \in Z-W$, when z' and y_1 lie in different components of $G-W$. Therefore $z \sim y_2$ for some vertex y_2 in $Y-W$, where y_1 and y_2 lie in different components of $G-W$. Since the parity between u and its non neighbour is even, it follows that $z \nsim Z-W$.

Suppose that $z \nsim y'$, where $y' \in Y-W$. Without loss of generality, let y' lie in the component C_1 containing the vertex y_1. Then, since G is a parity graph, $y_1 \nsim y'$ and $C_1 \cap (Z-W) \neq \emptyset$. Also $d(u,Z) = 2$ implies that each $y \in Y$ is adjacent to $N_Z(y)$. Since the parity among the nonadjacent vertices in $N_1(u)$ is even, there exist two nonadjacent vertices $v_1, v_2 \in C_1 \cap Y$ with a common neighbour $z' \in C_1 \cap Z$, where v_1 is nonadjacent to z (existence of v_1 is guaranteed by y') and v_2 is adjacent to z (existence of v_2 is guaranteed by y_1). Now the three vertices v_1, v_2, z' along with u, z and y_2 form a domino as induced subgraph of G (Figure 3.2), giving a contradiction. Hence z is adjacent to all the vertices in $Y-W$. Again, since $d(a,b) = 2$ for any two nonadjacent vertices a, b in $N_1(u)$, we will get $W \cap X \sim Y - W$ and $X - W = \emptyset$ (because $G-W$ is disconnected).

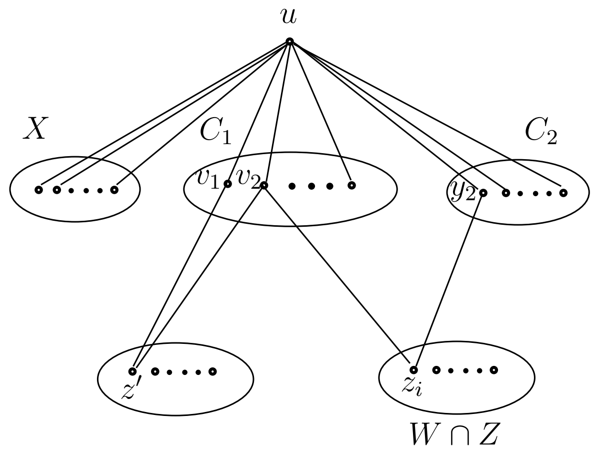

Figure 3.2. Structure of G under the case $u \in W$ and $W \cap Z \neq \emptyset$

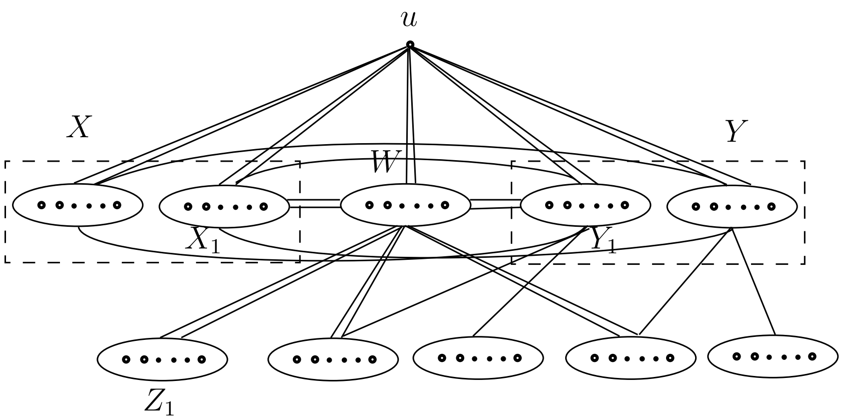

Figure 3.3. Structure of G under the case $W \subseteq Y$

If $W \cap Y \neq \emptyset$ and a vertex $y \in Y \cap W$ is adjacent to a vertex in $Y - W$, then $y \sim Y - W$. Therefore $y \not\sim\not\sim Z - W$ since G is a parity graph and the vertices of $Y - W$ lie in different components. If $y \not\sim\not\sim Y - W$, then each component of $G - W$ contains a vertex from Z, since $G - (W - y)$ is connected. Also y is adjacent to z_1 and z_2, where $z_1, z_2 \in Z - W$ and they lie in different components of $G - W$. Now the vertices u, y, z_1, z_2 along with the neighbours of z_1 and z_2 in $Y - W$ together form a domino as an induced subgraph, giving a contradiction. Hence the vertices in W have the same set of neighbours in $G - W$ (Figure 3.3; $X_1 = N(W) \cap X$, $Y_1 = N(W) \cap Y$, a double line joining two vertex subsets denotes every vertex in one set is adjacent to every vertex in the other set whereas a single line denotes a vertex in one set may or may not be adjacent to a vertex in the other set).

Finally, the only remaining case to be discussed is $W \subseteq Y$. Now $G - W$ has at least one component, say C', such that $V(C') \subseteq Z$. Also, since $d(u, Z) = 2$ and C' is connected, each vertex in W is adjacent to i all the vertices in C'. If a vertex $y \in Y - W$ is adjacent to a vertex in W, then $d(y, V(C')) = 2$. Suppose that

$y \not\sim W$. Then there exists $y_i \in W$ such that $y \not\sim y_i$. Now the induced path yuy_iz, where $z \in V(C')$, has length 3, giving a contradiction to G being a parity graph. Hence any two vertices in W have the same set of neighbours in $Y - W$. Again, from the fact that G is a parity graph, we have a vertex in $N_Z(N_{Y-W}(W))$ which is adjacent to all the vertices in W whenever it is adjacent to at least one vertex in W. In particular, if two vertices in W are adjacent, then they share the same set of neighbours outside W. Consider the case when two vertices $y_i, y_j \in W$ are nonadjacent. Suppose there exists a vertex $z \in Z$ such that $z \sim y_i$ and $z \not\sim y_j$. Let $y' \in Y - N(W)$ be a neighbour of z. Then clearly $y' \not\sim y_i$ and $y' \not\sim y_j$. Now the six vertices z, y', u, y_i, y_j and z_i, where $z_i \in V(C')$, together form a domino as induced subgraph, giving a contradiction to G. Thus, in all the cases, we have proved that the vertices in W have the same set of neighbours in $G - W$. □

Theorem 3.9. *All 2-connected domino-free parity graphs G with* $\operatorname{diam}(G) = \operatorname{diam}(\overline{G}) = 3$ *are reconstructible.*

Proof. Since the connectivity is a reconstructible parameter, recognizability of G follows by Theorem 2.2, Lemma 2.4 and Kelly's lemma [2].

Suppose that G contains a k-vertex-cut W with a vertex, say w, such that w is adjacent to the remaining $k - 1$ vertices in W (such a case arises when $W \subset N[u]$). Consider a card $G - v$ containing a $(k - 1)$-vertex-cut such that the degree of the deleted vertex is equal to the sum of $k - 1$ and the number of common neighbours of those $k - 1$ vertices outside the $(k - 1)$-vertex-cut. Now G can be obtained uniquely from $G - v$ by adding a new vertex to $G - v$ and making it adjacent to the common neighbours of the vertices in the $(k-1)$-vertex-cut and to those $k-1$ vertices. So we can assume that no k-vertex-cut of G contains a vertex adjacent to all other vertices in the vertex-cut.

If G contains at least two k-vertex-cuts, then all these vertex-cuts are disjoint because the vertices in a minimal vertex-cut have the same set of neighbours outside it. So we can determine the subgraph induced by any k-vertex-cut from the collection of all cards containing a $(k - 1)$-vertex-cut. Now G can be obtained uniquely from a card containing a $(k - 1)$-vertex-cut by replacing the $(k - 1)$-vertex-cut with the required induced subgraph already identified.

The only remaining case to be discussed is when G contains exactly one k-vertex-cut. Let W' be the unique k-vertex-cut of G. The proof of this case differs somewhat from the above cases; we reconstruct the subgraph $G[W']$ of G induced by W' from the deck of G by using mathematical induction on the order of G and then we replace a $(k - 1)$-vertex-cut by W' in a specified card. Assume that all 2-connected domino-free parity graphs of order strictly less than that of G and with diameter 2 are reconstructible. Now the deck of W' can be determined from the deck of G (by considering all the cards of G containing a $(k-1)$-vertex-cut). Since G is a domino-free parity graph, so is the subgraph $G[W']$ of G induced by W'. Again, since all the vertices in W' have the same set of neighbours in $G - W'$, the parity amongst any two nonadjacent vertices in W' is even. Now, we take $G[W']$ to be connected as otherwise it is reconstructible.

Suppose that $G[W']$ has a cut vertex, say w. Let C_1 be a component of $G[W']-w$ and let C_2 be the union of all other components of $G[W']-w$. If $d_{G[W']}(w,w_1)$ was equal to 2 for some vertex w_1 in C_1, then $d_{G[W']}(w_1,C_2)$ would be 3, giving a contradiction. Hence $w \sim C_1$ and similarly $w \sim C_2$, which imply $w \sim G[W']-w$. Therefore $\text{diam}(G[W'])=1$ and thus $G[W']$ is reconstructible.

Now let $G[W']$ be 2-connected and $\text{diam}(G[W'])>1$. Then $\text{diam}(G[W'])=2$ (as otherwise, $\text{diam}(G[W'])$ would be at least 3 and $G[W']$ would contain two nonadjacent vertices at distance 3, a contradiction). Thus $G[W']$ is a 2-connected domino-free parity graph of order strictly less than that of G and $\text{diam}(G[W'])=2$. By the induction assumption, W' is thus reconstructible. We now reconstruct G from any card $G-x$ containing a unique $(k-1)$-vertex-cut, say S, as follows. All graphs obtained from $G-x$, by replacing S with $G[W']$ and making the vertices in W' adjacent to the common neighbours of all the vertices of S but outside S, are isomorphic and they are G. □

4 Triangle-free graphs

Let G be a triangle-free graph with $\text{diam}(G)=\text{diam}(\overline{G})=3$. Consider a vertex u with eccentricity 3 in $\overline{G}$. Since G is triangle-free, the vertex independence number of $\overline{G}$ is at most 2. Therefore the vertices in $N_2(u)\cup N_3(u)$ (in $\overline{G}$) form a complete subgraph of G. Similarly, any two vertices in $N_1(u)$ (in $\overline{G}$) are adjacent in G. Thus G will be a bipartite graph with $\text{diam}(G)=\text{diam}(\overline{G})=3$, which is proved to be reconstructible in [12]. So we have the following theorem.

Theorem 4.1. *All 2-connected triangle-free graphs G with* $\text{diam}(G)=\text{diam}(\overline{G})=3$ *are reconstructible.*

Since graphs with diameter 2 and girth 5 are Moore graphs which are regular, the next theorem is evident.

Theorem 4.2. *All 2-connected triangle-free graphs G with* $\text{diam}(G)=2$ *and girth* 5 *are reconstructible.*

5 Conclusion

Let G be a triangle-free graph of diameter 2. Suppose that $\text{diam}(\overline{G})\geq 3$. Then, by arguing as above, we get that G is a bipartite graph of diameter 2 and hence G is complete bipartite, which is reconstructible. Also if the girth of G is not 5, then it must be 4 since G has diameter 2. Thus, the only remaining class of triangle-free graphs to reconstruct is all 2-connected triangle-free graphs G with $\text{diam}(G)=\text{diam}(\overline{G})=2$ and girth 4.

The next two theorems conclude all our results described above.

Theorem 5.1. *The class* $\mathscr{E}$ *of all* 2*-connected graphs* G *with* $\mathrm{diam}(G) = 2$ *is reconstructible if and only if one of the following subclasses of* $\mathscr{E}$ *is proved to be reconstructible.*

($\mathscr{E}_1$) *All non-parity graphs.*

($\mathscr{E}_2$) *Graphs with* C_3 *or* C_4 *as an induced subgraph.*

Proof. The necessity is obvious. For sufficiency, suppose that G is a 2-connected graph with $\mathrm{diam}(G) = 2$ but not in $\mathscr{E}_1 \cup \mathscr{E}_2$. Then G is a parity graph or a triangle-free graph with no C_4 as an induced subgraph. Hence G is reconstructible by Theorem 3.1 or Theorem 4.2. □

Theorem 5.2. *The class* $\mathscr{F}$ *of all* 2*-connected graphs* G *with* $\mathrm{diam}(G) = \mathrm{diam}(\overline{G}) = 3$ *is reconstructible if and only if one of the following subclasses of* $\mathscr{F}$ *is proved to be reconstructible.*

($\mathscr{F}_1$) *All non-parity graphs and graphs with domino as an induced subgraph.*

($\mathscr{F}_2$) *Graphs with triangles.*

Proof. The necessity is obvious. For sufficiency, suppose that G is a 2-connected graph with $\mathrm{diam}(G) = \mathrm{diam}(\overline{G}) = 3$ but not in $\mathscr{F}_1 \cup \mathscr{F}_2$. Then G is a domino-free parity graph or a triangle-free graph. Hence G is reconstructible by Theorem 3.9 or Theorem 4.1. □

It is known [14] that the RC is true if and only if all 2-connected graphs G in the $\mathscr{DR}$-class are reconstructible. Thus the RC is true if and only if one of the pairs $(\mathscr{E}_1, \mathscr{F}_1)$, $(\mathscr{E}_1, \mathscr{F}_2)$, $(\mathscr{E}_2, \mathscr{F}_1)$ or $(\mathscr{E}_2, \mathscr{F}_2)$ of classes of graphs is reconstructible.

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