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Regular 1-factorizations of complete graphs and decompositions into pairwise isomorphic rainbow spanning trees*

Gloria Rinaldi

Department of Sciences and Methods for Engineering
University of Modena and Reggio Emilia
via Amendola 2, 42122 Reggio Emilia
Italy
gloria.rinaldi@unimore.it

Abstract

A 1-factorization $\mathcal{F}$ of a complete graph K_{2n} is said to be G-regular, or regular under G, if G is an automorphism group of $\mathcal{F}$ acting sharply transitively on the vertex-set. The problem of determining which groups can realize such a situation dates back to a result by Hartman and Rosa (1985) on cyclic groups, and it is still open even though several other classes of groups were tested in the recent past. An attempt to obtain a fairly precise description of groups and 1-factorizations satisfying this symmetry constraint can be done by imposing further conditions. In this paper we prove that, regardless of the isomorphism type of G, the existence of a G-regular 1-factorization of K_{2n} together with a complete set of isomorphic rainbow spanning trees which are in the orbit of a single one is assured if and only if $n \geq 3$ is an odd number. Also, when n is even, we examine dihedral groups: for each dihedral group G of order $2n \geq 6$, it is possible to exhibit a G-regular 1-factorization of K_{2n} together with two non isomorphic rainbow spanning trees whose partial orbits give rise to a complete set. This extends some recent results obtained by Caughman et al. (2017) and by Mazzuoccolo et al. (2019) for the class of cyclic groups.

1 Introduction

A 1-*factorization* of a complete graph is a partition of the edge set into perfect matchings. Such a decomposition obviously exists only when the number of vertices is even, say $2n$. It has been known [27] that 1-factorizations of a complete graph K_{2n}

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exist for all integers n; nevertheless, the number of non-isomorphic ones explodes as n increases [9], and a general classification has not been possible. An attempt can be made if one imposes additional conditions either on the 1-factorization or on its automorphism group. For example, a precise description of the 1-factorization and of its automorphism group was given when the group is assumed to act multiply transitively on the vertex set [10]. A few years ago the following question was addressed:

> *Does there exist a 1-factorization of the complete graph K_{2n} admitting a prescribed group G as an automorphism group acting sharply transitively on the vertex-set of K_{2n}?*

In other words, G is assumed to be a regular permutation group on the vertex-set of K_{2n}, i.e., whenever u and v are vertices there is precisely one automorphism in G mapping u to v, and also the ismorphism type of G, which necessarily has order $2n$, is specialized. We also say that the 1-factorization is *regular under* G, or that it is G-*regular*, or simply that it is *regular* if we do not need to specify the isomorphism type of G. The question above is a restricted version of problem n.4 in the list of [30] (the word "sharply" does not appear there) and when n is odd this problem simplifies somewhat: G must be the semi-direct product of Z_2 with its normal complement and G always realizes a 1-factorization of K_{2n} upon which it acts sharply transitively on vertices, see [3, Remark 1]. When n is even, the complete answer is still unknown. Nevertheless, the answer was found for several classes of groups, see for example [17, 7, 3, 4, 28], which, respectively, consider the class of cyclic, abelian, dihedral, dicyclic and other nilpotent groups. In [6], the answer was found for the class of 2-groups with an elementary abelian Frattini subgroup and in [25] the first class of G-regular 1-factorizations for a non-solvable group G was given. Within the groups tested, the unique case in which the answer is negative is when G is cyclic and $2n$ is a power of 2 greater than 4 [17].

The existence of a G-regular 1-factorization of K_{2n} can be tested entirely in G using the so called *"starter method"* for groups of even order which was first introduced in [7] and which allows one to construct the 1-factorization starting from G. We will describe this method in the following Subsection 1.1. In an attempt to give a fairly precise description of G-regular 1-factorizations, further conditions can also be requested. For example some nonexistence results were achieved by assuming the existence of a fixed 1-factor [21, 28]; further results were obtained when the number of fixed 1-factors is as large as possible [3], or when the 1-factors satisfy some additional requests [5].

In the present paper we strengthen the symmetry conditions on the 1-factorization. More precisely, we consider G-regular 1-factorizations possessing a complete set of rainbow spanning trees which can be constructed via the automorphism group G.

We recall that a rainbow spanning tree is a spanning tree sharing exactly one edge with each 1-factor of the given 1-factorization. In other words, a 1-factorization of K_{2n} corresponds to a proper edge coloring of K_{2n} with precisely $2n-1$ colors: each color appears exactly n times and corresponds to a 1-factor. Therefore, a spanning tree is *rainbow* if its edges have distinct colors. It is also usual to say that such a tree is *orthogonal* to the 1-factorization. We also recall that if T is any subgraph of K_{2n} with exactly $2n-1$ edges, then T is a spanning tree if and only if T is a spanning connected graph, see for instance [31, p. 68].

A set of rainbow spanning trees is said to be *a complete set* if the trees form a partition of the edge set of K_{2n}. It is easy to prove that a complete set cannot exist in K_4, so we restrict our discussion to complete sets in K_{2n} with $n \geq 3$. Also, since each rainbow spanning tree has $2n-1$ edges, n is the number of disjoint trees in a complete set.

In Section 2, we prove that, regardless of the isomorphism type of G, a G-regular 1-factorization of K_{2n} together with a rainbow spanning tree whose orbit under a subgroup of G gives rise to a complete set exists if and only if $n \geq 3$ is an odd number. It is clear that the spanning trees are pairwise isomorphic in this case. In Section 3 we assume G to be a dihedral group. If the order of G is twice an even number, we can exhibit two non-isomorphic rainbow spanning trees whose partial orbits under G give rise to a complete set. This set is partitioned into two sets of isomorphic trees, with each set having cardinality $\frac{n}{2}$. These results extend those obtained in [23] for the class of cyclic groups, and in [12] where the group is assumed to be cyclic and with a 1-rotational action on the vertex set, i.e., the automorphism group acts sharply transitively on the set of all vertices except one, which is fixed by each element of the group.

The main interest of our paper fits in the general problem of characterizing G-regular 1-factorizations satisfying additional properties and we continue the analysis introduced in [23]. However, I recall that the problem of determining whether every given 1-factorization of a complete graph possesses a complete set of rainbow spanning trees dates back to the Brualdi and Hollingsworth conjecture [8], and to the Constantine conjecture when the trees are asked to be pairwise isomorphic as uncolored trees [13]. A recent asymptotic result settles both these conjectures for all sufficiently large n [16]. Nevertheless, the solution for each given n remains nontrivial even if one is allowed to choose the 1-factorization.

Most of the papers about these conjectures treat the general case by methods of extremal graph theory/probabilistic methods which can be applied for every 1-factorization of K_{2n}. The best known results hold for large n and mainly give lower-bounds on the number of rainbow spanning trees. Together with [16] we recall some other important papers in this direction: [1, 15, 18, 22, 24, 26]. The Brualdi-Hollingsworth conjecture was also extended in [20], by stating that edges of every properly colored K_n (not necessarily colored by a 1-factorization) can be partitioned into rainbow spanning trees. Results are, for example, contained in [2, 11, 24], and for large n, the results of [26] improved the best known bounds for the three conjectures in [8, 13, 20].

Some examples of 1-factorizations of K_{2n} satisfying the above conjectures without imposing conditions on n are available. Constantine himself proved the existence of a suitable 1-factorization satisfying his conjecture for the case $2n$ a power of 2 or five times a power of two [13].

Also, a first family of 1-factorizations for which the conjecture of Brualdi and Hollingsworth can be verified for each $n \geq 3$ was recently shown in [12]. In [23] a complete set of rainbow spanning trees was constructed in the family of 1-factorizations of [17] for each $n \geq 3$, except when $n = 2^s$, $s \geq 2$. In the same paper a complete set of isomorphic rainbow spanning trees was constructed in a suitable 1-factorization of K_{2n} whenever either $2n = 6$ or $2n$ is larger than 6 and belongs to $\{2^s d : s \geq 1, d \text{ odd } d \neq 3\}$.

Other examples will be constructed in Sections 2 and 3.

1.1 The starter method

Let G be a group of even order $2n$. We use a multiplicative notation for G and denote by 1_G its identity, we also use 1 if the group G is clear from the context. Let us denote by V and E the set of vertices and edges of K_{2n}, respectively. We identify the vertices of K_{2n} with the group elements of G. We shall denote by $[x, y]$ the edge with vertices x and y. Following [7] we always consider G in its right regular permutation representation. In other words, each group element $g \in G$ is identified with the permutation $V \to V$, $x \mapsto xg$. This action of G on V induces actions on the subsets of V and on sets of such subsets. Hence if $g \in G$ is an arbitrary group element and S is any subset of V then we write $Sg = \{xg : x \in S\}$. In particular, if $S = [x, y]$ is an edge, then $[x, y]g = [xg, yg]$. Furthermore, if U is a collection of subsets of V, then we write $Ug = \{Sg : S \in U\}$. In particular, if U is a collection of edges of K_{2n} then $Ug = \{[xg, yg] : [x, y] \in U\}$. The G-orbit of an edge $[x, y]$ has either length $2n$ or n and we speak of a *long* orbit or a *short* orbit, respectively, and we call $[x, y]$ a *long* edge or a *short* edge, respectively. If $[x, y]$ is a short edge, then there is a non-trivial group element g so that $[xg, yg] = [x, y]$. Such a g is unique ($g = x^{-1}y$) and is an involution; we call this g the involution *associated* with the short edge $[x, y]$.

It is easy to show that a 1-factor of K_{2n} which is fixed by G necessarily coincides with a short G-orbit of edges.

If e is an edge, respectively if S is a set of edges, we will denote by $\mathrm{Orb}_G(e)$, respectively by $\mathrm{Orb}_G(S)$, the orbit of e, respectively of the set S, under the action of G.

If H is a subgroup of G then a system of distinct representatives for the left cosets of H in G will be called a *left transversal* for H in G.

If $[x, y]$ is an edge in K_{2n} we define:

$$\partial([x,y]) = \begin{cases} \{xy^{-1}, yx^{-1}\} & \text{if } [x,y] \text{ is long} \\ \{xy^{-1}\} & \text{if } [x,y] \text{ is short} \end{cases}$$

$$\phi([x,y]) = \begin{cases} \{x,y\} & \text{if } [x,y] \text{ is long} \\ \\ \{x\} & \text{if } [x,y] \text{ is short.} \end{cases}$$

Roughly speaking, we also say that the edge $[x,y]$ has *difference set* $\partial([x,y])$, or that $\{xy^{-1}, yx^{-1}\}$ are the *differences* of $[x,y]$.

If S is a set of edges of K_{2n} we define:

$$\partial S = \bigcup_{e \in S} \partial(e) \qquad \phi(S) = \bigcup_{e \in S} \phi(e)$$

where, in either case, the union may contain repeated elements and so, in general, will return a multiset.

In [7, Definition 2.1] a *starter* in a group G of even order is a set $\Sigma = \{S_1, \ldots, S_k\}$ of subsets of E together with subgroups $H_1, \ldots, H_k$ which satisfy the following conditions:

(i) $\partial S_1 \cup \cdots \cup \partial S_k = G \setminus \{1_G\}$;

(ii) for $i = 1, \ldots, k$, the set $\phi(S_i)$ is a left transversal for H_i in G;

(iii) for $i = 1, \ldots, k$, H_i must contain the involutions associated with any short edge in S_i.

We note that $G - \{1_G\}$ is a set, so this definition implies that $\partial([x,y])$ are distinct for all $[x,y]$ in the multiset $S_1 \cup \ldots \cup S_k$. Hence it also follows S_i can have no edges in common with S_j for $i \neq j$. Moreover, each $\phi(S_i)$ is a set and then the edges of S_i are vertex disjoint.

It is proved in [7], that the existence of a starter in a finite group G of order $2n$ is equivalent to the existence of a G-regular 1-factorization of K_{2n}. Property (i) in previous definition ensures that every edge of K_{2n} will occur in exactly one G-orbit of an edge from $S_1 \cup \ldots \cup S_k$. Properties (ii) and (iii) ensure the union of the H_i-orbits of edges from S_i will form a 1-factor. Namely, for each index i, we form a 1-factor as $\cup_{e \in S_i} \mathrm{Orb}_{H_i}(e)$, whose stabilizer in G is the subgroup H_i; the G-orbit of this 1-factor, which has length $|G : H_i|$ (the index of H_i in G), is then included in the 1-factorization. Observe also that if the 1-factorization includes a 1-factor F_i which is fixed by G, then there exists a short edge e such that $F_i = \mathrm{Orb}_G(e)$ and the set $S_i = \{e\}$ is included in the starter.

2 Regular 1-factorizations and complete sets of rainbow spanning trees in a unique orbit

Let $\mathcal{F}$ be a G-regular 1-factorization of K_{2n}. We strengthen the symmetry conditions on $\mathcal{F}$ and ask for an automorphism group G having a nontrivial subgroup which preserves a spanning rainbow tree partition, and with trees in a unique orbit. This last request obviously ensures the rainbow trees to be pairwise isomorphic. In Proposition 2.1 some necessary conditions are pointed out.

Proposition 2.1. *Let $\mathcal{F}$ be a G-regular 1-factorization of K_{2n}. Let T be a rainbow spanning tree and let $\mathcal{T} = \mathrm{Orb}_S(T)$ be a complete set of rainbow spanning trees with S a (not necessarily proper) subgroup of G. The following properties hold:*

- *The integer n is odd;*
- *If G has more than one involution, then the subgroup S has index 2 in G and for each involution $j \in G$ we have $Tj \notin \mathcal{T}$;*
- *If G has exactly one involution, say j, there are two possibilities: either $Tj = T$, the 1-factor $F^* = \mathrm{Orb}_G([1, j])$ is in $\mathcal{F}$ and $\mathrm{Orb}_G(T) = \mathcal{T}$, or $Tj \notin \mathcal{T}$ and S has index 2 in G.*

Proof. Denote by H the stabilizer of T in S. The set $\mathrm{Orb}_S(T)$ contains n elements, so either $|S| = n$ and $H = \{1\}$ or $S = G$ and H has order 2. If the first case occurs, then the integer n is odd. In fact, if S contained an involution, say j, without loss of generality we can suppose $[1, j]$ to be an edge of T and then $Tj = T$ which contradicts $H = \{1\}$. If the second case occurs with $H = \{1, j\}$, then j is the unique involution of G and n is odd. In fact, again suppose $[1, j]$ to be an edge of T. Since T is connected and without cycles, $[1, j]$ is the unique edge of T which is fixed by j, also ∂T is a multiset covering all the elements of $G \setminus \{1\}$. So, if G contained an involution j_1 different from j, we should find at least two distinct edges e and $e' = ej$ in T such that $\partial e = \partial e' = \{j_1\}$ and at least $2n$ short edges with difference set $\{j_1\}$ are in $\mathrm{Orb}_S(T)$ which is a contradiction. Denote by F^* the 1-factor of $\mathcal{F}$ containing the edge $[1, j]$ of T, we have $F^* = \mathrm{Orb}_G([1, j])$. In fact, if this is not the case, we have a 1-factor $F_2 \in \mathcal{F}$, $F_2 \neq F^*$, containing at least one short edge which is necessarily fixed by the unique involution j. This implies $F_2 j = F_2$. Let e be the edge of $T \cap F_2$, then ej is an edge of $T \cap F_2$ and it is different from e since $[1, j]$ is the unique edge of T which is fixed by j. This yields a contradiction. Now let $\Sigma = \{S^*, S_1, \dots, S_t\}$ be a starter for $\mathcal{F}$ with $S^* = \{[1, j]\}$. No S_i, $i = 1, \dots, t$, contains short edges and $|\partial S_i| = |\phi(S_i)| = \frac{|G|}{|H_i|}$ where H_i is the subgroup of G associated with S_i. We know that H_i is the stabilizer in G of the 1-factor F_i arising from S_i; this ensures that H_i has odd order. In fact, if we suppose $j \in H_i$ and we take the edge e_i of T which is in F_i, we should have $e_i j$ different from e_i but still in $F_i \cap T$: a contradiction. Let d_i be the odd order of H_i. Since Σ is a starter in G we have: $\sum_{i=1}^{t} \frac{2n}{d_i} = |G \setminus \{1, j\}| = 2n-2$, and then $\sum_{i=1}^{t} \frac{n}{d_i} = n - 1$. This ensures that n is odd. Now the second and the third assertions listed in the statement easily follow by observing that an involution of G fixes T if and only if $\mathrm{Orb}_S(T) = \mathrm{Orb}_G(T)$. □

Let G be a group of order $2n$, n odd. Recall that G admits a unique subgroup H of index 2; see [19, I 6.7, p. 93]; furthermore if $j \in G \setminus H$ is an involution, then G is the semidirect product of H by the subgroup generated by j. When $h \in H$ we have that $jhj = h^{-1}$ if and only if jh is an involution (and hj is an involution as well). We denote by 1 the identity of G.

Let $\Phi = \{\{a_i, a_i^{-1}\} \mid i = 1, \dots, \frac{n-1}{2}\}$ be the *right patterned starter* of H [14], i.e., Φ is a partition of the set $H^* = H \setminus \{1\}$ into pairs $\{a_i, a_i^{-1}\}$.

Using the starter method, we describe a G-regular 1-factorization $\mathcal{F}$ of K_{2n} which was already obtained in [3].

Partition the elements of $G \setminus (H \cup \{j\})$ into three sets: A_0, A_1, A_2, where A_0 contains all the involutions of $G \setminus (H \cup \{j\})$, while $jh \in A_1$ if and only if $h^{-1}j \in A_2$.

Starter in G: $\Sigma = \{S\} \cup \{\overline{S}_h \mid jh \in A_0\} \cup \{\overline{S'}_h \mid jh \in A_1\}$.
With:
$S = \{[1,j],[a_i,a_i^{-1}] \mid \{a_i,a_i^{-1}\} \in \Phi\}\}$, $\quad \partial S = H^* \cup \{j\}$, $\quad \phi(S) = H$;
$\overline{S}_h = \{[1,jh]\}$, $\quad \partial\overline{S}_h = \{jh\}$, $\quad \phi(\overline{S}_h) = \{1\}$;
$\overline{S'}_h = \{[1,jh]\}$, $\quad \partial\overline{S'}_h = \{jh, h^{-1}j\}$, $\quad \phi(\overline{S'}_h) = \{1, jh\}$.

Now $\phi(S)$ and $\phi(\overline{S'}_h)$ are left transversals for $\{1,j\}$ and H respectively. With the starter above we construct the following 1-factors:

$F_1 = \mathrm{Orb}_{\{1,j\}}(S) = \{[1,j]\} \cup \{[a_i,a_i^{-1}],[a_ij,a_i^{-1}j] \mid \{a_i,a_i^{-1}\} \in \Phi\}$ with $F_1j = F_1$.

$\overline{F}_h = \{[h_i, jhh_i] \mid h_i \in H\}$, with $jh \in A_0$. This 1-factor is fixed by G.

$\overline{F'}_h = \mathrm{Orb}_H(\overline{S'}_h) = \{[h_i, jhh_i] \mid h_i \in H\}$, with $jh \in A_1$. It is fixed by H.

The orbits in G of these 1-factors give rise to the 1-factorization. For each $i \in \{1,\dots,\frac{n-1}{2}\}$, there exists $k \in \{1,\dots,\frac{n-1}{2}\}$ such that $a_ij = ja_k$ which implies also that $a_i^{-1}j = ja_k^{-1}$ and then, for each $h \in H$, the 1-factor $F_h = F_1h$ can be described as follows, and it is included in the 1-factorization:

$$F_h = \{[h,jh]\} \cup \{[a_ih, a_i^{-1}h],[ja_ih, ja_i^{-1}h] \mid \{a_i,a_i^{-1}\} \in \Phi\}.$$

For each $h \in H^*$ let

$$F_h^* = \{[h_i, jhh_i] \mid h_i \in H\}.$$

When $jh \in A_0$, we have that $\mathrm{Orb}_G(\overline{F}_h) = \overline{F}_h = F_h^*$. When $jh \in A_1$, we have $\mathrm{Orb}_G(\overline{F'}_h) = \{\overline{F'}_h, \overline{F'}_hj\}$ with $\overline{F'}_h = F_h^*$, $\overline{F'}_hj = F_{h_1}^*$, with $h^{-1}j = jh_1$, $jh_1 \in A_2$.

We conclude that:

$$\mathcal{F} = \{F_h \mid h \in H\} \cup \{F_h^* \mid h \in H^*\}.$$

We also have $F_h^*j = F_{h'}^*$ with $h' = jh^{-1}j$.

If G is cyclic, the 1-factorization described above is that already described in [17].

Moreover, if H is abelian and jh is an involution for each $h \in H$, i.e., $jhj = h^{-1}$ for each $h \in H$, then G is the *generalized dihedral group* $\mathrm{Dih}(H)$ [29, p. 210], and all $n-1$ of the 1-factors F_h^* are fixed by the entire group G.

We are now able to prove the following result.

Theorem 2.2. *Let G be a group of order $2n$, $n > 1$. There exists a G-regular 1-factorization $\mathcal{F}$, a rainbow spanning tree T and a subgroup H of G such that $\mathcal{T} = \mathrm{Orb}_H(T)$ is a complete set of rainbow spanning trees if and only if n is odd.*

Proof. The necessary condition on n was proved in Proposition 2.1. Therefore, it is sufficient to show that for each fixed group G of order $2n$, with n odd, a G-regular 1-factorization with the required property exists. In particular we will use the 1-factorization $\mathcal{F}$ of [3] which is described above, and we use the same notation adopted above. In particular, remember that we denoted by $\Phi = \{\{a_i, a_i^{-1}\} \mid i = 1, \ldots, \frac{n-1}{2}\}$ the right patterned starter of H which was involved in the construction of each 1-factor F_h.

Let T be the subgraph of K_{2n} induced by the following set of edges:

$$T = \{[1, j],\ [1, a_i^{-2}],\ [j, ja_i^2],\ [1, ja_i^{-2}],\ [ja_i^4, a_i^2] \mid i = 1, \ldots, \frac{n-1}{2}\}.$$

T is a rainbow spanning tree. In fact:

$[1, j] \in F_1$; and for each $i = 1, \ldots, \frac{n-1}{2}$, we have
$[1, a_i^{-2}] \in F_{a_i^{-1}}$ (because $[1, a_i^{-2}] = [a_i a_i^{-1}, a_i^{-1} a_i^{-1}]$);
$[j, ja_i^2] \in F_{a_i}$ (because $[j, ja_i^2] = [ja_i^{-1}, ja_i]a_i \in F_1 a_i = F_{a_i}$); $[1, ja_i^{-2}] \in F^*_{a_i^{-2}}$;
$[a_i^2, ja_i^4] \in F^*_{a_i^2}$ (because $[a_i^2, ja_i^4] = [1, ja_i^2]a_i^2 \in F^*_{a_i^2} a_i^2 = F^*_{a_i^2}$).
Moreover T contains no cycle because each vertex in H^* has degree one, and each vertex in jH^* is adjacent with either 1 or j and at most another vertex in H^*.

Finally, it is obvious that each Th, $h \in H$, is still rainbow and spanning.

Moreover, $\mathrm{Orb}_H(T) = \{Th \mid h \in H\}$ covers all edges of K_{2n} exactly once. In fact $\mathrm{Orb}_H([1, j])$ covers the h short edges with difference set $\{j\}$; $\mathrm{Orb}_H([1, ja_i^{-2}])$ covers all edges of $F^*_{a_i^{-2}}$; $\mathrm{Orb}_H([a_i^2, ja_i^4])$ covers all edges of $F^*_{a_i^2}$; $\mathrm{Orb}_H([1, a_i^{-2}])$ and $\mathrm{Orb}_H([j, ja_i^2])$ are disjoint and, since for each $i \in \{1, \ldots, \frac{n-1}{2}\}$ there exists $k \in \{1, \ldots, \frac{n-1}{2}\}$ such that $\{ja_i^2 j, ja_i^{-2} j\} = \{a_k, a_k^{-1}\}$, both these orbits cover exactly n distinct edges with difference set $\{a_i^2, a_i^{-2}\}$ for each $i = 1, \ldots, \frac{n-1}{2}$, and therefore all the $2n$ edges with difference set $\{a_i, a_i^{-1}\}$ are covered exactly once. $\square$

A first example of this situation is given below.

Example. Let $\mathbb{D}_6$ be the dihedral group of order 6, i.e., the group with defining relations $\langle \alpha, j : \alpha^3 = j^2 = 1, j\alpha j = \alpha^2 \rangle$, let $C_3 = \{1, a, a^2\}$ be the cyclic group of order 3 and denote by $G = D_6 C_3$ the direct product of D_6 and C_3. The unique subgroup of order 9 in G is $H = \{1, \alpha, \alpha^2, a, a^2, \alpha a, \alpha a^2, \alpha^2 a, \alpha^2 a^2\}$. Consider the 1-factorization $\mathcal{F}$ described above, in particular:

$\mathcal{F} = \{F_1, F_h, F^*_h \mid h \in H^*\}$ with $F_h = F_1 h$, $F^*_h = \mathrm{Orb}_H([1, h])$ and

$$F_1 = \{[1, j], [a, a^2], [\alpha, \alpha^2], [\alpha a, \alpha^2 a^2], [\alpha a^2, \alpha^2 a],$$
$$[ja, ja^2], [j\alpha, j\alpha^2], [j\alpha a, j\alpha^2 a^2], [j\alpha a^2, j\alpha^2 a]\}.$$

The rainbow tree T obtained in Theorem 2.2 is illustrated in Figure 1.

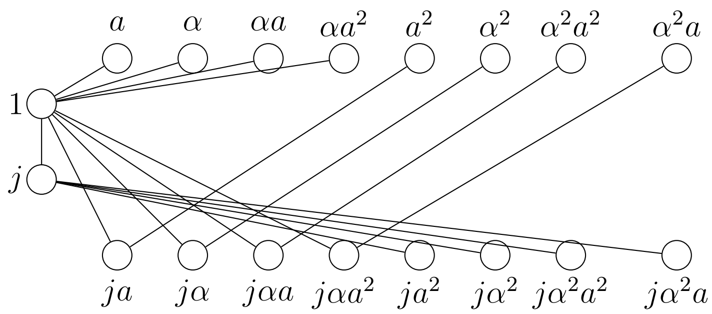

Figure 1: example with $G = \mathbb{D}_6 C_3$

3 Dihedral 1-factorizations and complete sets of rainbow spanning trees

It was proved in [23] that, when G is a cyclic group, a G-regular 1-factorization together with a complete set of rainbow spanning trees always exists except when the order of G is a power of 2 (the G-regular 1-factorization does not exist in this case [17]). In this section we enlarge this result to the class of dihedral groups and we prove the following.

Theorem 3.1. *Let G be a dihedral group of order $2n \geq 6$. There exists a G-regular 1-factorization of K_{2n} together with a complete set of rainbow spanning trees.*

When n is odd, the result follows from Theorem 2.2. The trees are pairwise isomorphic in this case, thus giving another example satisfying the Constantine conjecture. We consider the case n even and we exhibit a construction below.

We denote by $\mathbb{D}_{2n}$ the dihedral group of order $2n$. It has the following defining relations:

$$\mathbb{D}_{2n} = \langle \alpha, \beta : \alpha^n = \beta^2 = 1, \beta\alpha = \alpha^{n-1}\beta \rangle.$$

More precisely, we have $\mathbb{D}_{2n} = \{1, \alpha, \ldots, \alpha^{n-1}, \beta, \beta\alpha, \ldots, \beta\alpha^{n-1}\}$ with $\beta\alpha^i = \alpha^{n-i}\beta$, $i = 1, \ldots, n-1$.

We consider the $\mathbb{D}_{2n}$-regular 1-factorizations of K_{2n} constructed in [3] according to whether $n \equiv 0 \pmod 4$ or $n \equiv 2 \pmod 4$, and we describe them in terms of starters.

Starter in the case $n \equiv 0$ (mod 4):
$\Sigma = \{S, S', S^*\} \cup \{S_i \mid 1 \leq i \leq n-1, i \neq \frac{n}{2}\}$.
With:
$S = \{[\alpha^i, \alpha^{n-i+1}] \mid i = 1, \ldots, \frac{n}{4}\}$;
$S_i = \{[1, \beta\alpha^i]\}$, $i = 1, \ldots, n-1$, $i \neq \frac{n}{2}$;
$S' = \{[1, \alpha^{\frac{n}{2}}]\}$;

$S^* = \{[1, \beta], [\alpha^{\frac{n}{4}}, \beta\alpha^{\frac{n}{2}+\frac{n}{4}}], [\alpha^i, \alpha^{n-i}] \mid i = 1, \ldots, \frac{n}{4} - 1\}$ or $S^* = \{[1, \beta], [\alpha, \beta\alpha^3]\}$ according to whether $n > 4$ or $n = 4$.

Take the subgroup $K = \{1, \alpha^{\frac{n}{2}}, \beta, \beta\alpha^{\frac{n}{2}}\}$. We have:

$\partial S = \{\alpha^{2t+1},\ 1 \leq 2t+1 \leq n-1\}$ and $\phi(S)$ is a left transversal for K;

$\partial S_i = \{\beta\alpha^i\}$ and $\phi(S_i) = \{1\}$, $i = 1, \ldots, n-1$, $i \neq \frac{n}{2}$;

$\partial S' = \{\alpha^{\frac{n}{2}}\}$ and $\phi(S') = \{1\}$;

$\partial S^* = \{\alpha^{2t},\ 2 \leq 2t \leq n-2,\ 2t \neq \frac{n}{2}\} \cup \{\beta, \beta\alpha^{\frac{n}{2}}\}$ and $\phi(S^*)$ is a left transversal for K.

With the starter above, we construct the following 1-factors:

$F = \mathrm{Orb}_K(S)$, $\quad F^* = \mathrm{Orb}_K(S^*)$, $\quad F_{\alpha^{\frac{n}{2}}} = \mathrm{Orb}_{\mathbb{D}_{2n}}(S')$,

$F_{\beta\alpha^i} = \mathrm{Orb}_{\mathbb{D}_{2n}}(S_i) = \mathrm{Orb}_{\langle\alpha\rangle}(S_i)$, $i = 1, \ldots, n-1$, $i \neq \frac{n}{2}$.

Their orbits under D_{2n} give rise to the 1-factorization. Namely:

The 1-factor F is fixed by K. Its orbit under D_{2n} yields the 1-factors:

$$F, F\alpha, \ldots, F\alpha^{\frac{n}{2}-1}.$$

These 1-factors cover all the long edges with differences in the set $\{\alpha^{2t+1},\ 1 \leq 2t+1 \leq n-1\}$.

The 1-factor F^* is fixed by K. Its orbit under $\mathbb{D}_{2n}$ yields the 1-factors:

$$F^*, F^*\alpha, \ldots, F^*\alpha^{\frac{n}{2}-1}.$$

These 1-factors cover all the long edges with differences in the set $\{\alpha^{2t},\ 2 \leq 2t \leq n-2,\ 2t \neq \frac{n}{2}\}$ (which is empty whenever $n = 4$), together with all short edges with differences in the set $\{\beta, \beta\alpha^{\frac{n}{2}}\}$.

The other 1-factors are: $F_{\alpha^{\frac{n}{2}}}$ and $F_{\beta\alpha^i}$, $i = 1, \ldots, n-1, i \neq \frac{n}{2}$. All of them are fixed by $\mathbb{D}_{2n}$ and respectively cover all the short edges with differences in the set $\{\alpha^{\frac{n}{2}}\} \cup \{\beta\alpha^i, i = 1, \ldots, n-1, i \neq \frac{n}{2}\}$.

Starter in the case $n \equiv 2 \pmod 4$:

$\Sigma = \{S, S', S^*\} \cup \{S_i \mid 1 \leq i \leq n-1, i \neq \frac{n}{2}\}$.

With:

$S = \{[\alpha^i, \alpha^{n-i+1}] \mid i = 1, \ldots, \frac{n-2}{4}\} \cup \{[\beta\alpha^{\frac{n+2}{4}}, \alpha^{\frac{n}{2}+\frac{n+2}{4}}]\}$;

$S_i = \{[1, \beta\alpha^i]\}$, $i = 1, \ldots, n-1$, $i \neq \frac{n}{2}$;

$S' = \{[1, \alpha^{\frac{n}{2}}]\}$;

$S^* = \{[1, \beta], [\alpha^i, \alpha^{n-i}] \mid i = 1, \ldots, \frac{n-2}{4}\}$.

Take the subgroup $K = \{1, \alpha^{\frac{n}{2}}, \beta, \beta\alpha^{\frac{n}{2}}\}$. We have:

$\partial S = \{\alpha^{2t+1},\ 1 \leq 2t+1 \leq n-1,\ 2t+1 \neq \frac{n}{2}\} \cup \{\beta\alpha^{\frac{n}{2}}\}$ and $\phi(S)$ is a left transversal for K;

$\partial S_i = \{\beta\alpha^i\}$ and $\phi(S_i) = \{1\}$, $i = 1, \ldots, n-1$, $i \neq \frac{n}{2}$;

$\partial S' = \{\alpha^{\frac{n}{2}}\}$ and $\phi(S') = \{1\}$;

$\partial S^* = \{\alpha^{2t},\ 2 \leq 2t \leq n-2\} \cup \{\beta\}$ and $\phi(S^*)$ is a left transversal for K.
With the starter above, we construct the following 1-factors:
$F = \mathrm{Orb}_K(S)$, $F^* = \mathrm{Orb}_K(S^*)$, $F_{\alpha^{\frac{n}{2}}} = \mathrm{Orb}_{\mathbb{D}_{2n}}(S')$,
$F_{\beta\alpha^i} = \mathrm{Orb}_{\mathbb{D}_{2n}}(S_i)$, $i = 1, \ldots, n-1$, $i \neq \frac{n}{2}$.
Their orbits under D_{2n} give rise to the 1-factorization. Namely:
The 1-factor F is fixed by K. Its orbit under $\mathbb{D}_{2n}$ yields the 1-factors:

$$F, F\alpha, \ldots, F\alpha^{\frac{n}{2}-1}.$$

These 1-factors cover all the long edges with differences in the set $\{\alpha^{2t+1},\ 1 \leq 2t+1 \leq n-1,\ 2t+1 \neq \frac{n}{2}\}$ together with all the short edges with difference $\beta\alpha^{\frac{n}{2}}$.
The 1-factor F^* is fixed by K. Its orbit under $\mathbb{D}_{2n}$ yields the 1-factors:

$$F^*, F^*\alpha, \ldots, F^*\alpha^{\frac{n}{2}-1}.$$

These 1-factors cover all the long edges with differences in the set $\{\alpha^{2t},\ 2 \leq 2t \leq n-2\}$ together with all the short edges with difference β.
The other 1-factors are $F_{\alpha^{\frac{n}{2}}}$ and $F_{\beta\alpha^i}$, $i = 1, \ldots, n-1, i \neq \frac{n}{2}$. All of them are fixed by $\mathbb{D}_{2n}$ and respectively cover all the short edges with differences in the set $\{\alpha^{\frac{n}{2}}\} \cup \{\beta\alpha^i, i = 1, \ldots, n-1, i \neq \frac{n}{2}\}$.
We are now able to construct a complete set of rainbow spanning trees in both of these two cases. We know from Theorem 2.2 that such a set cannot be obtained as the orbit of a single tree and we use a slightly different strategy. More precisely, we will construct two spanning rainbow trees T_1 and T_2 in such a way that half the orbit of T_1 together with half the orbit of T_2 under the action of the group $\langle\alpha\rangle$ form the complete set. This set is partitioned into two isomorphic classes with $\frac{n}{2}$ trees each.

3.1 Case $n \equiv 0 \pmod 4$

Let $\mathcal{F}$ be the dihedral regular 1-factorization of K_{2n} described above when $n \equiv 0 \pmod 4$.
Consider the forest T' induced by the following set of edges:

$$T' = \{[1, \alpha^{4t+1}],\quad [\beta, \beta\alpha^{4t+1}],\quad 0 \leq t \leq \tfrac{n-4}{4}\}.$$

We have $[1, \alpha^{4t+1}] \in F\alpha^{2t}$ and $[\beta, \beta\alpha^{4t+1}] \in F\alpha^{2t+1}$, $0 \leq t \leq \frac{n-4}{4}$.
In fact $[1, \alpha^{4t+1}] = [\alpha^{2t+1}, \alpha^{n-2t}]\alpha^{2t} \in F\alpha^{2t}$ and

$$[\beta, \beta\alpha^{4t+1}] = [\beta\alpha^{n-2t-1}, \beta\alpha^{2t}]\alpha^{2t+1} = [\alpha^{2t+1}, \alpha^{n-2t}]\beta\alpha^{2t+1} \in F\beta\alpha^{2t+1} = F\alpha^{2t+1}$$

since $F\beta = F$.

We conclude that T' is rainbow as it has exactly one edge in each 1-factor $F\alpha^i$, $i = 0, \ldots, \frac{n}{2}-1$. It will be useful to observe that $\mathrm{Orb}_{\langle\alpha\rangle}(T')$ gives a set of n disjoint rainbow forests whose edges all together cover exactly once all the edges of the 1-factors F, $F\alpha$, $\ldots$, $F\alpha^{\frac{n}{2}-1}$.

Consider the tree T'' induced by the following sets of edges according to whether $n = 4$ or $n > 4$:

$$T'' = \{[1, \beta],\ [1, \beta\alpha^2]\},\ \text{or}$$

$$T'' = \{[1, \beta],\ [1, \beta\alpha^{\frac{n}{2}}],\ [1, \alpha^{2t}],\ [\beta\alpha^{\frac{n}{2}}, \beta\alpha^{2t}],\ \ 1 \le t \le \tfrac{n}{4} - 1\}.$$

We have $[1, \beta] \in F^*$ and $[1, \beta\alpha^{\frac{n}{2}}] = [\alpha^{\frac{n}{4}}, \beta\alpha^{\frac{n}{2}+\frac{n}{4}}]\alpha^{\frac{3}{4}n} \in F^*\alpha^{\frac{3}{4}n} = F^*\alpha^{\frac{n}{4}}$ since $F^*\alpha^{\frac{n}{2}} = F^*$.

Moreover, if $n > 4$, $[1, \alpha^{2t}] \in F^*\alpha^t$ and $[\beta\alpha^{\frac{n}{2}}, \beta\alpha^{2t}] \in F^*\alpha^{\frac{n}{4}+t}$, with $1 \le t \le \frac{n}{4} - 1$. In fact $[1, \alpha^{2t}] = [\alpha^{n-t}, \alpha^t]\alpha^t \in F^*\alpha^t$ and

$$\begin{aligned}
[\beta\alpha^{\frac{n}{2}}, \beta\alpha^{2t}] &= [\beta\alpha^{\frac{3}{4}n-t}, \beta\alpha^{\frac{n}{4}+t}]\alpha^{-\frac{n}{4}+t} = [\alpha^{\frac{n}{4}+t}\beta, \alpha^{\frac{3}{4}n-t}\beta]\alpha^{-\frac{n}{4}+t} \\
&= [\alpha^{\frac{n}{4}+t}, \alpha^{\frac{3}{4}n-t}]\beta\alpha^{-\frac{n}{4}+t} \in F^*\beta\alpha^{-\frac{n}{4}+t} = F^*\beta\alpha^{\frac{n}{2}}\alpha^{\frac{n}{4}+t} = F^*\alpha^{\frac{n}{4}+t}
\end{aligned}$$

since $F^*\beta\alpha^{\frac{n}{2}} = F^*$.

We conclude that T'' is a rainbow tree as it has exactly one edge in each 1-factor $F^*\alpha^i$, $i = 0, \ldots, \frac{n}{2} - 1$. We can also observe that $\mathrm{Orb}_{\langle\alpha\rangle}(T'')$ gives a set of n disjoint rainbow trees whose edges all together cover exactly once all edges of the 1-factors F^*, $F^*\alpha$, $\ldots$, $F^*\alpha^{\frac{n}{2}-1}$.

Observe that $T' \cup T''$ is rainbow itself. Also, T' and T'' have just 1 and β as common vertices and the edge $[1, \beta]$ of T'' connects the two components of T'. Therefore $T' \cup T''$ is a rainbow tree.

Now let $n = 4$. Let R be the subgraph of K_{2n} induced by the edges of $T' \cup T''$ together with the edges $\{[\beta\alpha^2, \alpha^3], [\alpha^2, \beta\alpha^3]\}$. Consider the tree T_1 induced by the edges of R together with the edge $[1, \alpha^2]$ and the tree T_2 induced by the edges of $R\alpha^2$ together with the edge $[\beta\alpha^3, \beta\alpha]$; see Figure 2, where T, T' and $\{[\beta\alpha^2, \alpha^3], [\alpha^2, \beta\alpha^3]\}$ are pictured assigning a color to each of them.

Observe that $[\beta\alpha^2, \alpha^3] \in F_{\beta\alpha^3}$, $[\alpha^2, \beta\alpha^3] \in F_{\beta\alpha}$, $[1, \alpha^2] \in F_{\alpha^2}$; then T_1 is a rainbow spanning tree. In the same manner one can observe that T_2 is a rainbow spanning tree itself.

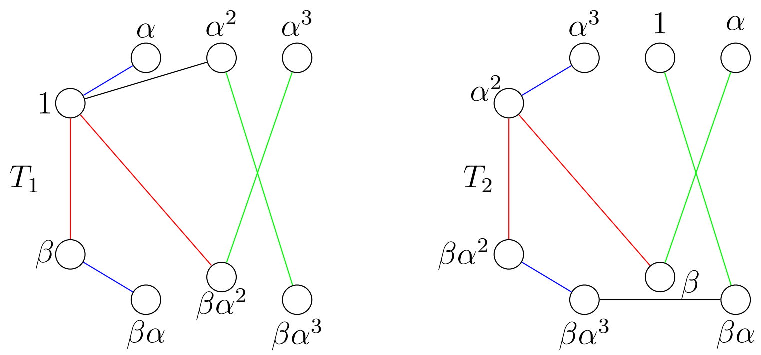

Figure 2: case $n = 4$

The set $\{T_1, T_1\alpha, T_2, T_2\alpha\}$ is a complete set of spanning trees orthogonal to the 1-factorization $\mathcal{F}$.

Now let $n > 4$. The set $\overline{V}$ of vertices of K_{2n} which do not belong to $T' \cup T''$ is the following:
$\overline{V} = \{\alpha^{\frac{n}{2}}\} \cup \{\alpha^{2t}, \beta\alpha^{2t}, \frac{n}{2} < 2t \leq n-2\} \cup \{\alpha^i, \beta\alpha^i, i \equiv 3 \pmod 4, 3 \leq i \leq n-1\}$.

To cover $\overline{V}$, we now construct T'''. Namely, let T''' be the subgraph of K_{2n} induced by the following set of edges:

$A = \{[\beta\alpha^{n-2}, \alpha^i], [\alpha^{n-2}, \beta\alpha^i], 3 \leq i \leq n-5, i \equiv 3 \pmod 4\}$,

$B = \{[\beta\alpha^{n-2}, \alpha^{n-1}], [\alpha^{n-2}, \beta\alpha^{n-1}], [\beta\alpha^{n-2}, \alpha^{\frac{n}{2}}], [\alpha^{n-2}, \beta\alpha^{\frac{n}{2}}]\}$,

$C = \{[\beta\alpha^{n-2}, \alpha^{2t}], [\alpha^{n-2}, \beta\alpha^{2t}], \frac{n}{4} + 1 \leq t \leq \frac{n}{2} - 2\}$,

where $C = \emptyset$ whenever $n = 8$.

Observe that A contains exactly one edge for each 1-factor $F_{\beta\alpha^r}$ with $3 \leq r \leq n-3$, r odd.
In fact $[\beta\alpha^{n-2}, \alpha^i] \in F_{\beta\alpha^{n-i-2}}$ and $n - i - 2 \equiv 3 \pmod 4$ varies from 3 to $n-5$. Moreover, $[\alpha^{n-2}, \beta\alpha^i] \in F_{\beta\alpha^{i+2}}$ and $i + 2 \equiv 1 \pmod 4$ varies from 5 to $n-3$.

We can also observe that $\mathrm{Orb}_{\langle\alpha\rangle}(A)$ covers exactly once all the edges in these 1-factors.

The four edges of B are respectively contained in $F_{\beta\alpha^{n-1}}$, $F_{\beta\alpha}$, $F_{\beta\alpha^{\frac{n}{2}-2}}$, $F_{\beta\alpha^{\frac{n}{2}+2}}$ and $\mathrm{Orb}_{\langle\alpha\rangle}(B)$ covers exactly once all the edges in these 1-factors.

When $n > 8$, the set C contains exactly one edge in each $F_{\beta\alpha^{2s}}$, with $2 \leq 2s \leq n-2$ and $2s \notin \{\frac{n}{2} - 2, \frac{n}{2}, \frac{n}{2} + 2\}$.
In fact $[\beta\alpha^{n-2}, \alpha^{2t}] \in F_{\beta\alpha^{n-2t-2}}$ and $[\alpha^{n-2}, \beta\alpha^{2t}] \in F_{\beta\alpha^{2t+2}}$ with $\frac{n}{4} + 1 \leq t \leq \frac{n}{2} - 2$.

We also have that $\mathrm{Orb}_{\langle\alpha\rangle}(C)$ covers exactly once all the edges in these 1-factors.

Observe that T''' is rainbow and it is the disjoint union of two stars: one at vertex $\beta\alpha^{n-2}$ and one at vertex α^{n-2}. The star at vertex α^{n-2} is connected to $T' \cup T''$ via the edge $[\alpha^{n-2}, \beta\alpha^{\frac{n}{2}}]$. Therefore, the graph $R = T' \cup T'' \cup T'''$ is a spanning forest with two connected components: one is the star at $\beta\alpha^{n-2}$ and the other is the tree induced by all edges not containing $\beta\alpha^{n-2}$. We can also observe that R is rainbow as it contains exactly one edge in each 1-factor of $\mathcal{F} - \{F_{\alpha^{\frac{n}{2}}}\}$; furthermore, $\mathrm{Orb}_{\langle\alpha\rangle}(R)$ covers exactly once all the edges in these 1-factors. The two connected components of R can be joined by the edge $[1, \alpha^{\frac{n}{2}}] \in F_{\alpha^{\frac{n}{2}}}$. Therefore the graph $T_1 = R \cup \{[1, \alpha^{\frac{n}{2}}]\}$ is orthogonal to $\mathcal{F}$; also it is spanning and connected and so it is a spanning tree orthogonal to $\mathcal{F}$. Observe also that $R\alpha^{\frac{n}{2}}$ contains exactly one edge in each 1-factor of $\mathcal{F} - \{F_{\alpha^{\frac{n}{2}}}\}$ and it has two connected components: one is the star at vertex $\beta\alpha^{\frac{n}{2}-2}$ and the other is the tree induced by all the edges not containing this vertex. As before, we connect the two components using the edge $[\beta\alpha^{n-2}, \beta\alpha^{\frac{n}{2}-2}] \in F_{\alpha^{\frac{n}{2}}}$ and $T_2 = R\alpha^{\frac{n}{2}} \cup \{[\beta\alpha^{n-2}, \beta\alpha^{\frac{n}{2}-2}]\}$ is a spanning tree orthogonal to $\mathcal{F}$.

The set $\mathcal{T} = \{T_1, T_1\alpha, \ldots, T_1\alpha^{\frac{n}{2}-1}, T_2, T_2\alpha, \ldots, T_2\alpha^{\frac{n}{2}-1}\}$ is a complete set of spanning trees orthogonal to $\mathcal{F}$.

For the reader's convenience, in Figures 3 and 4 we show T_1 when $n = 8$ and $n = 12$, respectively. We picture the sets T', T'', A, B, C assigning a color to each of them.

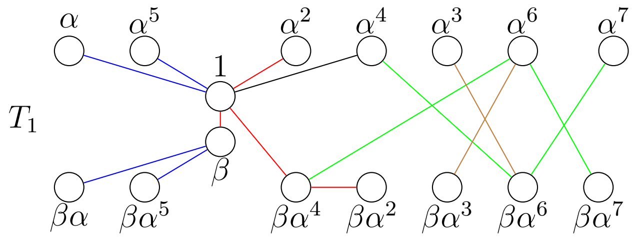

Figure 3: case $n = 8$

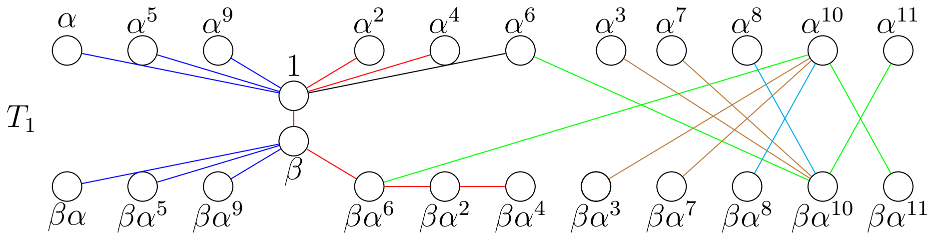

Figure 4: case $n = 12$

3.2 Case $n \equiv 2 \pmod 4$

Let $\mathcal{F}$ be the dihedral regular 1-factorization of K_{2n} described above when $n \equiv 2 \pmod 4$.

Consider the tree T' induced by the following set of edges:

$$T' \;=\; \{[1, \alpha^{4t+2}],\ [\beta, \beta\alpha^{n-(4t+2)}],\ [1, \beta],\quad 0 \le t \le \tfrac{n-6}{4}\}.$$

We have $[1, \beta] \in F^*$, and for each t, $0 \le t \le \frac{n-6}{4}$, we have: $[1, \alpha^{4t+2}] \in F^*\alpha^{2t+1}$ and $[\beta, \beta\alpha^{n-(4t+2)}] \in F^*\alpha^{\frac{n}{2}-(2t+1)}$.

In fact $[1, \alpha^{4t+2}] = [\alpha^{2t+1}, \alpha^{n-(2t+1)}]\alpha^{2t+1} \in F^*\alpha^{2t+1}$ and $[\beta, \beta\alpha^{n-(4t+2)}] = [\beta, \alpha^{4t+2}\beta] = [1, \alpha^{4t+2}]\beta$ and $[1, \alpha^{4t+2}]\beta \in F^*\alpha^{2t+1}\beta = F^*\beta\alpha^{n-(2t+1)} = F^*\alpha^{\frac{n}{2}-(2t+1)}$.

We conclude that T' is rainbow; in fact it has exactly one edge in each 1-factor $F^*\alpha^i$, $0 \le i \le \frac{n}{2} - 1$. It will be useful to observe that $\mathrm{Orb}_{\langle\alpha\rangle}(T')$ gives a set of n disjoint rainbow trees whose edges all together cover exactly once all the edges of the 1-factors F^*, $F^*\alpha$, …, $F^*\alpha^{\frac{n}{2}-1}$.

Consider the forest T'' induced by the following sets of edges:

$$T'' = \quad \{[\beta, \alpha^{\frac{n}{2}}],\ [1, \alpha^{2t-1}],\ [\beta\alpha^{\frac{n-2}{2}}, \beta\alpha^{2t-1}],\quad 1 \le t \le \tfrac{n-2}{4}\}.$$

We have $[\beta, \alpha^{\frac{n}{2}}] \in F\alpha^{\frac{n-2}{4}}$, and for each t, $1 \le t \le \frac{n-2}{4}$, we have $[1, \alpha^{2t-1}] \in F\alpha^{t-1}$ and $[\beta\alpha^{\frac{n-2}{2}}, \beta\alpha^{2t-1}] \in F\alpha^{\frac{n-2}{4}+t}$.

In fact $[\beta, \alpha^{\frac{n}{2}}] = [\beta\alpha^{\frac{n}{2}+\frac{n+2}{4}}, \alpha^{\frac{n+2}{4}}]\alpha^{\frac{n-2}{4}} \in F\alpha^{\frac{n-2}{4}}$ since

$$[\beta\alpha^{\frac{n}{2}+\frac{n+2}{4}}, \alpha^{\frac{n+2}{4}}] = [\beta\alpha^{\frac{n+2}{4}}, \alpha^{\frac{n+2}{4}+\frac{n}{2}}]\alpha^{\frac{n}{2}} \in F\alpha^{\frac{n}{2}} = F$$

and $[1,\alpha^{2t-1}]=[\alpha^t,\alpha^{n-t+1}]\alpha^{t-1}\in F\alpha^{t-1}$. Moreover,

$$[\beta\alpha^{\frac{n-2}{2}},\beta\alpha^{2t-1}]=[\beta\alpha^{\frac{n-2}{2}+\frac{n-2}{4}-t+1},\beta\alpha^{\frac{n-2}{2}-\frac{n-2}{4}+t}]\alpha^{-\frac{n-2}{4}+t-1}$$

and we have

$$[\beta\alpha^{\frac{n-2}{2}+\frac{n-2}{4}-t+1},\beta\alpha^{\frac{n-2}{2}-\frac{n-2}{4}+t}]=[\alpha^{n-\frac{n-2}{4}+t},\alpha^{1+\frac{n-2}{4}-t}]\alpha^{\frac{n}{2}}\beta\in F\alpha^{\frac{n}{2}}\beta=F\alpha^{\frac{n}{2}}$$

which implies that $[\beta\alpha^{\frac{n-2}{2}},\beta\alpha^{2t-1}]\in F\alpha^{\frac{n}{2}-\frac{n-2}{4}+t-1}=F\alpha^{\frac{n-2}{4}+t}$.

We conclude that T'' is rainbow; in fact it has exactly one edge in each 1-factor $F\alpha^i$, $0\le i\le\frac{n}{2}-1$. It will be useful to observe that $\mathrm{Orb}_{\langle\alpha\rangle}(T'')$ gives a set of n disjoint graphs whose edges all together cover exactly once all the edges of the 1-factors F, $F\alpha$, …, $F\alpha^{\frac{n}{2}-1}$.

The graph $T'\cup T''$ is rainbow and it is induced by three stars: one at vertex 1, one at vertex β and one at vertex $\beta\alpha^{\frac{n-2}{2}}$. Since $[1,\beta]\in T'\cup T''$, there are two possibilities according to whether $\frac{n}{2}\equiv 1 \pmod 4$ or $\frac{n}{2}\equiv 3 \pmod 4$. If the first case occurs, $T'\cup T''$ is a rainbow tree because the vertex $\beta\alpha^{\frac{n-2}{2}}$ is a vertex of both T' and T''. If the second case occur, $T'\cup T''$ is a rainbow forest with two connected components because $\beta\alpha^{\frac{n-2}{2}}$ is not a vertex in T'. Furthermore, the set $\overline{V}$ of vertices of K_{2n} which do not belong to $T'\cup T''$ can be described as follows:

$\{\beta\alpha^{\frac{n}{2}}\}\cup\{\alpha^{4t},\beta\alpha^{4t-2},\alpha^{\frac{n}{2}+2t},\beta\alpha^{\frac{n}{2}+2t},1\le t\le\frac{n-2}{4}\}$ or

$(\{\beta\alpha^{\frac{n}{2}}\}\cup\{\alpha^{4t},\beta\alpha^{4t-2},\alpha^{\frac{n}{2}+2t},\beta\alpha^{\frac{n}{2}+2t},1\le t\le\frac{n-2}{4}\})\setminus\{\beta\alpha^{\frac{n-2}{2}}\}$ according to whether $\frac{n}{2}\equiv 1 \pmod 4$ or $\frac{n}{2}\equiv 3 \pmod 4$. To cover $\overline{V}$, we now construct the rainbow graph T'''.

Namely, let $T'''=A\cup B\cup C$ be the subgraph of K_{2n} induced by the following set of edges:

$$A=\{[\beta\alpha^{n-1},\alpha^{4t}],1\le t\le\tfrac{n-2}{4},t\ne\tfrac{n-2}{8}\}\cup\{[\alpha^{n-1},\beta\alpha^{4t-2}],1\le t\le\tfrac{n-2}{4}\}$$

or

$$A=\{[\beta\alpha^{n-1},\alpha^{4t}],1\le t\le\tfrac{n-2}{4}\}\cup\{[\alpha^{n-1},\beta\alpha^{4t-2}],1\le t\le\tfrac{n-2}{4},t\ne\tfrac{n+2}{8}\},$$

according to whether $\frac{n}{2}\equiv 1 \pmod 4$ or $\frac{n}{2}\equiv 3 \pmod 4$;

$$B=\{[\beta\alpha^{\frac{n}{2}-2},\alpha^{\frac{n-2}{2}}],[\alpha,\beta\alpha^{\frac{n}{2}}],[\alpha^{n-1},\beta\alpha^{\frac{n}{2}}]\};$$

$$C=\{[\beta\alpha^{n-1},\alpha^{\frac{n}{2}+2t}],[\alpha^{n-1},\beta\alpha^{\frac{n}{2}+2t}],1\le t\le\tfrac{n-6}{4}\},$$

where $C=\emptyset$ whenever $n=6$.

Observe that A contains exactly one edge for each 1-factor $F_{\beta\alpha^r}$ with $1\le r\le n-3$, r odd, $r\ne\frac{n}{2}$.

We can also observe that $\mathrm{Orb}_{\langle\alpha\rangle}(A)$ covers exactly once all the edges in these 1-factors.

The three edges of B are respectively contained in $F_{\beta\alpha^{n-1}}$, $F_{\beta\alpha^{\frac{n}{2}-1}}$, $F_{\beta\alpha^{\frac{n}{2}+1}}$ and $\mathrm{Orb}_{\langle\alpha\rangle}(B)$ covers exactly once all the edges in these 1-factors.

When $n > 6$, the set C contains exactly one edge in each $F_{\beta\alpha^s}$, with s even, $2 \leq s \leq n-2$, $s \notin \{\frac{n}{2}-1, \frac{n}{2}, \frac{n}{2}+1\}$.

We also have that $\mathrm{Orb}_{\langle\alpha\rangle}(C)$ covers exactly once all the edges in these 1-factors.

Observe that $T' \cup T''$ and $A \cup C$ are disjoint. Furthermore, $T' \cup T'' \cup B$ is connected, with no cycles. The graph $A \cup C$ has no cycles as well. Also, if we have $n = 6$ then $T' \cup T'' \cup B$ and A are disjoint, while if we have $n > 6$ then $T' \cup T'' \cup B$ and $A \cup C$ just have the unique vertex α^{n-1} in common. We conclude that the graph $R = T' \cup T'' \cup T'''$ is acyclic and it covers all the vertices of K_{2n}. Observe that R has two connected components: the star at $\beta\alpha^{n-1}$ induced by all the edges of $A \cup C$ through $\beta\alpha^{n-1}$, and the tree induced by all the edges of $T' \cup T'' \cup B$ together with all the edges of $A \cup C$ through α^{n-1}. Finally, R is rainbow because T', T'', A, B, C are rainbow and do not share colors with each other. In particular, we point out that R contains exactly one edge in each 1-factor of $\mathcal{F} - \{F_{\alpha^{\frac{n}{2}}}\}$ and $\mathrm{Orb}_{\langle\alpha\rangle}(R)$ covers exactly once all the edges in these 1-factors. To connect the two components of R we can take the edge $[\alpha^{n-2}, \alpha^{\frac{n}{2}-2}] \in F_{\alpha^{\frac{n}{2}}}$, so that $T_1 = R \cup \{[\alpha^{n-2}, \alpha^{\frac{n}{2}-2}]\}$ is connected, spanning and rainbow. Therefore it is a spanning tree orthogonal to $\mathcal{F}$. Now take the graph $R\alpha^{\frac{n}{2}}$. It contains exactly one edge in each 1-factor of $\mathcal{F} - \{F_{\alpha^{\frac{n}{2}}}\}$ and it also has two connected components induced, respectively, by all the edges containing the vertex $\beta\alpha^{\frac{n}{2}-1}$ and by all the edges not containing it. We can take the edge $[\beta\alpha^{\frac{n}{2}-1}, \beta\alpha^{n-1}] \in F_{\alpha^{\frac{n}{2}}}$ to connect these two components and $T_2 = R\alpha^{\frac{n}{2}} \cup \{[\beta\alpha^{\frac{n}{2}-1}, \beta\alpha^{n-1}]\}$ is a spanning tree orthogonal to $\mathcal{F}$.

The set $\mathcal{T} = \{T_1, T_1\alpha, \ldots, T_1\alpha^{\frac{n}{2}-1}, T_2, T_2\alpha, \ldots, T_2\alpha^{\frac{n}{2}-1}\}$ is a complete set of spanning trees orthogonal to $\mathcal{F}$.

For the reader's convenience, in Figures 5 and 6 we show T_1 when $n = 6$ and $n = 10$, respectively. We picture the sets T', T'', A, B, C assigning a color to each of them.

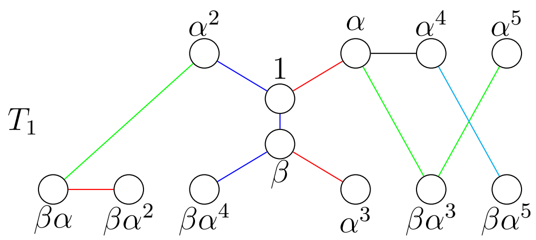

Figure 5: case $n = 6$

For each even integer n we have exhibited a $\mathbb{D}_{2n}$-regular 1-factorization of K_{2n} with a complete set of rainbow spanning trees. This, together with the result of Theorem 2.2, ends the proof of Theorem 3.1. $\square$

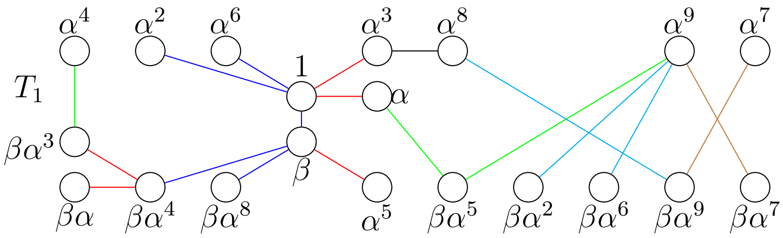

Figure 6: case $n = 10$

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