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On tight 9-cycle decompositions of complete 3-uniform hypergraphs

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Abstract

The complete 3-uniform hypergraph of order v, denoted by $K_v^{(3)}$, has a set V of size v as its vertex set and the set of all 3-element subsets of V as its edge set. A 3-uniform tight 9-cycle has vertex set $\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9\}$ and edge set $\big\{\{v_1, v_2, v_3\}, \{v_2, v_3, v_4\}, \{v_3, v_4, v_5\}, \{v_4, v_5, v_6\}, \{v_5, v_6, v_7\}, \{v_6, v_7, v_8\}, \{v_7, v_8, v_9\}, \{v_8, v_9, v_1\}, \{v_9, v_1, v_2\}\big\}$. We show there exists a tight 9-cycle decomposition of $K_v^{(3)}$ if and only if $v \equiv 1$ or $2 \pmod{27}$.

1 Introduction

A commonly studied problem in combinatorics concerns decompositions of complete graphs or other similar structures into isomorphic copies of other smaller graphs. Some of the best known unsolved problems in combinatorics relate to this area. For example, a projective plane of order v is equivalent to a decomposition of the complete graph K_{v^2+v+1} into isomorphic copies of K_{v+1}. One of the more celebrated decomposition problems for graphs pertains to decomposing K_v into m-cycles. This cycle decomposition problem originated with Walecki's solution of the problem of decomposing K_v into Hamiltonian cycles in the 1890s (see Lucas [13] or the more recent article by Alspach [2]). After decades of partial results by numerous authors, the cycle decomposition problem was fully settled by Alspach and Gavlas [3] and by Šajna [16] in the early 2000s.

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Because k-uniform hypergraphs are generalizations of the concept of a graph, authors have investigated the corresponding decomposition problems for hypergraphs. A *hypergraph* H consists of a finite nonempty set V of *vertices* and a set E of nonempty subsets of V called *hyperedges* or simply *edges*. If for each $e \in E$, we have $|e| = t$, then H is said to be t-*uniform*. Thus graphs are 2-uniform hypergraphs. Let V be a nonempty set and let $t \geq 2$ be an integer. Let $K_V^{(t)}$ denote the hypergraph with vertex set V and edge set the set of all t-element subsets of V. If $|V| = v$, we use $K_v^{(t)}$ denote any hypergraph isomorphic to $K_V^{(t)}$, and we refer to $K_v^{(t)}$ as the *complete* t-*uniform hypergraph of order* v.

A *decomposition* of a hypergraph K is a set $\Delta = \{H_1, H_2, \ldots, H_s\}$ of subhypergraphs of K such that $E(H_1) \cup E(H_2) \cup \cdots \cup E(H_s) = E(K)$ and $E(H_i) \cap E(H_j) = \varnothing$ for all $1 \leq i < j \leq s$. If each element H_i of Δ is isomorphic to a fixed hypergraph H, then H_i is called an H-*block*, and Δ is called an H-*decomposition* of K. We may in this case say that H *decomposes* K. An H-decomposition of $K_v^{(t)}$ is also known as an H-*design of order* v. The problem of determining all values of v for which there exists an H-design of order v is known as the *spectrum problem* for H.

We note that a $K_k^{(t)}$-design of order v is a generalization of Steiner systems and is equivalent to an $S(t, k, v)$-design. A summary of results on $S(t, k, v)$-designs appears in [5]. Keevash [11] has recently shown that for all t and k the obvious necessary conditions for the existence of an $S(t, k, v)$-design are sufficient for sufficiently large values of v. Similar results were obtained by Glock, Kühn, Lo, and Osthus [7, 8] and extended to include the corresponding asymptotic results for H-designs of order v for all uniform hypergraphs H. These results for t-uniform hypergraphs mirror the celebrated results of Wilson [17] for graphs. Although these asymptotic results assure the existence of H-designs for sufficiently large values of v for any uniform hypergraph H, the spectrum problem has been settled for very few hypergraphs of uniformity larger than 2.

There are several ways of defining an m-cycle in a t-uniform hypergraph. We focus on tight m-cycles, which generalize the Katona-Kierstead [10] definition of a Hamilton cycle (also called a Hamiltonian chain in [10]). For $m > t \geq 2$, let $\mathbb{Z}_m$ denote the group of integers modulo m and let $TC_m^{(t)}$ denote the t-uniform hypergraph with vertex set $\mathbb{Z}_m$ and (hyper)edge-set $\{\{i, i+1, \ldots, i+t-1\} : i \in \mathbb{Z}_m\}$. Any hypergraph isomorphic to $TC_m^{(t)}$ is a t-*uniform tight* m-*cycle*.

The problem of decomposing $K_v^{(3)}$ into $TC_v^{(3)}$'s was first investigated by Bailey and Stevens in [4]. Meszka and Rosa [15] added to the results from [4] and introduced the idea of $TC_m^{(3)}$-decompositions of $K_v^{(3)}$ with particular focus on the case $m = 5$. It is noted in [15] that as a consequence of Hanani's classic result on the existence of Steiner quadruple systems [9], there exists a $TC_4^{(3)}$-decomposition of $K_v^{(3)}$ if and only if $v \equiv 2$ or $4 \pmod 6$. More recently, several authors have given partial results on $TC_5^{(3)}$- and $TC_7^{(3)}$-decompositions of $K_v^{(3)}$ (see [12], [6], and [14]). For $m > 4$, the first complete results on $TC_m^{(3)}$-decompositions of $K_v^{(3)}$ were given in [1], where it is shown that there exists a $TC_6^{(3)}$-decomposition of $K_v^{(3)}$ if and only if $v \equiv 1, 2, 10, 20, 28,$ or $29 \pmod{36}$.

In this work, we focus on the case $m = 9$ and show that there exists a tight

9-cycle decomposition of $K_v^{(3)}$ if and only if $v \equiv 1$ or $2 \pmod{27}$.

2 Additional Notation and Terminology

If H is a hypergraph and r is a nonnegative integer, we let rH denote the edge-disjoint union of r copies of H. We next define some notation for certain types of 3-uniform hypergraphs.

Let A, B, C be pairwise-disjoint sets. The hypergraph with vertex set $A \cup B \cup C$ and edge set consisting of all 3-element sets having exactly one vertex in each of A, B, C is denoted by $K_{A,B,C}^{(3)}$. The hypergraph with vertex set $A \cup B$ and edge set consisting of all 3-element sets having at most 2 vertices in each of A and B is denoted by $L_{A,B}^{(3)}$. If $|A| = a$, $|B| = b$, and $|C| = c$, we may use $K_{a,b,c}^{(3)}$ to denote any hypergraph that is isomorphic to $K_{A,B,C}^{(3)}$ and $L_{a,b}^{(3)}$ to denote any hypergraph that is isomorphic to $L_{A,B}^{(3)}$. We use $K_{a,\bar{b},\bar{c}}^{(3)} \cup L_{\bar{b},\bar{c}}^{(3)}$ to denote any hypergraph isomorphic to $K_{A,B,C}^{(3)} \cup L_{B,C}^{(3)}$.

It is simple to observe that if A, B, B', and C are pairwise-disjoint, then $K_{A,B\cup B',C}^{(3)} = K_{A,B,C}^{(3)} \cup K_{A,B',C}^{(3)}$ and $L_{A,B\cup B'}^{(3)} = L_{A,B}^{(3)} \cup L_{A,B'}^{(3)} \cup K_{A,B,B'}^{(3)}$. Thus we have the following basic lemmas.

Lemma 1. *If a, b, b', c, and z are positive integers, then*

$$K_{a,b+b',zc}^{(3)} = z\Big(K_{a,b,c}^{(3)} \cup K_{a,b',c}^{(3)}\Big).$$

Lemma 2. *If a, b, x, and y are positive integers, then*

$$L_{xa,yb}^{(3)} = xyL_{a,b}^{(3)} \cup \binom{x}{2} yK_{a,a,b}^{(3)} \cup x\binom{y}{2} K_{a,b,b}^{(3)}.$$

3 Some Small Examples

Because all our tight 9-cycles are 3-uniform, we will henceforth use TC_9 in place of $TC_9^{(3)}$. Moreover, we will use $[a, b, c, d, e, f, g, h, i]$ to denote any hypergraph isomorphic to the TC_9 with vertex set $\{a, b, c, d, e, f, g, h, i\}$ and edge set $\big\{\{a, b, c\}, \{b, c, d\}, \{c, d, e\}, \{d, e, f\}, \{e, f, g\}, \{f, g, h\}, \{g, h, i\}, \{h, i, a\}, \{i, a, b\}\big\}$ as seen in Figure 1.

Next, we give several examples of TC_9-decompositions that are used in proving our main result.

Example 1. Let $V\Big(K_{28}^{(3)}\Big) = \mathbb{Z}_{28}$ and let

$$\begin{aligned} B = \big\{&[0,1,2,16,25,27,3,5,14], [0,26,24,4,21,5,19,3,20], [0,3,6,20,2,5,1,4,14],\\ &[0,4,8,14,7,9,27,1,22], [0,5,10,23,4,16,22,6,15], [0,6,12,7,1,24,9,4,5],\\ &[0,7,14,18,5,6,8,9,24], [0,8,16,23,26,4,12,18,21], [0,9,18,17,8,15,3,10,1],\\ &[0,10,20,22,25,6,14,23,26], [0,17,6,16,5,26,2,23,12],\\ &[0,12,24,27,4,8,16,20,25], [0,13,26,9,24,3,23,2,17]\big\}. \end{aligned}$$

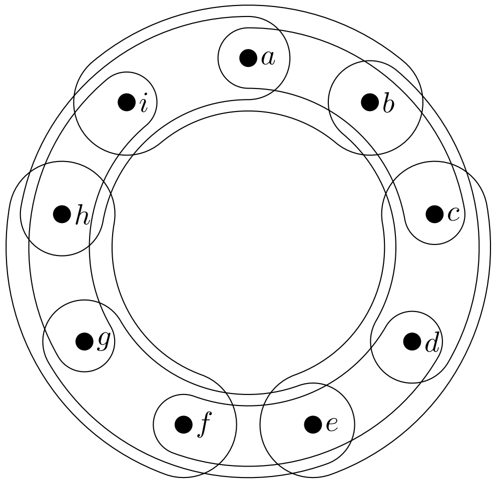

Figure 1: The 3-uniform tight 9-cycle TC_9 denoted $[a, b, c, d, e, f, g, h, i]$.

Then a TC_9-decomposition of $K_{28}^{(3)}$ consists of the orbits of the TC_9-blocks in B under the action of the map $j \mapsto j + 1 \pmod{28}$.

Example 2. Let $V\left(K_{29}^{(3)}\right) = \mathbb{Z}_{29}$ and let

$$\begin{aligned} B = \big\{&[0,8,16,17,21,22,24,25,28], && [0,16,3,5,13,15,19,21,27],\\ &[0,24,19,22,5,8,14,17,26], && [0,3,6,10,26,1,9,13,25],\\ &[0,11,22,27,18,23,4,9,24], && [0,19,9,15,10,16,28,5,23],\\ &[0,27,25,3,2,9,23,1,22], && [0,6,12,20,23,2,18,26,21],\\ &[0,14,28,8,15,24,13,22,20], && [0,22,15,25,7,17,8,18,19],\\ &[0,1,2,13,28,10,3,14,18], && [0,9,18,1,20,3,27,10,17],\\ &[0,17,5,18,12,25,22,6,16], && [0,25,21,6,4,18,17,2,15]\big\}. \end{aligned}$$

Then a TC_9-decomposition of $K_{29}^{(3)}$ consists of the orbits of the TC_9-blocks in B under the action of the map $j \mapsto j + 1 \pmod{29}$.

Example 3. Let $V\left(K_{1,9,9}^{(3)}\right) = \{\infty\} \cup \mathbb{Z}_{18}$ with vertex partition

$$\big\{\{\infty\}, \{0,2,\ldots,16\}, \{1,3,\ldots,17\}\big\}$$

and let $V\left(L_{9,9}^{(3)}\right) = \mathbb{Z}_{18}$ with vertex partition $\big\{\{0,2,\ldots,16\}, \{1,3,\ldots,17\}\big\}$. Let

$$\begin{aligned} B_0 = \big\{&[0,5,\infty,12,15,14,11,2,9], && [0,11,7,6,17,3,4,12,5],\\ &[0,1,2,8,11,16,6,3,10], && [0,3,15,4,2,7,8,13,11]\big\},\\ B_1 = \big\{&[0,1,\infty,10,9,12,8,17,3], && [1,2,\infty,11,10,13,9,0,4],\\ &[2,3,\infty,12,11,14,10,1,5], && [3,4,\infty,13,12,15,11,2,6],\\ &[4,5,\infty,14,13,16,12,3,7], && [5,6,\infty,15,14,17,13,4,8],\\ &[6,7,\infty,16,15,0,14,5,9], && [7,8,\infty,17,16,1,15,6,10],\\ &[8,9,\infty,0,17,2,16,7,11]\big\}. \end{aligned}$$

Then a TC_9-decomposition of $L^{(3)}_{\overline{9},\overline{9}} \cup K^{(3)}_{1,\overline{9},\overline{9}}$ consists of the TC_9-blocks in B_1 and the orbits of the TC_9-blocks in B_0 under the action of the map $\infty \mapsto \infty$ and $j \mapsto j+1 \pmod{18}$, for $j \in \mathbb{Z}_{18}$.

Example 4. Let $V\left(K^{(3)}_{2,9,9}\right) = \{\infty_1, \infty_2\} \cup \mathbb{Z}_{18}$ with vertex partition $\big\{\{\infty_1, \infty_2\}, \{0, 2, \ldots, 16\}, \{1, 3, \ldots, 17\}\big\}$ and let $V\left(L^{(3)}_{9,9}\right) = \mathbb{Z}_{18}$ with vertex partition $\big\{\{0, 2, \ldots, 16\}, \{1, 3, \ldots, 17\}\big\}$. Let

$$\begin{aligned}
B_0 = \big\{&[0,5,\infty_2,12,11,10,2,7,15], && [0,5,\infty_1,12,15,14,11,2,9],\\
&[0,11,7,6,17,3,4,12,5], && [0,3,15,4,2,7,8,13,11]\big\},\\
B_1 = \big\{&[0,1,\infty_1,10,9,12,8,17,3], && [1,2,\infty_1,11,10,13,9,0,4],\\
&[2,3,\infty_1,12,11,14,10,1,5], && [3,4,\infty_1,13,12,15,11,2,6],\\
&[4,5,\infty_1,14,13,16,12,3,7], && [5,6,\infty_1,15,14,17,13,4,8],\\
&[6,7,\infty_1,16,15,0,14,5,9], && [7,8,\infty_1,17,16,1,15,6,10],\\
&[8,9,\infty_1,0,17,2,16,7,11], && [0,3,\infty_2,12,9,16,10,1,7],\\
&[1,4,\infty_2,13,10,17,11,2,8], && [2,5,\infty_2,14,11,0,12,3,9],\\
&[3,6,\infty_2,15,12,1,13,4,10], && [4,7,\infty_2,16,13,2,14,5,11],\\
&[5,8,\infty_2,17,14,3,15,6,12], && [6,9,\infty_2,0,15,4,16,7,13],\\
&[7,10,\infty_2,1,16,5,17,8,14], && [8,11,\infty_2,2,17,6,0,9,15]\big\}.
\end{aligned}$$

Then a TC_9-decomposition of $L^{(3)}_{\overline{9},\overline{9}} \cup K^{(3)}_{2,\overline{9},\overline{9}}$ consists of the TC_9-blocks in B_1 and the orbits of the TC_9-blocks in B_0 under the action of the map $\infty_i \mapsto \infty_i$, for $i \in \{1,2\}$, and $j \mapsto j+1 \pmod{18}$, for $j \in \mathbb{Z}_{18}$.

Example 5. Let $V\left(K^{(3)}_{3,3,3}\right) = \mathbb{Z}_9$ and let

$$B = \big\{[0,1,2,3,4,5,6,7,8], [0,2,4,6,8,1,3,5,7], [0,4,8,3,7,2,6,1,5]\big\}.$$

Then B is a TC_9-decomposition of $K^{(3)}_{3,3,3}$.

4 Main Results

We give necessary and sufficient conditions for the existence of a TC_9-decomposition of $K^{(3)}_v$. We begin with the necessary conditions.

Lemma 3. *There exists a TC_9-decomposition of $K^{(3)}_v$ only if $v \equiv 1$ or $2 \pmod{27}$.*

Proof. Consider that TC_9 has size 9 and is 3-regular; whereas, $K^{(3)}_v$ has size $\binom{v}{3}$ and is $\binom{v-1}{2}$-regular. Hence, for a TC_9-decomposition of $K^{(3)}_v$ to exist, we must have $9 \mid v(v-1)(v-2)/6$ and $3 \mid (v-1)(v-2)/2$. Thus, $v \equiv 1$ or $2 \pmod{27}$. □

We prove that the above conditions are sufficient by showing how to construct a TC_9-decomposition for $K^{(3)}_v$ for all admissible values of v. These constructions are dependent on the small examples we gave in Section 3. First, we prove a lemma that is fundamental to our constructions.

Lemma 4. *Let $v = 27x + r$ for some positive integer x and for $r \in \{1, 2\}$. Then there exists a decomposition of $K_v^{(3)}$ that is comprised of isomorphic copies of each of the following hypergraphs under the given conditions:*

- *one copy of $K_{27+r}^{(3)}$,*
- *$\binom{x}{2}$ copies of $K_{r,\overline{27},\overline{27}}^{(3)} \cup L_{\overline{27},\overline{27}}^{(3)}$ if $x \geq 2$,*
- *$\binom{x}{3}$ copies of $K_{27,27,27}^{(3)}$ if $x \geq 3$.*

Proof. Let $R, V_1, \ldots, V_x$ be pairwise-disjoint sets of vertices with $|R| = r$ and $|V_1| = |V_2| = \cdots = |V_x| = 27$ and let $V = R \cup V_1 \cup \cdots \cup V_x$. Then, the result follows from the fact that $K_V^{(3)}$ can be viewed as the edge-disjoint union

$$K_{V_1 \cup R}^{(3)} \cup \bigcup_{1 \leq i < j \leq x} \left(K_{R,V_i,V_j}^{(3)} \cup L_{V_i,V_j}^{(3)} \right) \cup \bigcup_{1 \leq i < j < k \leq x} \left(K_{V_i,V_j,V_k}^{(3)} \right). \qquad \square$$

Before proceeding to our main result, we prove the following basic lemma.

Lemma 5. *There exists a TC_9-decomposition of $K_{r,\overline{27},\overline{27}}^{(3)} \cup L_{\overline{27},\overline{27}}^{(3)}$ for $r \in \{1, 2\}$.*

Proof. Let $R, A_1, A_2, A_3, B_1, B_2, B_3$ be pairwise-disjoint sets of 9 vertices each except that $|R| \in \{1, 2\}$ and let $A = A_1 \cup A_2 \cup A_3$ and $B = B_1 \cup B_2 \cup B_3$. Then

$$K_{R,A,B}^{(3)} = \bigcup_{\substack{1 \leq j \leq 3 \\ 1 \leq k \leq 3}} K_{R,A_j,B_k}^{(3)}$$

and

$$L_{A,B}^{(3)} = \bigcup_{\substack{1 \leq j \leq 3 \\ 1 \leq k \leq 3}} \left(L_{A_j,B_k}^{(3)} \right) \cup \bigcup_{\substack{1 \leq i < j \leq 3 \\ 1 \leq k \leq 3}} \left(K_{A_i,A_j,B_k}^{(3)} \right) \cup \bigcup_{\substack{1 \leq i \leq 3 \\ 1 \leq j < k \leq 3}} \left(K_{A_i,B_j,B_k}^{(3)} \right).$$

Thus,

$$K_{R,A,B}^{(3)} \cup L_{A,B}^{(3)} = \bigcup_{\substack{1 \leq j \leq 3 \\ 1 \leq k \leq 3}} \left(K_{R,A_j,B_k}^{(3)} \cup L_{A_j,B_k}^{(3)} \right) \cup \bigcup_{\substack{1 \leq i < j \leq 3 \\ 1 \leq k \leq 3}} \left(K_{A_i,A_j,B_k}^{(3)} \right) \cup \bigcup_{\substack{1 \leq i \leq 3 \\ 1 \leq j < k \leq 3}} \left(K_{A_i,B_j,B_k}^{(3)} \right).$$

Therefore, for $r \in \{1, 2\}$, $K_{r,\overline{27},\overline{27}}^{(3)} \cup L_{\overline{27},\overline{27}}^{(3)}$ can be decomposed into copies of $K_{r,\overline{9},\overline{9}}^{(3)} \cup L_{\overline{9},\overline{9}}^{(3)}$ and copies of $K_{9,9,9}^{(3)}$. By Examples 3 and 4, TC_9 decomposes $K_{1,\overline{9},\overline{9}}^{(3)} \cup L_{\overline{9},\overline{9}}^{(3)}$ and $K_{2,\overline{9},\overline{9}}^{(3)} \cup L_{\overline{9},\overline{9}}^{(3)}$, respectively. Thus, TC_9 decomposes $K_{r,\overline{27},\overline{27}}^{(3)} \cup L_{\overline{27},\overline{27}}^{(3)}$, where $r \in \{1, 2\}$. $\square$

Finally, we give our main result.

Theorem 6. *There exists a TC_9-decomposition of $K_v^{(3)}$ if and only if $v \equiv 1$ or $2 \pmod{27}$.*

Proof. The necessary conditions for the existence of a TC_9-decomposition of $K_v^{(3)}$ are established in Lemma 3. Thus, we need only establish their sufficiency.

Suppose $v \equiv 1$ or $2 \pmod{27}$. If $v \leq 2$, the result is vacuously true. Otherwise, let $v = 27x+r$ where $r \in \{1, 2\}$. By Lemma 4, it suffices to find TC_9-decompositions of $K_{27+r}^{(3)}$, $K_{r,\overline{27},\overline{27}}^{(3)} \cup L_{\overline{27},\overline{27}}^{(3)}$, and $K_{27,27,27}^{(3)}$. We give TC_9-decompositions of $K_{28}^{(3)}$ and of $K_{29}^{(3)}$ in Examples 1 and 2, respectively. By Lemma 5 we have that TC_9 decomposes $K_{r,\overline{27},\overline{27}}^{(3)} \cup L_{\overline{27},\overline{27}}^{(3)}$ for $r \in \{1, 2\}$, and by Lemma 1 we have that $K_{3,3,3}^{(3)}$ decomposes $K_{27,27,27}^{(3)}$. Since TC_9 decomposes $K_{3,3,3}^{(3)}$ (by Example 5), the result then follows. □

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References

[1] M. Akin, R. C. Bunge, S. I. El-Zanati, J. Hamilton, B. Kolle, S. Lehmann and L. Neiburger, On tight 6-cycle decompositions of complete 3-uniform hypergraphs, (preprint).

[2] B. Alspach, The wonderful Walecki construction, *Bull. Inst. Combin. Appl.* **52** (2008), 7–20.

[3] B. Alspach and H. Gavlas, Cycle decompositions of K_n and $K_n - I$, *J. Combin. Theory, Ser. B* **81** (2001), 77–99.

[4] R. F. Bailey and B. Stevens, Hamilton decompositions of complete k-uniform hypergraphs, *Discrete Math.* **310** (2010), 3088–3095.

[5] C. J. Colbourn and R. Mathon, Steiner systems, in *The CRC Handbook of Combinatorial Designs*, 2nd edition, (Eds. C.J. Colbourn and J.H. Dinitz), CRC Press, Boca Raton (2007), 102–110.

[6] M. Gionfriddo, L. Milazzo and Z. Tuza, Hypercycle systems, *Australas. J. Combin.* **77** (2020), 336–354.

[7] S. Glock, D. Kühn, A. Lo and D. Osthus, The existence of designs via iterative absorption, arXiv:1611.06827v2, (2017), 63 pp.

[8] S. Glock, D. Kühn, A. Lo and D. Osthus, Hypergraph F-designs for arbitrary F, arXiv:1706.01800, (2017), 72 pp.

[9] H. Hanani, On quadruple systems, *Canad. J. Math.* **12** (1960), 145–157.

[10] G. Y. Katona and H. A. Kierstead, Hamiltonian chains in hypergraphs, *J. Graph Theory* **30** (1999), 205–212.

[11] P. Keevash, The existence of designs, arXiv:1401.3665v2, (2018), 39 pp.

[12] G. R. Li, Y. M. Lei, L. Q. Zhao, Y. S. Yang and Jirimutu, Decomposing the complete 3-uniform hypergraph $K_n^{(3)}$ into 5-cycles, *J. Math. Res. Appl.* **36** (2016), 9–14.

[13] E. Lucas, Récreations Mathématiqués, Vol. II, Paris, 1892.

[14] G. M. Meihua and Jirimutu, Decomposing complete 3-uniform hypergraph $K_n^{(3)}$ into 7-cycles, *Opuscula Math.* **39** (2019), 383–393.

[15] M. Meszka and A. Rosa, Decomposing complete 3-uniform hypergraphs into Hamiltonian cycles, *Australas. J. Combin.* **45** (2009), 291–302.

[16] M. Šajna, Cycle decompositions III: complete graphs and fixed length cycles, *J. Combin. Des.* **10** (2002), 27–78.

[17] R. M. Wilson, Decompositions of Complete Graphs into Subgraphs Isomorphic to a Given Graph, in "Proc. Fifth British Combinatorial Conference" (C. St. J. A. Nash-Williams and J. Sheehan, Eds.), *Congr. Numer.* XV (1975), 647–659.

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