AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **80(2)** (2021), Pages 241–248

A note on uniformly resolvable $\{P_4, C_6\}$-designs

Salvatore Milici

Dipartimento di Matematica e Informatica
Università di Catania
Italy
milici@dmi.unict.it

Dedicated to my friends Prof. Carmelo Mammana and Prof. Biagio Micale, recently passed away

Abstract

Given a collection of graphs $\mathcal{H}$, a uniformly resolvable $\mathcal{H}$*-design* of order v is a decomposition of the edges of K_v into isomorphic copies of graphs from $\mathcal{H}$ (also called *blocks*) in such a way that all blocks in a given parallel class are isomorphic to the same graph from $\mathcal{H}$. We consider the case $\mathcal{H} = \{P_4, C_6\}$, and prove that the necessary conditions on the existence of such designs are also sufficient.

1 Introduction

Given a collection of graphs $\mathcal{H}$, an $\mathcal{H}$*-design* of order v (also called an $\mathcal{H}$-decomposition of K_v) is a decomposition of the edges of K_v into isomorphic copies of graphs from $\mathcal{H}$; the copies of $H \in \mathcal{H}$ in the decomposition are called *blocks*. An $\mathcal{H}$-design is called *resolvable* if it is possible to partition the blocks into *classes* $\mathcal{P}_i$ such that every point of K_v appears exactly once in some block of each $\mathcal{P}_i$.

A resolvable $\mathcal{H}$-decomposition of K_v is sometimes also referred to as an $\mathcal{H}$*-factorization of* K_v, and a class can be called an $\mathcal{H}$*-factor of* K_v. A resolvable $\mathcal{H}$-design is called *uniform* if every block of the class is isomorphic to the same graph from $\mathcal{H}$. Uniformly resolvable decompositions of K_v have also been studied in [4, 7–14, 16]. In what follows, we will denote by $(a_1, a_2, \ldots, a_n)$ the n-cycle on $\{a_1, a_2, \ldots,$
$a_n\}$ with edge-set $\{\{a_1, a_2\}, \{a_2, a_3\}, \ldots, \{a_{n-1}, a_n\}, \{a_n, a_1\}\}$ and by $[a_1, \ldots, a_n]$, $n \geq 2$, the path P_n having vertex set $\{a_1, \ldots, a_n\}$ and edge set $\{\{a_1, a_2\}, \{a_2, a_3\}, \ldots,$ $\{a_{n-1}, a_n\}\}$. In this paper we study the existence of uniformly resolvable decompositions into paths P_4 and cycles C_6 for the complete graph K_v.

The existence of resolvable decompositions for each of P_k and C_k has been studied separately, some time ago.

ISSN: 2202-3518

- There exists a resolvable C_k-decomposition of $K_v - I$ if and only if $v \equiv 0 \pmod 2$ and k divides v (see [5]).
- There exists a resolvable P_k-decomposition of λK_v if and only if $v \equiv 0 \pmod k$ and $\lambda k(v-1) \equiv 0 \pmod{2(k-1)}$ (see [1, 6]).

A uniformly resolvable (P_4, C_6)-decomposition of K_v into exactly r P_4-factors and s C_6-factors is abbreviated (P_4, C_6)-URD$(v; r, s)$. Since the results for the extremal cases $r = 0$ and $s = 0$ are known (see, for instance, [1, 5, 6]) we deal with (P_4, C_6)-URD$(v; r, s)$ where $r, s > 0$. For $v \equiv 0 \pmod{12}$, we define the set

$$J(v) = \left\{ (\frac{2(v-3)}{3} - 4x, 1 + 3x), \quad x = 0, 1, \ldots, \frac{v-6}{6} \right\}.$$

In this paper we completely solve the existence problem of a (P_4, C_6)-URD$(v; r, s)$ of K_v by proving the following result:

Main Theorem. *Let $v \equiv 0 \pmod{12}$. There exists a (P_4, C_6)-URD$(v; r, s)$ of K_v if and only if $(r, s) \in J(v)$.*

2 Necessary conditions

Lemma 2.1. *If there exists a (P_4, C_6)-URD$(v; r, s)$, then $v \equiv 0 \pmod{12}$ and $(r, s) \in J(v)$.*

Proof. The condition $v \equiv 0 \pmod{12}$ is trivial. Assume that there exists a (P_4, C_6)-URD$(v; r, s)$. By resolvability, it follows that

$$\frac{3rv}{4} + \frac{6sv}{6} = \frac{v(v-1)}{2}$$

and hence

$$3r + 4s = 2(v-1). \tag{1}$$

This equation implies that $3r \equiv 2(v-1) \pmod 4$ and $4s \equiv 2(v-1) \pmod 3$. Then we obtain $r \equiv 2 \pmod 4$ and $s \equiv 1 \pmod 3$. Now letting $s = 1 + 3x$, the equation (1) yields $r = \frac{2(v-3)}{3} - 4x$. Since r and s cannot be negative, and x is an integer, the value of x has to be in the range as given in the definition of $J(v)$. This completes the proof. □

3 Preliminaries and constructions

An $\mathcal{H}$-decomposition of the complete multipartite graph with u parts each of size g is known as a group divisible design $\mathcal{H}$-GDD of type g^u, and the parts of size g are called the groups of the design. When $\mathcal{H} = \{H\}$, we simply write H-GDD and when $H = K_n$ we refer to such a group divisible design as an n-GDD. We denote a (uniformly) resolvable $\mathcal{H}$-GDD by $\mathcal{H}$-(U)RGDD. It is easy to deduce that the number of parallel classes of an H-RGDD is $\frac{g(u-1)|V(H)|}{2|E(H)|}$. A (P_4, C_6)-URGDD (r, s) of type g^u is a uniformly resolvable decomposition of the complete multipartite graph with u parts each of size g into r classes containing only copies of P_4-paths and s classes containing only copies of C_6-cycles .

If the blocks of an n-GDD of type g^u can be partitioned into partial parallel classes, each of them containing all points except those of one group, we refer to the decomposition as an *n-frame*. It is easy to deduce that the number of partial factors missing a specified group is $\frac{g}{n-1}$ ([3]). It is well–known that a 2-frame of type g^u exists if and only if $u \geq 3$ and $g(u-1) \equiv 0 \pmod 2$ ([3]).

An incomplete resolvable (P_4, C_6)-decomposition of K_v with a hole of size h is a (P_4, C_6)-decomposition of $K_{v+h} - K_h$ in which there are two types of classes, full classes and partial classes which cover every point except those in the hole (the points of K_h are referred to as the hole). Specifically, a (P_4, C_6)-IURD$(v + h, h; [r_1, s_1], [\bar{r}_1, \bar{s}_1])$ is a uniformly resolvable (P_4, C_6)-decomposition of $K_{v+h} - K_h$ with r_1 partial classes of paths P_4 and s_1 partial classes of cycles C_6 which cover only the points not in the hole, $\bar{r}_1$ full classes of paths P_4 and $\bar{s}_1$ full classes cycles C_6 which cover every point of K_{v+h}.

We also recall the following definitions. Let (s_1, t_1) and (s_2, t_2) be two pairs of non-negative integers. Define $(s_1, t_1) + (s_2, t_2) = (s_1 + s_2, t_1 + t_2)$. If X and Y are two sets of pairs of non-negative integers, then $X + Y$ denotes the set $\{(s_1, t_1) + (s_2, t_2) : (s_1, t_1) \in X, (s_2, t_2) \in Y\}$. If X is a set of pairs of non-negative integers and h is a positive integer, then $h * X$ denotes the set of all pairs of non-negative integers which can be obtained by adding any h elements of X together (repetitions of elements of X are allowed).

The following three constructions have been proved in a more general setting in [7]. For the ease of the reader, since we will make use of them, we adapt their proofs in our case.

Construction 3.1. *Let t be a positive integer and $\mathcal{G}$ be an n-RGDD of type g^u. If there exists a (P_4, C_6)-URGDD$(\bar{r}, \bar{s})$ of type t^n for each $(\bar{r}, \bar{s}) \in J_1$, then so does a (P_4, C_6)-URGDD(r, s) of type $(gt)^u$ for each $(r, s) \in h * J$, where $h = \frac{g(u-1)}{n-1}$.*

Proof. Let $\mathcal{G}$ be an n-RGDD of type g^u, with u groups G_i, $i = 1, 2, \ldots, u$, of size g; let $R_1, R_2, \ldots, R_h$, $h = \frac{g(u-1)}{n-1}$, be the parallel classes of this n-RGDD. Expand t times each point and for each block b of a given resolution class of $\mathcal{G}$ place on $b \times \{1, 2, \ldots, t\}$ a copy of a (P_4, C_6)-URGDD(r_1, s_1) of type t^n with $(r_1, s_1) \in J_1$. Thus we obtain a (P_4, C_6)-URGDD(r, s) of type $(gt)^u$ with $(r, s) \in h * J_1$. □

Construction 3.2. *Let v, g, t and u be non-negative integers such that $v = gtu$. If there exist*

(1) *an n-RGDD of type g^u;*

(2) *a (P_4, C_6)-URGDD(r_1, s_1) of type t^n with $(r_1, s_1) \in J_1$;*

(3) *a* (P_4, C_6)*-URD*$(gt; r_2, s_2)$*, with* $(r_2, s_2) \in J_2$*;*

then there exists a (P_4, C_6)*-URD*$(v; r, s)$ *for each* $(r, s) \in J_2 + h * J_1$*, where* $h = \frac{g(u-1)}{n-1}$ *is the number of parallel classes of an* n*-RGDD of type* g^u*.*

Proof. Let $\mathcal{G}$ be an n-RGDD of type g^u, with u groups G_i, $i = 1, 2, \ldots, u$, of size g with $h = \frac{g(u-1)}{n-1}$ parallel classes. Expand each point t times and for each block b of a given resolution class of $\mathcal{G}$ place on $b \times \{1, 2, \ldots, t\}$ a copy of a (P_4, C_6)-URGDD(r_1, s_1) of type t^n with $(r_1, s_1) \in J_1$. For each $i = 1, 2, \ldots, u$, place on $G_i \times \{1, 2, \ldots, t\}$ a copy of a (P_4, C_6)-URD$(gt; r_2, s_2)$ with $(r_2, s_2) \in J_2$. The result is a $(K_2, K_{1,3})$-URD$(v; r, s)$ with $(r, s) \in J_2 + h * J_1$. □

Construction 3.3. *Let* v*,* g*,* t*,* h *and* u *be non-negative integers such that* $v = gtu + h$*. If there exist*

(1) *a 2-frame* $\mathcal{F}$ *of type* g^u*;*

(2) *a* (P_4, C_6)*-URD*$(h; r_1, s_1)$ *with* $(r_1, s_1) \in J_1$*;*

(3) *a* (P_4, C_6)*-URGDD*(r_2, s_2) *of type* t^2 *with* $(r_2, s_2) \in J_2$*;*

(4) *a* (P_4, C_6)*-IURD*$(gt + h, h; [r_1, s_1], [r_3, s_3])$ *with* $(r_1, s_1) \in J_1$ *and* $(r_3, s_3) \in J_3 = g * J_2$*;*

then there exists a (P_4, C_6)*-URD*$(v; r, s)$ *for each* $(r, s) \in J_1 + u * J_3$*.*

Proof. Let $\mathcal{F}$ be a 2-frame of type g^u with groups G_i, $i = 1, 2, \ldots, u$; expand each point t times and add a set $H = \{a_1, a_2, \ldots, a_h\}$. For $j = 1, 2$, let $p_{i,j}$ be the j-th partial parallel class which miss the group G_i; for each $b \in p_{i,j}$, place on $b \times \{1, 2, \ldots, t\}$ a copy $D^b_{i,j}$ of a (P_4, C_6)-URGDD(r_2, s_2) of type t^2, with $(r_2, s_2) \in J_2$; place on $G_i \times \{1, 2, \ldots, t\} \cup H$ a copy D_i of a (P_4, C_6)-IURD$(gt + h, h; [r_1, s_1], [r_3, s_3])$ with H as hole, $(r_1, s_1) \in J_1$ and $(r_3, s_3) \in J_3 = g * J_2$. Now combine all together the parallel classes of $D^b_{i,j}$, $b \in p_{i,j}$, along with the full classes of D_i. We obtain r_3 classes of paths P_4 and s_3 classes of 6-cycles, $(r_3, s_3) \in J_3$, on $\cup_{i=1}^u G_i \times \{1, 2, \ldots, t\} \cup H$. Fill the hole H with a copy D of (P_4, C_6)-URD$(h; r_1, s_1)$ with $(r_1, s_1) \in J_1$ and combine the classes of D with the partial classes of D_i. Then we obtain r_1 classes of paths P_4 and s_1 classes of 6-cycles, on $\cup_{i=1}^u G_i \times \{1, 2, \ldots, t\} \cup H$. The result is a (P_4, C_6)-URD$(v; r, s)$ for each $(r, s) \in J_1 + u * J_3$.

□

We also recall the following two results that we use to prove the main theorem.

Lemma 3.4. ([2]) *For* $l \geq 3$ *and* $u \geq 2$*, there exists a* C_l*-RGDD of type* g^u *if and only if* $g(u-1) \equiv 0 \pmod 2$*,* $gu \equiv 0 \pmod l$*,* $l \equiv 0 \pmod 2$ *if* $u = 2$*, and* $(g, u, l) \notin \{(2, 3, 3), (6, 3, 3), (2, 6, 3), (6, 2, 6)\}$*.*

Lemma 3.5. ([15]) $K_{m,n}$ *has a* P_{2k}*-factorization if and only if* $m = n$ *and* $m \equiv 0 \pmod{k(2k-1)}$*.*

4 Small cases

Lemma 4.1. *A (P_4, C_6)-URGDD(r, s) of type 4^3 exists for every $(r, s) \in \{(4, 1), (0, 4)\}$.*

Proof. The case $(0, 4)$ follows by Lemma 3.4. For the case $(4, 1)$ take the groups to be $\{1, 2, 3, 4\}, \{5, 6, 7, 8\}, \{x, y, z, t\}$ and the following factors:

$\{(1, x, 2, 6, y, 5), (3, t, 4, 7, z, 8)\}$,

$\{[y, 1, 6, t], [7, 2, 8, x], [4, z, 5, 3]\}$, $\{[1, 7, x, 4], [t, 5, 2, z], [8, y, 3, 6]\}$,

$\{[y, 7, t, 2], [1, 8, 4, 5], [3, z, 6, x]\}$,$\{[7, 3, x, 5], [6, 4, y, 2], [z, 1, t, 8]\}$.

□

Lemma 4.2. *A (P_4, C_6)-URD$(12; r, s)$ exists for every $(r, s) \in J(12)$.*

Proof. Take a (P_4, C_6)-URGDD(r, s) of type 4^3 with $(r, s) \in \{(4, 1), (0, 4)\}$, which exist from Lemma 4.1. Place on each group of size 4 a copy of a (P_4, C_6)-URD$(4; 2, 0)$. This gives a (P_4, C_6)-URD$(12; r, s)$ for each $(r, s) \in \{(2, 0) + \{(0, 4), (4, 1)\}\} = \{(6, 1), (2, 4)\} = J(12)$. □

Lemma 4.3. *A (P_4, C_6)-URGDD(r, s) of type 12^2 exists for every $(r, s) \in \{(8, 0), (4, 3), (0, 6)\}$.*

Proof. The cases $(0, 6)$ and $(8, 0)$ are covered by Lemmas 3.4 and 3.5, respectively. For the case $(4, 3)$, we take the groups to be

$\{a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4, c_1, c_2, c_3, c_4\}$, $\{x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4, z_1, z_2, z_3, z_4\}$

and the following factors :

$\{(a_i, x_{1+i}, b_i, y_{1+i}, c_i, z_{1+i}), i = 1, 2, 3, 4\}$,

$\{(a_i, x_{2+i}, b_i, y_{2+i}, c_i, z_{2+i}), i = 1, 2, 3, 4\}$,

$\{(a_i, x_{3+i}, b_i, y_{3+i}, c_i, z_{3+i}), i = 1, 2, 3, 4\}$,

$\{[y_2, a_1, y_1, a_4], [a_2, y_3, a_3, y_4], [z_4, b_1, z_3, b_4], [b_2, z_1, b_3, z_2], [x_4, c_3, x_3, c_2], [c_4, x_1, c_1, x_2]\}$,

$\{[a_1, y_4, a_4, y_3], [y_1, a_2, y_2, a_3], [b_1, z_2, b_4, z_1], [z_3, b_2, z_4, b_3], [c_3, x_2, c_2, x_1], [x_3, c_4, x_4, c_1]\}$,

$\{[z_1, a_1, x_1, b_1], [z_2, a_2, x_2, b_2], [x_3, c_1, y_1, a_3], [x_4, c_2, y_2, a_4], [y_3, b_3, z_3, c_3], [y_4, b_4, z_4, c_4]\}$,

$\{[a_1, y_3, c_3, x_1], [a_2, y_4, c_4, x_2], [b_3, x_3, a_3, z_3], [b_4, x_4, a_4, z_4], [y_1, b_1, z_1, c_1], [y_2, b_2, z_2, c_2]\}$.

□

Lemma 4.4. *A (P_4, C_6)-URGDD(r, s) of type 12^3 exists for every $(r, s) \in \{(16 - 4x, 3x),\ x = 0, 1, 2, 3, 4\}$.*

Proof. For the case $(16, 0)$, we apply Construction 3.1 with $t = 6$ to a 2-RGDD of type 2^3 (with 4 parallel classes) to obtain a (P_4, C_6)-URGDD$(16, 0)$ of type 12^3. For the remaining cases we apply Construction 3.1 with $t = 4$ to a 3-RGDD of type 3^3 (with 3 parallel classes) to obtain a (P_4, C_6)-URGDD$(\bar{r}, \bar{s})$ of type 12^3 for each $(\bar{r}, \bar{s}) \in 3 * \{(4, 1), (0, 4)\}$=$\{(16 - 4y, 3y), y = 1, 2, 3, 4\}$. The input designs are given by Lemma 4.1. □

Lemma 4.5. *A (P_4, C_6)-URD$(36; r, s)$ exists for every $(r, s) \in J(36)$.*

Proof. Construction 3.2 applied to a (P_4, C_6)-URGDD(r_1, s_1) of type 12^3 with $(r_1, s_1) \in \{(16-4y, 3y),\ y = 0, 1, 2, 3, 4\}$ (from Lemma 4.4) gives a (P_4, C_6)-URD$(36; r, s)$ for each (r, s) with

$$\begin{aligned}(r,s) &\in J(12) + \{(16-4x, 3x),\ x = 0,1,2,3,4\} \\ &= \{\{(6,1),(2,4)\} + \{(16-4x,3x),\ x = 0,1,2,3,4\}\} \\ &= \{(22-4x, 1+3x), x = 0,1,2,3,4,5\} \\ &= J(36).\end{aligned}$$

The input designs are given by Lemmas 4.2 and 4.4. □

Lemma 4.6. *A (P_4, C_6)-URGDD(r, s) of type 12^5 exists for every $(r, s) \in \{(32 - 4x, 3x), x = 0, 2, 3, 4, 5, 6, 7, 8\}$.*

Proof. For the case $(32, 0)$ apply Construction 3.1 with $t = 6$ to a 2-RGDD of type 2^5 (with 8 parallel classes) to obtain a (P_4, C_6)-URGDD$(32, 0)$ of type 12^5. For the remaining cases apply Construction 3.1 with $t = 4$ to a 3-RGDD of type 3^5 (with 6 parallel classes) to obtain a (P_4, C_6)-URGDD$(\bar{r}, \bar{s})$ of type 12^5 for each $(\bar{r}, \bar{s}) \in 6 * \{(4, 1), (0, 4)\}=\{(32-4y, 3y), y = 2, 3, 4, 5, 6, 7, 8\}$. The input designs are given by Lemma 4.1. □

Lemma 4.7. *A (P_4, C_6)-URD$(60; r, s)$ exists for every $(r, s) \in J(60)$.*

Proof. Construction 3.2 applied to a (P_4, C_6)-URGDD(r_1, s_1) of type 12^5 with $(r_1, s_1) \in \{(32-4x, 3x), x = 0, 2, 3, 4, 5, 6, 7, 8\}$ (from Lemma 4.6) gives a (P_4, C_6)-URD$(36; r, s)$ for each (r, s) with

$$\begin{aligned}(r,s) &\in J(12) + \{(32-4y, 3y),\ y = 0,2,3,4,5,6,7,8\} \\ &= \{\{(6,1),(2,4)\} + \{(32-4y,3y),\ y = 0,2,3,4,5,6,7,8\}\} \\ &= \{(38-4x, 1+3x),\ x = 0,1,2,3,4,5,6,7,8\} \\ &= J(60).\end{aligned}$$

The input designs are given by Lemmas 4.2 and 4.6. □

5 Proof of Main Result

Lemma 5.1. *Let $v \equiv 0 \pmod{24}$. Then a (P_4, C_6)-URD$(v; r, s)$ exists for every $(r, s) \in J(v)$.*

Proof. Let $v = 24t$. Apply Construction 3.1 with $t = 12$ to a 2-RGDD of type $12^{\frac{v}{12}}$ with $\frac{v-12}{12}$ parallel classes to obtain a (P_4, C_6)-URGDD$(\bar{r}, \bar{s})$ of type $12^{\frac{v}{12}}$ for each $(\bar{r}, \bar{s}) \in \frac{v-12}{12} * \{(8, 0), (4, 3), (0, 6)\}\}$ (the input designs are given by Lemma 4.3). Now fill the groups with a (P_4, C_6)-URD$(12; r_1, s_1)$ for each $(r_1, s_1) \in \{(6, 1), (2, 4)\}$

(see Lemma 4.2). Apply Construction 3.2 to get a (P_4, C_6)-URD$(v; r, s)$ of K_v for each $(r, s) \in J(12) + \frac{v-12}{12} * \{(8, 0), (4, 3), (0, 6)\}\} = \{\{(6, 1), (2, 4)\} + \{(\frac{2(v-12)}{3} - 4x, 3x), x = 0, 1, \ldots, \frac{v-12}{6}\}\} = \{(\frac{2(v-3)}{3} - 4x, 1 + 3x), x = 0, 1, \ldots, \frac{v-6}{6}\} = J(v)$. □

Lemma 5.2. *Let* $v \equiv 12 \pmod{24}$. *Then a* (P_4, C_6)*-URD*$(v; r, s)$ *exists for every* $(r, s) \in J(v)$.

Proof. Let $v = 12 + 24t$. The cases $v = 12, 36, 60$ follow by Lemmas 4.2,4.5 and 4.7. For $t \geq 3$ apply Construction 3.3 with $t = 12$ and $h = 12$ to a 2-frame of type $2^{\frac{v-12}{24}}$ to obtain a (P_4, C_6)-URD $(v; r, s)$ for each $(r, s) \in J(12) + \frac{v-12}{24} * \{(16 - 4y, 3y), y = 0, 1, 2, 3, 4\}\} = \{\{(6, 1), (2, 4)\} + \{(\frac{2(v-12)}{3} - 4x, 3x), x = 0, 1, \ldots, \frac{v-12}{6}\}\} = \{(\frac{2(v-3)}{3} - 4x, 1 + 3x), x = 0, 1, \ldots, \frac{v-6}{6}\} = J(v)$. The input designs are given by Lemmas 4.1, 4.2, 4.5 and 4.7. □

As a consequence of Lemmas 2.1, 5.1, and 5.2 our main result immediately follows.

Theorem 5.3. *A* (P_4, C_6)*-URD*$(v; r, s)$*, with* $r, s > 0$*, exists if and only if* $v \equiv 0 \pmod{12}$ *and* $(r, s) \in J(v)$.

Remark 5.4. Note that the existence of uniformly resolvable $\{P_{2t}, C_{2(2t-1)}\}$-designs with $t \geq 3$ is currently under investigation.

References

[1] J. C. Bermond, K. Heinrich and M. L. Yu, Existence of resolvable paths designs, *Europ. J. Combin.* **11** (1990), 205–211.

[2] H. Cao, M. Niu and C. Tang, On the existence of cycle frames and almost resolvable cycle systems, *Discrete Math.* **311** (2011), 2220–2232.

[3] C. J. Colbourn and J. H. Dinitz, *The CRC Handbook of Combinatorial Designs*, Chapman and Hall/CRC, Boca Raton, FL (2007); online updates at `www.emba.uvm.edu/~dinitz/newresults.html`.

[4] M. Gionfriddo and S. Milici, On the existence of uniformly resolvable decompositions of K_v and $K_v - I$ into paths and kites, *Discrete Math.* **313** (2013), 2830–2834.

[5] D. G. Hoffman and P. J. Schellenberg, The existence of C_k-factorizations of $K_{2n} - I$, *Discrete Math.* **97** (1991), 243–250.

[6] D. G. Horton, Resolvable paths designs, *J. Combin. Theory Ser. A* **39** (1985), 117–131.

[7] M. S. Keranen, D. L. Kreher, S. Milici and A. Tripodi, Uniformly resolvable decompositions of K_v into 1-factors and 4-stars, *Australas. J. Combin.* **76** (2020), 55–72.

[8] G. Lo Faro, S. Milici and A. Tripodi, Uniformly resolvable decompositions of into paths on two, three and four vertices, *Discrete Math.* **338** (2015), 2212–2219.

[9] S. Milici, A note on uniformly resolvable decompositions of K_v and $K_v - I$ into 2-star and 4-cycles, *Australas. J. Combin.* **56** (2013), 195–200.

[10] S. Milici and Zs. Tuza, Uniformly resolvable decompositions of K_v into P_3 and K_3 graphs, *Discrete Math.* **331** (2014), 137–141.

[11] R. Rees, Uniformly resolvable pairwise balanced designs with block sizes two and three, *J. Combin. Theory Ser. A* **45** (1987), 207–225.

[12] E. Schuster, Uniformly resolvable designs with index one and block sizes three and four—with three or five parallel classes of block size four, *Discrete Math.* **309** (2009), 2452–2465.

[13] E. Schuster, Uniformly resolvable designs with index one and block sizes three and five and up to five with blocks of size five, *Discrete Math.* **309** (2009), 4435–4442.

[14] E. Schuster and G. Ge, On uniformly resolvable designs with block sizes 3 and 4, *Des. Codes Cryptogr.* **57** (2010), 47–69.

[15] H. Wang, P_{2p}-factorization of a complete bipartite graph, *Discrete Math.* **120** (1993), 307–308.

[16] H. Wei and G. Ge, Uniformly resolvable designs with block sizes 3 and 4, *Discrete Math.* **339** (2016), 1069–1085.

(Received 16 Jan 2021)