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Upper bounds on the coalition number

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Abstract

A *dominating set* in a graph $G = (V, E)$ is a set $S \subseteq V$ such that every vertex not in S is adjacent to at least one vertex in S. A *coalition* in a graph G consists of two disjoint sets $V_1, V_2 \subset V$ neither of which is a dominating set but whose union $V_1 \cup V_2$ is a dominating set. A vertex partition $\pi = \{V_1, V_2, \ldots, V_k\}$ such that every set V_i is either a dominating set consisting of a single vertex, or is not a dominating set but forms a

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coalition with another set V_j which is not a dominating set, is called a *coalition partition*. The maximum order of a coalition partition is called the *coalition number* of G. In this paper we obtain a tight upper bound on the coalition number of any graph G in terms of the maximum degree of G. We also give a tight upper bound on the coalition number in terms of both maximum degree and minimum degree of G.

1 Introduction

The term *coalition* is used to describe a situation in which two or more parties negotiate and reach an agreement on a temporary course of action that is viewed as mutually beneficial, a common example arising in parliamentary systems of government, when in a general election no political party achieves a majority. Although parliamentary coalitions typically involve agreements between more than two political parties, the graph theory model presented in this paper represents situations in which coalitions are formed by only two groups.

We will need the following definitions. Given a graph $G = (V, E)$, with vertex set V of *order* $n = |V|$, the *open neighborhood* of a vertex $v \in V$ is the set $N(v) = \{u \mid uv \in E\}$, and its *closed neighborhood* is $N[v] = N(v) \cup \{v\}$. Each vertex $u \in N(v)$ is called a *neighbor* of v, and $|N(v)|$ is called the *degree* of v, denoted $\deg(v)$. We let $\delta(G)$ and $\Delta(G)$ denote the minimum and maximum degree, respectively, over all degrees of vertices in G. For a set S of vertices, we denote the *subgraph induced by* S by $G[S]$. A set $S \subseteq V$ is a *dominating set* of a graph G if every vertex in $V - S$ has at least one neighbhor in S. A set $S \subseteq V$ is a *vertex cover* of a graph G if every edge in $E(G)$ is incident to at least one vertex of S. The minimum cardinality of any vertex cover of G is the *vertex cover number*, denoted by $\beta(G)$. For an integer k, we use the standard notation $i \in [k]$ to mean that i is an integer and $1 \leq i \leq k$.

We denote the family of paths, cycles, and complete graphs of order n by P_n, C_n, and K_n, respectively, and the complete bipartite graph having r vertices in one partite set and s vertices in the other by $K_{r,s}$. The *union* $G \cup H$ of two disjoint graphs G and H is the disconnected graph with components G and H. Let $G - e$ denote the graph obtained by removing an arbitrary edge from G.

In a graph G of order n, a vertex of degree $n-1$ is called a *full* vertex. A subset V_i is called a *singleton set* if $|V_i| = 1$. Note that any full vertex forms a singleton dominating set.

The concept of coalitions in graphs was introduced by the authors in 2020 [3] as follows.

Definition 1 A *coalition* in a graph G consists of two disjoint sets of vertices $V_1, V_2 \subset V$, neither of which is a dominating set but whose union $V_1 \cup V_2$ is a dominating set. We say that the sets V_1 and V_2 *form a coalition* and are *coalition partners*.

Definition 2 A *coalition partition*, henceforth called a *c-partition*, in a graph G is a vertex partition $\pi = \{V_1, V_2, \ldots, V_k\}$ such that every set V_i of π is either a singleton dominating set or forms a coalition with another set V_j in π. The *coalition number* $C(G)$ equals the maximum order k of a c-partition of G, and a c-partition of G having order $C(G)$ is called a *$C(G)$-partition*.

Definition 3 Let G be a graph of order n with vertex set $V = \{v_1, v_2, \ldots, v_n\}$. The *singleton partition*, denoted π_1, of G is the partition of V into n singleton sets, that is, $\pi_1 = \{\{v_1\}, \{v_2\}, \ldots, \{v_n\}\}$.

Note that if G has no full vertices, then no set V_i in a c-partition is a dominating set, and, hence, must form a coalition with another set V_j in the partition.

Coalition graphs were defined in [1, 3] as follows.

Definition 4 Given a c-partition $\pi = \{V_1, V_2, \ldots, V_k\}$ of order k of a graph $G = (V, E)$, the *coalition graph* $CG(G, \pi)$ is the graph whose k vertices correspond one-to-one with the sets of π, and two vertices V_i and V_j are adjacent in $CG(G, \pi)$ if and only if their corresponding sets V_i and V_j form a coalition in G.

Note that in Definition 4, we abuse notation slightly by letting V_i represent both a set in π and a vertex in $CG(G, \pi)$. For simplicity, we will continue this throughout the paper, depending on context to make it clear. Note also that for any graph G and a $C(G)$-partition π, there will be a corresponding coalition graph $CG(G, \pi)$ having $C(G)$ vertices.

A few examples will illustrate these definitions. Consider first the path $P_6 = (v_1, v_2, v_3, v_4, v_5, v_6)$. The partition $\pi = \{\{v_1, v_6\}, \{v_2\}, \{v_3\}, \{v_4\}, \{v_5\}\}$ is a c-partition of P_6. No set of π is a dominating set, but $\{v_2\}$ and $\{v_5\}$ form a coalition; $\{v_1, v_6\}$ and $\{v_3\}$ form a coalition; and $\{v_1, v_6\}$ and $\{v_4\}$ form a coalition. Thus, every set forms a coalition with at least one other set. From this it follows that the coalition number of the path P_6, satisfies $C(P_6) \geq 5$.

To see that $C(P_6) = 5$, we note that the only partition of $V(P_6)$ of larger order is the singleton partition $\pi_1 = \{\{v_1\}, \{v_2\}, \{v_3\}, \{v_4\}, \{v_5\} \{v_6\}\}$. Since no dominating set of P_6 contains v_1 and one other vertex, the set $\{v_1\}$ does not form a coalition with any other set in π_1, and therefore the singleton partition π_1 of P_6 is not a c-partition. Hence, $C(P_6) = 5$ and $\pi = \{\{v_1, v_6\}, \{v_2\}, \{v_3\}, \{v_4\}, \{v_5\}\}$ is a $C(P_6)$-partition.

Consider next the cycle C_6. The following partitions are c-partitions of C_6 of orders 2, 3, 4, 5, and 6:

$\pi_2 = \{\{v_1, v_2, v_3\}, \{v_4, v_5, v_6\}\}$,
$\pi_3 = \{\{v_1, v_2\}, \{v_3, v_5\}, \{v_4, v_6\}\}$,
$\pi_4 = \{\{v_1\}, \{v_2\}, \{v_3, v_5\}, \{v_4, v_6\}\}$,
$\pi_5 = \{\{v_1\}, \{v_2\}, \{v_3\}, \{v_4\}, \{v_5, v_6\}\}$,
$\pi_6 = \{\{v_1\}, \{v_2\}, \{v_3\}, \{v_4\}, \{v_5\} \{v_6\}\}$.

Note as well that these five c-partitions of C_6 result in the following coalition graphs:

$CG(C_6, \pi_2) \simeq K_2,$
$CG(C_6, \pi_3) \simeq K_3,$
$CG(C_6, \pi_4) \simeq K_4 - e,$
$CG(C_6, \pi_5) \simeq K_2 \cup P_3,$
$CG(C_6, \pi_6) \simeq 3K_2.$

In [3] the authors show that every graph G has a c-partition. Hence, we have the following straightforward bounds on the coalition number.

Corollary 1.1 *If G is a graph of order n, then $1 \leq C(G) \leq n$.*

It is easy to see that the trivial graph K_1 is the only graph which attains the lower bound of Corollary 1.1, while the complete graphs K_n and the complete bipartite graphs $K_{r,s}$, with $2 \leq r \leq s$, among other graphs, attain the upper bound.

In Section 2 we provide a tight upper bound on $C(G)$ for all graphs G in terms of maximum degree $\Delta(G)$, and construct families of graphs achieving this upper bound. In Section 3, we give an improved upper bound on $C(G)$ for some graphs in terms of minimum and maximum degree of G. We also construct a family of graphs achieving this bound.

We will use the following known results.

Lemma 1.1 ([2]) *For any graph G with c-partition π,*

$$\Delta(CG(G, \pi)) \leq \Delta(G) + 1.$$

Lemma 1.2 ([4]) *For any graph G with c-partition π, the vertex cover number*

$$\beta(CG(G, \pi)) \leq \delta(G) + 1.$$

2 Upper Bound in Terms of Maximum Degree

In this section, we give an upper bound on $C(G)$ for any graph G in terms of the maximum degree $\Delta(G)$ and give a construction of graphs attaining the bound.

2.1 Upper Bound

We are now ready to present our main result.

Theorem 2.1 *For any graph G, $C(G) \leq (\Delta(G) + 3)^2/4$.*

Proof. The theorem obviously holds for any graph G of order n with a full vertex, since $C(G) \leq n < (n + 2)^2/4 = (\Delta(G) + 3)^2/4$. So we can assume that G is a graph with no full vertices.

Let $\pi = \{V_1, V_2, \ldots, V_k\}$ be a $C(G)$-partition for G, and let $H = CG(G, \pi)$ be the coalition graph of G and π. Then H has order $k = C(G)$. Since G has no full vertices, every set in π is in a coalition with another set in π, and so H has no isolated vertices. In order to obtain an upper bound on $C(G) = k$, we obtain a bound on the order k of H.

Let S be a minimum vertex cover for H, that is, $|S| = \beta(H)$. Let $\overline{S}$ denote the set $V(H) - S$, and let $N_{\overline{S}}(v) = N_H(v) \cap \overline{S}$. Note that the set $\overline{S}$ is an independent set in H.

If $|S| = 1$, then every edge of H is incident to the only vertex, say V_i, of S. Since H has no isolated vertices, V_i is a full vertex of H. Thus, $k = 1+|N_H(V_i)| \leq 1+\Delta(H)$. By Lemma 1.1, $k \leq 1 + \Delta(H) \leq 1 + (\Delta(G) + 1) = \Delta(G) + 2$, and the theorem holds since $C(G) = k \leq \Delta(G) + 2 < (\Delta(G) + 3)^2/4$.

Assume next that $|S| = 2$, and let $S = \{V_i, V_j\}$. If $V_iV_j \notin E(H)$, then the set $V_i \cup V_j$ is not a dominating set of G. Thus, there exists a vertex u in G not dominated by $V_i \cup V_j$. Since S is a vertex cover of H, every vertex in $\overline{S}$ is adjacent to at least one of V_i and V_j in H, that is, their corresponding sets must form a coalition with V_i or V_j in π. The only sets of π that can form a coalition with V_i or V_j in G must contain at least one member of $N_G[u]$. Since $|N_G[u]| \leq \Delta(G) + 1$, it follows that $k \leq 2 + (\Delta(G) + 1) = \Delta(G) + 3$, and $\Delta(G) + 3 \leq (\Delta(G) + 3)^2/4$, as desired. If $V_iV_j \in E(H)$, then it follows that $k \leq 2 + 2(\Delta(H) - 1) = 2\Delta(H)$. By Lemma 1.1, $C(G) = k \leq 2\Delta(H) \leq 2(\Delta(G) + 1) = 2\Delta(G) + 2 \leq (\Delta(G) + 3)^2/4$. Thus, the theorem holds if $|S| = 2$.

Henceforth, we may assume that $|S| \geq 3$. By Lemma 1.2, $|S| \leq \delta(G) + 1 \leq \Delta(G) + 1$. Among all vertices in S, let V_i be one having the maximum number of neighbors in $\overline{S}$, that is, $|N_{\overline{S}}(V_i)|$ is maximized. Let $|N_{\overline{S}}(V_i)| = m$.

Recall that no set of π dominates G. Let u be a vertex in G that is not dominated by the set V_i. Let U be the subset of π whose sets contain members of $N_G[u]$. Note that every coalition of π must include a member of U to dominate u in G. As before, abusing notation slightly, we use U to refer to the collection of sets of π in G and to the set of their corresponding vertices in H. Notice that the sets corresponding to the m vertices in $N_{\overline{S}}(V_i)$ must be in U, since each of these sets forms a coalition with V_i in G. Since $|N_G[u]| \leq \Delta(G) + 1$, there are at most $\Delta(G) + 1 - m$ other sets in U.

By our choice of u, each vertex in $S \cap U$ has at most m neighbors in $\overline{S}$. Since S is a vertex cover of H, we deduce that the vertices in $\overline{S}$ in H are the m vertices in $N_{\overline{S}}(V_i)$, along with at most $\Delta(G) + 1 - m - |S \cap U|$ other vertices in $U \cap \overline{S}$ and at most $m \cdot |S \cap U|$ vertices that are adjacent to vertices in $S \cap U$. Thus, $|\overline{S}| \leq m + \Delta(G) + 1 - m - |S \cap U| + m \cdot |S \cap U| = \Delta(G) + 1 + (m-1)|S \cap U|$. Since $|S| \leq \Delta(G) + 1$, we have $k = |S| + |\overline{S}| \leq (\Delta(G) + 1) + \Delta(G) + 1 + (m-1)|S \cap U| = 2\Delta(G) + 2 + (m-1)|S \cap U|$.

Recall that $|S \cap U| \leq \Delta(G) + 1 - m$. If $|S \cap U| < \Delta(G) + 1 - m$, then $k \leq 2\Delta(G) + 2 + (m-1)(\Delta(G) - m) = \Delta(G) + 2 + m(\Delta(G) + 1) - m^2$. This value is maximized when $m = (\Delta(G) + 1)/2$, and so $k \leq (\Delta(G) + 3)^2/4$. Hence, the result holds for $|S \cap U| < \Delta(G) + 1 - m$.

Henceforth, we may assume that $|S \cap U| = \Delta(G) + 1 - m$. Let $\Delta = \Delta(G)$. We prove two claims.

Claim 1 *If there exist two vertices in $S \cap U$, say V_1 and V_2, such that $|(N_{\overline{S}}(V_1) \cup N_{\overline{S}}(V_2)| \le m$, then $k < (\Delta + 3)^2/4$.*

Proof. Suppose that there exist two vertices in $S \cap U$, say V_1 and V_2, such that $|(N_{\overline{S}}(V_1) \cup N_{\overline{S}}(V_2)| \le m$. Then the vertices in $\overline{S}$ in H are the m vertices in $N_{\overline{S}}(V_i)$, the vertices adjacent to V_1 or V_2, and the vertices adjacent to the vertices in $(S \cap U) - \{V_1, V_2\}$. Since $|S \cap U| = \Delta + 1 - m$ and each vertex in $S \cap U$ has at most m neighbors in $\overline{S}$, it follows that $|\overline{S}| \le m + m + m|(S \cap U) - \{V_1, V_2\}| = 2m + m(\Delta + 1 - m - 2)$. Thus, $k = |S| + |\overline{S}| \le (\Delta + 1) + 2m + m(\Delta - m - 1) \le \Delta + 1 + m + m\Delta - m^2 = \Delta + 1 + m(\Delta + 1) - m^2$. This value is maximized when $m = (\Delta + 1)/2$, and so $k \le (\Delta + 1)(\Delta + 5)/4 < (\Delta + 3)^2/4$, as desired. ($\Box$)

Hence, we may assume every pair of vertices in $S \cap U$ have greater than m vertices in the union of their neighborhoods in $\overline{S}$ in H, else the result holds by Claim 1.

Claim 2 *If every vertex in $S \cap U$ has fewer than m neighbors in $\overline{S} - N_{\overline{S}}(V_i)$, then $k \le (\Delta + 3)^2/4$.*

Proof. Suppose that every vertex in $S \cap U$ has fewer than m neighbors in $\overline{S} - N_{\overline{S}}(V_i)$. Then the vertices of $\overline{S}$ are the m vertices of $N_{\overline{S}}(V_i)$ and at most $(m - 1)|S \cap U|$ vertices adjacent to the vertices of $S \cap U$. Since $|S \cap U| = \Delta + 1 - m$, we have $k = |S| + |\overline{S}| \le (\Delta + 1) + m + (m - 1)|S \cap U| = \Delta + 1 + m + (m - 1)(\Delta + 1 - m)$. This value is maximized when $m = (\Delta + 3)/2$, and so $k \le (\Delta + 3)^2/4$. ($\Box$)

Henceforth, we may assume that there is at least one vertex, say V_j, in $S \cap U$ with m neighbors in $\overline{S} - N_{\overline{S}}(V_i)$, else the result holds by Claim 2. Note that $V_j \ne V_i$. Further, $N_G[u] \cap V_j \ne \emptyset$ in G, since $V_j \in S \cap U$ in H.

Since the set V_j does not dominate G, there is a vertex x of G that is not dominated by V_j in G. Let X be the subset of π whose sets contain members of $N_G[x]$. Again, we use X to refer to the collection of sets of π in G and to the set of their corresponding vertices in H.

Note that V_j could have been chosen instead of V_i, so the arguments that hold for V_i also hold for V_j. In particular, we may assume that $|X \cap S| = \Delta + 1 - m$ and that the vertices of $\overline{S} \cap X$ are precisely the m vertices adjacent to V_j. Furthermore, since V_i and V_j have no common neighbors in $\overline{S}$, none of the m vertices in $N_{\overline{S}}(V_i)$ are in X. Since vertex $V_r \in N_{\overline{S}}(V_i)$ in H represents a coalition partner of the set V_i in G and no such set V_r dominates x in G, we deduce that $V_i \in X \cap S$.

If there is a vertex $V_p \in (S - (U \cup \{V_i\})$, then V_p must have a neighbor in $\overline{S}$, otherwise $S - \{V_p\}$ is a vertex cover of H with cardinality less than $|S| = \beta(H)$, a contradiction. Moreover, every neighbor of V_p in H must be in U since $V_p \notin U$. Thus, $N_{\overline{S}}(V_p) \subseteq N_{\overline{S}}(V_i)$, implying that $V_p \in X$ and $|(N_{\overline{S}}(V_p) \cup N_{\overline{S}}(V_i)| \le m$. Now suppose we had chosen V_j instead of V_i. Then Claim 1 implies that every two vertices in $S \cap X$ must have more than m vertices of $\overline{S}$ in the union of their neighborhoods. But V_i and V_p are in $S \cap X$ and $|(N_{\overline{S}}(V_p) \cup N_{\overline{S}}(V_i)| \leq m$, a contradiction.

Thus, no such vertex V_p exists. Hence, $S-(U \cup \{V_i\}) = \emptyset$, and so $|S| = |S \cap U|+1$. Now the vertices of $\overline{S}$ are the m vertices of $N_{\overline{S}}(V_i)$ and at most $m|S \cap U|$ vertices adjacent to the vertices of $S \cap U$. Thus, $k = |S|+|\overline{S}| \leq |S \cap U|+1+m+m|S \cap U| = (\Delta+1-m)+1+m+m(\Delta+1-m) = \Delta+2+m(\Delta+1-m) = \Delta+2+m(\Delta+1)-m^2$. This value is maximized when $m = (\Delta+1)/2$, and so $k \leq (\Delta+3)^2/4$, completing the proof. □

The upper bound of Theorem 2.1 is sharp. For example, a graph G with $\Delta(G) = 3$ and $C(G) = 9 = (\Delta(G)+3)^2/4$ is illustrated in Figure 1, where a $C(G)$-partition is given by $\pi = \{V_1, V_2, \ldots, V_9\}$, where V_i is the set of vertices labeled i for $i \in [9]$. In the next section, we construct graphs G achieving this tight bound for all $\Delta(G) \geq 0$. An open problem is to characterize the graphs attaining the bound of Theorem 2.1.

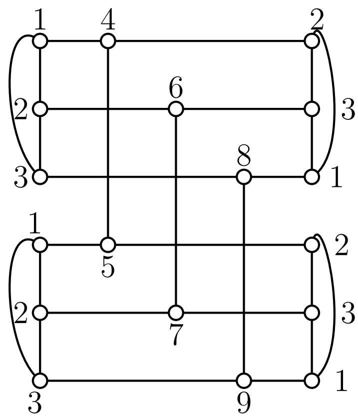

Figure 1: $C(G) = 9$

2.2 Graphs Achieving the Upper Bound

In this section, we construct families of graphs achieving the upper bound of Theorem 2.1.

Theorem 2.2 *For every non-negative integer Δ, there exists a graph G with $\Delta(G) = \Delta$, for which $C(G) = \frac{(\Delta+2)(\Delta+4)}{4}$ if Δ is even, and $C(G) = \frac{(\Delta+3)^2}{4}$ if Δ is odd.*

Proof. For every non-negative integer Δ, we construct a graph G for which $\Delta(G) = \Delta$ and $C(G) = \frac{(\Delta+2)(\Delta+4)}{4}$ if Δ is even, and $C(G) = \frac{(\Delta+3)^2}{4}$ if Δ is odd.

Assume first that Δ is even. If $\Delta = 0$, then the empty graph $\overline{K}_2$ with its singleton c-partition has $C(G) = 2 = \frac{(\Delta+2)(\Delta+4)}{4}$. And the singleton c-partition for the cycle C_6 shows that there is a graph with $\Delta = 2$ and $C(G) = 6 = \frac{(\Delta+2)(\Delta+4)}{4}$.

For even $\Delta \geq 4$, we the build a Δ-regular graph G with a labeling of the vertices of G that produces a c-partition π and the desired coalition number.

Let $p = (\Delta + 2)/2$. To build G, begin with two groups of p complete graphs K_p. Thus, we start with $(2p)K_p$. To aid in our discussion, we refer to the i^{th} clique in a group for $i \in [p]$, that is, we associate a unique label from 1 to p for each complete graph in Group 1 and similarly for the p graphs in Group 2. For each of the cliques in Group 1, label its vertices a unique number from $[p]$, that is, the vertices of each K_p in Group 1 are labeled from 1 to p. For Clique i in Group 2, label its vertices from $ip+1, ip+2, \ldots, ip+p$. It follows that every vertex in Group 2 has a different label from $p+1$ to p^2+p. Finally, we add edges to finish building G as follows:

For each Clique i in Group 2 and each vertex v labeled $ip+j$ in Clique i, add edges from v to every vertex, except the vertex labeled i, in Clique j of Group 1. Then G is a Δ-regular graph. Figure 2 illustrates the construction for $\Delta = 4$.

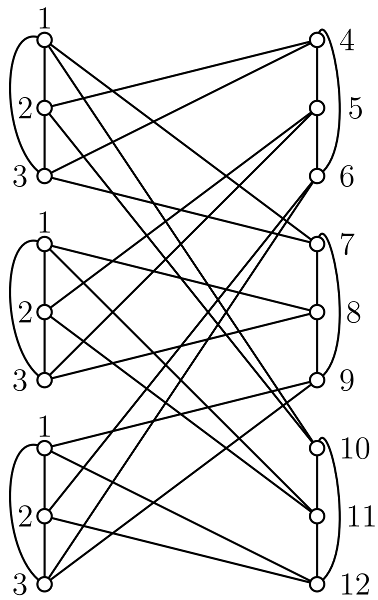

Figure 2: $\Delta = 4$ and $C(G) = 12$

Define partition $\pi = \{V_1, V_2, \ldots, V_k\}$, where V_i is the set of vertices labeled i in G. We note that every vertex in the cliques of Group 2 is in a singleton set of π. Note also that π has order $p^2 + p = (\frac{\Delta+2}{2})^2 + \frac{\Delta+2}{2} = \frac{(\Delta+2)(\Delta+4)}{4}$.

To see that π is a c-partition of G, note that no $V_s \in \pi$ dominates G. We need to show that every $V_s \in \pi$ forms a coalition with another set of π. If $V_s = \{v\}$ is a singleton set containing a vertex from a clique of Group 2, then $s = ip + j$, where the vertex v is in Clique i of Group 2. Thus, V_s forms a coalition with V_i in π, and the result holds for all s, where $p+1 \le s \le p^2+p$. Moreover, since there exists an $s = ip + j$ for all $i \in [p]$, we have that each V_i forms a coalition with a set in π. Hence, π is a c-coalition of G, implying that $C(G) \ge k = \frac{(\Delta+2)(\Delta+4)}{4}$. By Theorem 2.1, $C(G) \le \frac{(\Delta+3)^2}{4}$, and so $C(G) = \frac{(\Delta+2)(\Delta+4)}{4}$.

Next assume that Δ is odd. The graph $G = 2K_2$ has $\Delta = 1$ and $C(G) = 4 = \frac{(\Delta+3)^2}{4}$.

For odd $\Delta \geq 3$, we the build a graph G with $\Delta(G) = \Delta$ and give a labeling of the vertices of G that produces a c-partition π and the desired coalition number as follows.

Let $p = (\Delta + 3)/2$, $q = (\Delta + 1)/2$, and $r = (\Delta + 5)/2$. To build G, begin with two groups, such that Group 1 is the union of r complete graphs K_p, and Group 2 is the union p complete graphs K_q. As before, we refer to the i^{th} clique, associating a unique label i from 1 to r with each K_p in Group 1 and associating a unique label from 1 to p with each K_q in Group 2. For each of the r cliques in Group 1, label its vertices a unique number from $[p]$. In other words, the vertices of each K_p in Group 1 are labeled from 1 to p. For the pq vertices in Group 2, give each vertex a different label from $p + 1$ to $p + pq$.

Finally, we add edges to finish building G as follows: for each vertex v in Clique i, for $i \in [p]$, of Group 2, add edges from v to $p - 1$ vertices in Group 1, such that each of these $p - 1$ neighbors of v has a different label from $\{1, 2, \ldots, p\} - \{i\}$ and no vertex of Group 1 is adjacent to more than $\Delta - p + 1$ vertices of Group 2. Now each vertex of Group 2 has degree $q - 1 + p - 1 = q + p - 2 = \Delta = \Delta(G)$, and each vertex of Group 1 has degree at most $p - 1 + \Delta - p + 1 = \Delta$. Figure 3 illustrates the construction for $\Delta = 5$.

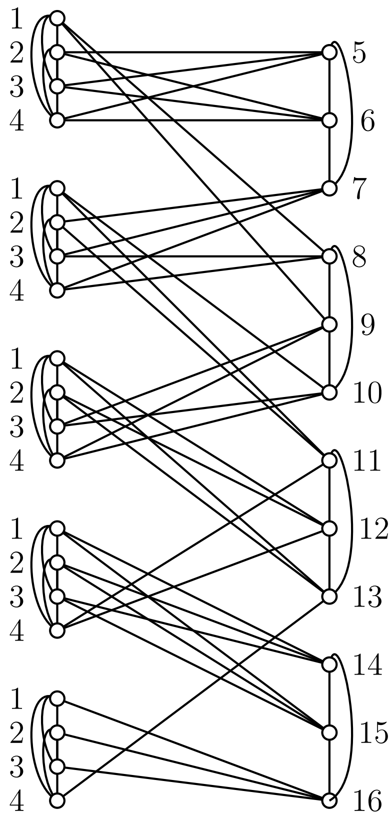

Figure 3: $\Delta = 5$ and $C(G) = 16$

Define partition $\pi = \{V_1, V_2, \ldots, V_k\}$, where V_i is the set of vertices labeled i in G. We note that π has order $p + pq = (\Delta + 3)/2 + (\Delta + 3)(\Delta + 1)/2 = (\Delta + 3)^2/4$, and that every vertex in a clique of Group 2 is in a singleton set of π.

It is straightforward to see that no $V_s \in \pi$ dominates G. We need to show that every $V_s \in \pi$ forms a coalition with another set of π. If $V_s = \{v\}$ is a singleton set containing a vertex from Clique i, for $i \in [p]$, of Group 2, then V_s forms a coalition with V_i in π. Thus, every singleton set V_s containing a vertex from Group 2 is in a coalition, and also every V_i, for $i \in [p]$, is in a coalition with a singleton set of π. Hence, π is a c-coalition of G, implying that $C(G) \geq (\Delta + 3)^2/4$. By Theorem 2.1, $C(G) = (\Delta + 3)^2/4$. □

3 Upper Bound in Terms of Minimum and Maximum Degree

In this section, we give an upper bound on $C(G)$ for some graphs G in terms of the minimum degree $\delta(G)$ and maximum degree $\Delta(G)$. We also construct graphs attaining this bound.

Theorem 3.1 *If G is a graph with no full vertices and $\delta(G) < \Delta(G)/2$, then*

$$C(G) \leq (\delta(G) + 1)(\Delta(G) - \delta(G) + 2).$$

Proof. Let $\pi = \{V_1, V_2, \ldots, V_k\}$ be a c-partition for G, where $k = C(G)$, and let $H = CG(G, \pi)$. Then H has order k. Since G has no full vertices, every set in π forms a coalition with another set in π, and so H has no isolated vertices. In order to obtain an upper bound on $C(G) = k$, we obtain a bound on the order k of H.

Let S be a minimum vertex cover for H, that is, $|S| = \beta(H)$. By Lemma 1.2, $|S| = \beta(H) \leq \delta(G) + 1$. Let $\overline{S}$ denote the set $V(H) - S$, and let $N_{\overline{S}}(v) = N_H(v) \cap \overline{S}$. Let V_i be a vertex in S that is adjacent to the maximum number of vertices in $\overline{S}$.

If $|N_{\overline{S}}(V_i)| \leq \Delta(G) - \delta(G) + 1$, then $k = |S| + |\overline{S}| \leq |S| + |S|(\Delta(G) - \delta(G) + 1) \leq (\delta(G) + 1) + (\delta(G) + 1)(\Delta(G) - \delta(G) + 1) = (\delta(G) + 1)(\Delta(G) - \delta(G) + 2)$, as desired.

Thus, we may assume that $|N_{\overline{S}}(V_i)| > \Delta(G) - \delta(G) + 1$, that is, $|N_{\overline{S}}(V_i)| = \Delta(G) - \delta(G) + 1 + q$, for some $q \geq 1$.

Recall that no set of π dominates G. Let u be a vertex in G that is not dominated by the set V_i. Let U be the subset of π whose sets contain vertices in $N_G[u]$. Note that $N_{\overline{S}}(V_i) \subseteq U$ in H. Since $|U| \leq |N_G[u]| \leq \Delta(G) + 1$, there are at most $(\Delta(G) + 1) - (\Delta(G) - \delta(G) + 1 + q) = \delta(G) - q$ vertices in $U - N_{\overline{S}}(V_i)$ in H.

Since every edge in H must be incident to a vertex in U and S is a vertex cover, it follows that $k = |S| + |\overline{S}| \leq (\delta(G) + 1) + (\Delta(G) - \delta(G) + 1 + q) + |S \cap U|(\Delta(G) - \delta(G) + 1 + q) \leq (\delta(G) + 1) + (\Delta(G) - \delta(G) + 1 + q) + (\delta(G) - q)(\Delta(G) - \delta(G) + 1 + q)$. Simplifying, we have

$$k \leq (\delta(G) + 1)(\Delta(G) - \delta(G) + 2) - q(\Delta(G) - 2\delta(G) + q).$$

Since $\Delta(G) > 2\delta(G)$, $k < (\delta(G) + 1)(\Delta(G) - \delta(G) + 2)$, proving the theorem. □

To see that the upper bound of Theorem 3.1 is sharp, we construct graphs G with $\Delta(G) = \Delta$ and $\delta(G) = \delta$, such that $\delta < \Delta/2$, attaining the bound as follows:

Let $m = \Delta - \delta + 1$. Begin with two groups, such that Group 1 is the union of $\delta + 1$ complete graphs $K_{\delta+1}$, and Group 2 is the union of $\delta + 1$ complete graphs K_m. As before, we refer to the i^{th} clique in each group for $i \in [\delta + 1]$. For each of the cliques in Group 1, label its vertices from 1 to $\delta + 1$. Label every vertex in Group 2 a different label from $\delta + 2$ to $(\delta + 1)(m + 1)$.

Finally, we add edges to finish building G as follows: for each vertex v in Clique i, for $i \in [\delta + 1]$, of Group 2, add edges from v to δ vertices in Clique i in Group 1, such that none of the δ vertices have label i. Note that every vertex in Group 2 has maximum degree $\Delta(G)$, while the vertex labeled i in Clique i of Group 2 has degree $\delta(G)$. An argument similar to the proof of Theorem 2.2 shows that $C(G) = (\delta(G) + 1)(\Delta(G) - \delta(G) + 2)$. Figure 4 illustrates the construction for $\Delta = 5$ and $\delta = 2$.

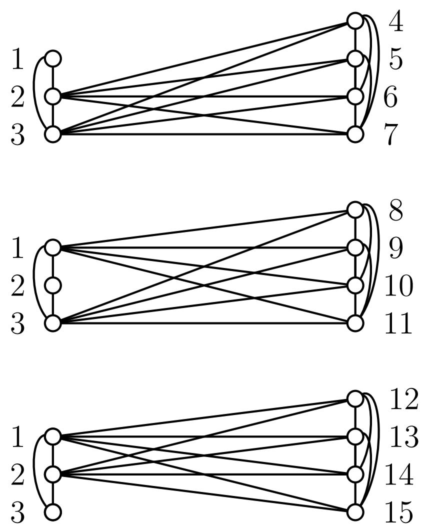

Figure 4: $\delta = 2$, $\Delta = 5$, and $C(G) = 15$

We conclude this section with a corollary to Theorem 3.1. The coalition number of paths P_n is given in [3] as follows.

Theorem 3.2 *For the path P_n,*

$$C(P_n) = \begin{cases} n & \textit{if} \quad n \leq 4 \\ 4 & \textit{if} \quad n = 5 \\ 5 & \textit{if} \quad 6 \leq n \leq 9 \\ 6 & \textit{if} \quad n \geq 10. \end{cases}$$

Corollary 3.1 *For any tree T, $C(T) \leq 2\Delta(T) + 2$.*

Proof. If T is the trivial graph K_1, then $C(T) = 1 < 2 = 2\Delta(T) + 2$. If $T \simeq K_2$, then $C(T) = 2 < 4 = 2\Delta(T) + 2$. If T is a star having order at least three, then

$C(T) = 3 < 2\Delta(T) + 2$. Thus, we may assume that T is not a star, T has order at least 3, and $\Delta(T) \geq 2$. If $\Delta(T) = 2$, then T is a path and the result follows from Theorem 3.2 since $C(T) \leq 6 = 2\Delta(T) + 2$.

If $\Delta(T) \geq 3$, then $1 = \delta(T) < \Delta(T)/2$ and the result follows from Theorem 3.1. □

Corollary 3.1 is sharp for paths P_n, for $n \geq 10$.

We conclude this section with two open problems:

1. Characterize the graphs attaining the bound of Theorem 3.1.
2. Characterize the trees attaining the bound of Corollary 3.1.

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