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Enumerations on polyominoes determined by Fuss-Catalan words

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Abstract

In this paper we introduce the concept of s-Fuss-Catalan words. This new family of words generalizes the Catalan words (taking $s = 1$), which are a particular case of growth-restricted words. Here we enumerate the polyominoes or bargraphs associated with the s-Fuss-Catalan words according to the semiperimeter and area statistics. Additionally, we obtain combinatorial formulas to count the s-Fuss-Catalan bargraphs according of these statistics.

1 Introduction

Given a positive integer s, an *s-Fuss-Catalan path* of length $(s + 1)n$ is a lattice path in the first quadrant of the xy-plane from $(0, 0)$ to the point $((s+1)n, 0)$ using up-steps $U_s = (1, s)$ and down-steps $D = (1, -1)$. For $s = 1$ we recover the concept of the classical Dyck path of length $2n$ enumerated by the famous Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$. The number of s-Fuss-Catalan paths of length $(s + 1)n$ is given by the Fuss-Catalan numbers $C_{n,s} = \frac{1}{sn+1}\binom{(s+1)n}{n}$. There are several combinatorial interpretations for both the Catalan numbers and for Fuss-Catalan numbers (see, for example, [14] and [9]).

For an s-Fuss-Catalan path of length $(s + 1)n$, we associate the word formed by the subtraction $s - 1$ from the y-coordinate of each final point of the U_s steps. This family of words is called *s-Fuss-Catalan words*. See Figure 1 for an example.

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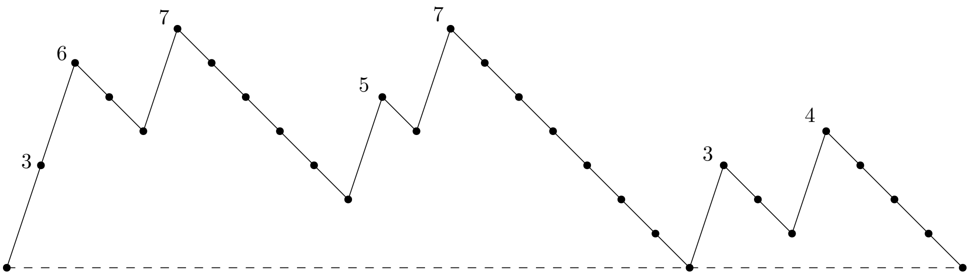

Figure 1: The 3-Dyck path corresponding to the 3-Fuss-Catalan word `1453512`.

The *s-Fuss-Catalan words* can be characterized as the words $w = w_1w_2 \cdots w_n$ over the set of positive integers satisfying $w_1 = 1$ and $1 \leq w_i \leq w_{i-1} + s$ for $i = 2, \ldots, n$. Denote by $\mathcal{C}_n^{(s)}$ the set of s-Fuss-Catalan words of length n. It is clear that the cardinality of $\mathcal{C}_n^{(s)}$ is given by the Fuss-Catalan number $C_{n,s}$. An s-Fuss-Catalan word $w = w_1w_2 \cdots w_n$ can be represented as a *polyomino* P of n columns, also called a *bargraph*, whose i-th column contains w_i cells for $1 \leq i \leq n$. See Figure 2 for an example.

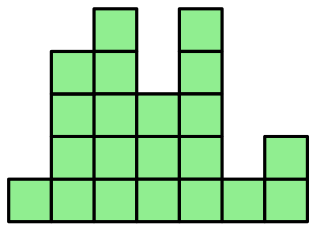

Figure 2: Polyomino corresponding to the 3-Fuss-Catalan word `1453512`.

The s-Fuss-Catalan words generalize the concept of Catalan words (taking $s = 1$). Catalan words have been studied in the context of exhaustive generation of Gray codes for growth-restricted words [12]. Recently, Baril et al. [2, 3] studied the distribution of descents on the set of Catalan words avoiding a pattern of length at most three and pair of patterns of length three. Callan and the two authors of this paper [7] started the study of the combinatorial properties of the polyominoes associated with the Catalan words. For example, in [7] it is possible to find formulas for the generating functions enumerating area and semiperimeter. Additionally, the authors in [11] study the number of interior lattice vertices lying strictly within the polygon determined by the polyomino. We remark that polyominoes provide a rich source of combinatorial ideas and have been studied in connection with several discrete structures such as words, set partitions, polyominoes, permutations, graphs, among others (see for example [4, 5, 6, 8, 10] and references contained therein).

The goal of this paper is to enumerate the area and semiperimeter of the family of polyominoes determined by the s-Fuss-Catalan words. So a property that is true in this generalization immediately holds for the polyominoes associated to Catalan words (taking $s = 1$). The results given in this paper were found using generating functions and the kernel method. In particular, we give a functional equation satisfied by the generating function of the polyominoes determined by s-Fuss-Catalan words according to the area and the semiperimeter statistics. Then we can derive generating functions to the total distribution of both statistics and give some combinatorial expressions.

2 Area and Semiperimeter Statistics

A *bargraph* is a self-avoiding lattice path in the first quadrant with steps up $u = (0, 1)$, horizontal $h = (1, 0)$, and down $d = (0, -1)$ that starts at the origin and ends on the x-axis. The bargraphs are a particular family of polyominoes (cf. [8]). We define the *area* of a bargraph as the number of cells. The *semiperimeter* of a bargraph is the sum of the number of up and horizontal steps. Let P_w be the bargraph associated with the s-Fuss-Catalan word w. We denote by $\texttt{area}(P_w)$ and $\texttt{sper}(P_w)$ the area and semiperimeter of P_w, respectively. Hence, for the bargraphs in Figure 2, $\texttt{area}(P_w) = 21$ and $\texttt{sper}(P_w) = 15$.

Let $\mathcal{C}_n^{(s)}$ denote the set of s-Fuss-Catalan words of length n, and $\mathcal{C}^{(s)} = \bigcup_{n\geq 0} \mathcal{C}_n^{(s)}$. Let $\mathcal{C}_{n,i}^{(s)}$ denote the set of words in $\mathcal{C}_n^{(s)}$ having last letter equal to i, and let $c_s(n, i) = |\mathcal{C}_{n,i}^{(s)}|$. Yang and Wang [15] studied the sequence $c_s(n, i)$ in the context of the Enumerating Combinatorial Objects (ECO) method. The sequence $c_s(n, j)$ satisfies the recurrence relation

$$c_s(n, i) = c_s(n-1, i-(s+1)+1) + c_s(n-1, i-(s+1)+2) + \cdots + c_s(n-1, (n-1)s),$$

for all $n, i \geq 1$, with the initial conditions $c_s(1, 1) = 1$ and $c_s(1, i) = 0$ for all $i > 1$. For example, the first few rows for the matrix $[c_2(n, i)]_{n,i\geq 1}$ are

$$[c_2(n,i)]_{n,i\geq 1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 3 & 3 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 12 & 12 & 12 & 9 & 6 & 3 & 1 & 0 & 0 & 0 & 0 \\ 55 & 55 & 55 & 43 & 31 & 19 & 10 & 4 & 1 & 0 & 0 \\ 273 & 273 & 273 & 218 & 163 & 108 & 65 & 34 & 15 & 5 & 1 \end{pmatrix}.$$

We introduce the following generating functions according to the above parameters:

$$A_i^{(s)}(x; p, q) := \sum_{n\geq 1} x^n \sum_{w\in\mathcal{C}_{n,i}^{(s)}} p^{\texttt{sper}(P_w)} q^{\texttt{area}(P_w)}.$$

That is $A_i(x; p, q)$ is the generating function for the s-Fuss-Catalan words (or Catalan bargraphs) ending in i with respect to the area and semiperimeter. Moreover, define the multivariate generating function

$$A^{(s)}(x; p, q; v) := \sum_{i\geq 1} A_i^{(s)}(x; p, q) v^{i-1}.$$

In Theorem 2.1 we give a functional expression for the generating function $A^{(s)}(x;p,q;v)$.

Theorem 2.1. *The generating function $A^{(s)}(x;p,q;v)$ satisfies the functional equation*

$$A^{(s)}(x;p,q;v) = p^2qx + \frac{pqx}{1-qv}A^{(s)}(x;p,q;1) + \left(\frac{pq^2xv(1-(pqv)^s)}{1-pqv} - \frac{pq^2xv}{1-qv}\right)A^{(s)}(x;p,q;qv). \quad (1)$$

Proof. From the definition of an s-Fuss-Catalan word, we have, for $i = 1$, the following relation:

$$A_1^{(s)}(x;p,q) = p^2qx + pqx\sum_{j\geq 1} A_j^{(s)}(x;p,q). \quad (2)$$

See Figure 2 for a graphical representation of this decomposition.

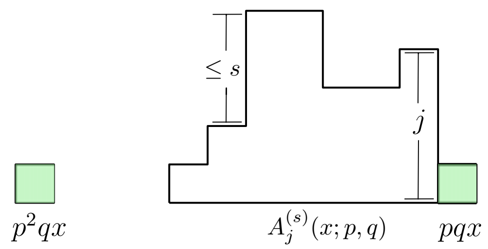

Figure 3: Decomposition of the s-Fuss-Catalan words in $\mathcal{C}_{n,1}^{(s)}$.

For $2 \leq i \leq s$ we have (see Figure 4)

$$A_i^{(s)}(x;p,q) = \sum_{k=1}^{i-1} p^{i-k+1}q^ixA_k^{(s)}(x;p,q) + pq^ix\sum_{\ell\geq i} A_\ell^{(s)}(x;p,q); \quad (3)$$

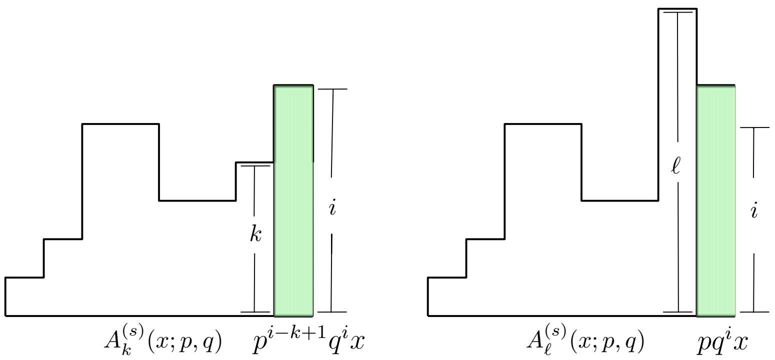

Figure 4: Decomposition of the s-Fuss-Catalan words in $\mathcal{C}_{n,i}^{(s)}$, for $2 \leq i \leq s$.

and for $i > s$ we obtain the recursion

$$A_i^{(s)}(x;p,q) = pq^i x \sum_{k=0}^{s-1} p^{s-k} A_{i-s+k}^{(s)}(x;p,q) + pq^i x \sum_{\ell \geq i} A_\ell^{(s)}(x;p,q). \tag{4}$$

Multiplying (4) by v^{i-1}, summing over $i \geq s+1$ and using (2) and (3), we have

$$A_1^{(s)}(x;p,q) = p^2qx + pqxA^{(s)}(x;p,q;1),$$
$$A_i^{(s)}(x;p,q) = \sum_{k=0}^{i-2} p^{i-k}q^i x A_{k+1}^{(s)}(x;p,q) + pq^i x \left(A^{(s)}(x;p,q,1) - \sum_{k=1}^{i-1} A_k^{(s)}(x;p,q) \right), \quad 2 \leq i \leq s$$

and

$$\begin{aligned}
A^{(s)}(x;p,q;v) &- \sum_{k=1}^{s} A_k^{(s)}(x;p,q)v^{k-1} \\
&= \left(x \sum_{k=1}^{s} (pq)^{k+1}v^k - \frac{pq^2xv}{1-qv} \right) A^{(s)}(x;p,q;qv) \\
&- \left(x \sum_{k=1}^{s-1} p^{k+1}q^{k+1}v^k + \frac{pq^{s+1}xv^s}{1-qv} - \frac{pq^2xv}{1-qv} \right) A_1^{(s)}(x;p,q) \\
&- \left(x \sum_{k=1}^{s-2} p^{k+1}q^{k+2}v^{k+1} + \frac{pq^{s+1}xv^s}{1-qv} - \frac{pq^3xv^2}{1-qv} \right) A_2^{(s)}(x;p,q) \\
&- \cdots - \left(p^2q^sxv^{s-1} + \frac{pq^{s+1}xv^s}{1-qv} - \frac{pq^sxv^{s-1}}{1-qv} \right) A_{s-1}^{(s)}(x;p,q) \\
&+ \frac{pq^{s+1}xv^s}{1-qv} A^{(s)}(x;p,q;1).
\end{aligned}$$

Notice that

$$\begin{aligned}
\sum_{k=1}^{s} A_k^{(s)}(x;p,q)v^{k-1} &= p^2qx + \sum_{k=1}^{s} pq^kxv^{k-1}A^{(s)}(x;p,q;1) \\
&+ A_1^{(s)}(x;p,q) \sum_{k=1}^{s-1} (p^{k+1}q^{k+1} - pq^{k+1})xv^k + A_2^{(s)}(x;p,q) \sum_{k=1}^{s-2} (p^{k+1}q^{k+2} - pq^{k+2})xv^{k+1} \\
&+ \cdots + A_{s-1}^{(s)}(x;p,q)(p^2q^s - pq^s)xv^{s-1},
\end{aligned}$$

which leads to

$$\begin{aligned}
A^{(s)}(x;p,q;v) &= p^2qx + \frac{pqx}{1-qv} A^{(s)}(x;p,q;1) \\
&+ \left(\sum_{k=1}^{s} (pq)^{k+1}xv^k - \frac{pq^2xv}{1-qv} \right) A^{(s)}(x;p,q;qv).
\end{aligned}$$

□

3 The Area Statistic

The goal of this section is to analyze the area statistic. By setting $p = 1$ in Theorem 2.1 we obtain the functional equation

$$A^{(s)}(x;1,q;v) = qx + \frac{qx}{1-qv}A^{(s)}(x;1,q;1) - \frac{q^{s+2}xv^{s+1}}{1-qv}A^{(s)}(x;1,q;qv). \qquad (5)$$

Let $T_s(v) := -\frac{q^{s+2}xv^{s+1}}{1-qv}$; then by iterating this equation an infinite number of times (here, we may assume $|x|< 1$ or $|q|< 1$), we obtain the equality

$$A^{(s)}(x;1,q;v) = qx\left(1+\sum_{i\geq 0}\prod_{\ell=0}^{i}T_s(q^{\ell}v)\right) + \sum_{i\geq 1}\frac{qx}{1-q^iv}\prod_{\ell=1}^{i-1}T_s(q^{\ell-1}v)A^{(s)}(x;1,q,1).$$

By setting $v = 1$, and solving for $A^{(s)}(x;1,q;1)$, we may state the following result.

Theorem 3.1. *The generating function enumerating the polyominoes associated with the nonempty s-Fuss-Catalan words according to their length and area is given by*

$$A^{(s)}(x;1,q;1) = \frac{qx + qx\sum_{i\geq 1}\frac{(-1)^iq^{i((s+1)i+s+3)/2}x^i}{\prod_{\ell=1}^{i}(1-q^{\ell})}}{1-qx\sum_{i\geq 0}\frac{(-1)^iq^{i((s+1)i+s+3)/2}x^i}{\prod_{\ell=1}^{i+1}(1-q^{\ell})}}.$$

For example, for $s = 2, 3$ we have the series

$$\begin{aligned}A^{(2)}(x;1,q;1) &= qx + \left(q^4+q^3+q^2\right)x^2 + \left(q^9+q^8+2q^7+2q^6+3q^5+2q^4+q^3\right)x^3\\ &\quad + \left(q^{16}+q^{15}+2q^{14}+3q^{13}+4q^{12}+5q^{11}+7q^{10}+7q^9+8q^8\right.\\ &\quad \left.+7q^7+6q^6+3q^5+q^4\right)x^4+\cdots\end{aligned}$$

and

$$\begin{aligned}A^{(3)}(x;1,q;1) &= qx + q^2\left(q^3+q^2+q+1\right)x^2\\ &\quad + q^3\left(q^9+q^8+2q^7+2q^6+3q^5+3q^4+4q^3+3q^2+2q+1\right)x^3\\ &\quad + q^4(q^{18}+q^{17}+2q^{16}+3q^{15}+4q^{14}+5q^{13}+7q^{12}+8q^{11}+10q^{10}+12q^9+13q^8\\ &\quad +14q^7+14q^6+14q^5+12q^4+10q^3+6q^2+3q+1)x^4+\cdots.\end{aligned}$$

Figure 5 shows the weights of the polyominoes associated with the 2-Fuss-Catalan words members of $\mathcal{C}_3^{(2)}$. Notice that the sum of the weights of this example corresponds to the coefficient $[x^3]A^{(2)}(x;1,q;1)$.

Define $A_u^{(s)}(v) = \frac{d}{du}A^{(s)}(x;1,q;v)$ with $u \in \{q,v\}$ and $A^{(s)}(v) = A^{(s)}(x;1,q;v)$. Then from (5), we have

$$\begin{aligned}\left(1+\frac{xv^{s+1}}{1-v}\right)A_q^{(s)}(v)\mid_{q=1} &= x + \frac{x}{(1-v)^2}A^{(s)}(1)\mid_{q=1} + \frac{x}{1-v}A_q^{(s)}(1)\mid_{q=1}\\ &\quad - \frac{(s+2)xv^{s+1}-(s+1)xv^{s+2}}{(1-v)^2}A^{(s)}(v)\mid_{q=1}\\ &\quad - \frac{xv^{s+2}}{1-v}A_v^{(2)}(v)\mid_{q=1}.\end{aligned}$$

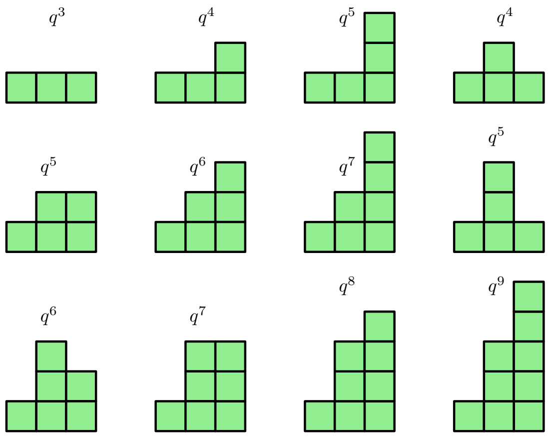

Figure 5: Weights for the polyominoes associated with the words in $\mathcal{C}_3^{(2)}$.

This type of functional equation can be solved systematically using the kernel method (see [1]). Let $v_0 = \sum_{n\geq 0} \frac{1}{sn+1}\binom{(s+1)n}{n}x^n$ be the root of the equation $v_0 = 1 + xv_0^{s+1}$, which is the generating function for the sequence $|\mathcal{C}_n^{(s)}|$. Note that $A^{(s)}(x;1,1;1) = v_0 - 1$. Thus, by taking $v = v_0$, then we have

$$A_q^{(s)}(1)\mid_{q=1} = v_0 + \frac{v_0^{s+1}(s+2-(s+1)v_0)}{1-v_0}A^{(s)}(v_0)\mid_{q=1} + v_0^{s+2}A_v^{(s)}(v)\mid_{q=1,v=v_0}. \quad (6)$$

Note that from (5) we have

$$A^{(s)}(v_0)\mid_{q=1} = \frac{x}{1-(s+1)xv_0^s} \quad (7)$$

and

$$A_v^{(s)}(v)\mid_{q=1,v=v_0} = \frac{\binom{s+1}{2}x^2v_0^{s-1}}{(1-(s+1)xv_0^s)^2}. \quad (8)$$

Hence, by (6), (7), (8), and the fact that $v_0 = 1 + xv_0^{s+1}$, we obtain the following result.

Theorem 3.2. *The generating function for the total area over the polyominoes associated with the members of* $\mathcal{C}_n^{(s)}$ *is given by*

$$\begin{aligned} A_q^{(s)}(1)\mid_{q=1} &= v_0 - \frac{s+2-(s+1)v_0}{1-(s+1)xv_0^s} + \frac{\binom{s+1}{2}(v_0-1-xv_0^s)}{(1-(s+1)xv_0^s)^2} \\ &= x\frac{dv_0}{dx} + \binom{s+1}{2}\frac{1}{v_0}\left(x\frac{dv_0}{dx}\right)^2, \end{aligned}$$

where $v_0 = \sum_{n\geq 0} \frac{1}{sn+1}\binom{(s+1)n}{n}x^n$.

Note that if $v_0 = \sum_{n\geq 0} \frac{1}{sn+1}\binom{(s+1)n}{n}x^n$ (solution of $v_0 = 1 + xv_0^{s+1}$), then $\frac{1}{v_0} = 1 - \sum_{n\geq 0} \frac{1}{n+1}\binom{(s+1)n+s-1}{n}x^{n+1}$. Hence, by Theorem 3.2

$$A_q^{(s)}(1)\mid_{q=1} = \sum_{j\geq 0} \frac{j}{sj+1}\binom{(s+1)j}{j}x^j$$
$$+\binom{s+1}{2}\left(1 - \sum_{j\geq 0}\frac{1}{j+1}\binom{(s+1)j+s-1}{j}x^{j+1}\right)\left(\sum_{j\geq 0}\frac{j}{sj+1}\binom{(s+1)j}{j}x^j\right)^2,$$

from which, by comparing the coefficient of x^n on both sides, we obtain the following result.

Theorem 3.3. *The total area over the polyominoes associated with the members of $\mathcal{C}_n^{(s)}$ is given by*

$$\frac{n}{sn+1}\binom{(s+1)n}{n} + \binom{s+1}{2}\sum_{j=0}^{n}\frac{j(n-j)}{(sj+1)(s(n-j)+1)}\binom{(s+1)j}{j}\binom{(s+1)(n-j)}{n-j}$$
$$-\binom{s+1}{2}\sum_{j=0}^{n-1}\sum_{i=0}^{j}\frac{i(j-i)}{(si+1)(s(j-i)+1)(n-j)}$$
$$\times\binom{(s+1)i}{i}\binom{(s+1)(j-i)}{j-i}\binom{(s+1)(n-j)-2}{n-1-j}.$$

Let $a_s(n)$ denote the total area of the polyominoes associated with the members of $\mathcal{C}_n^{(s)}$. For $s = 1$ the combinatorial formula given in Theorem 3.3 can be simplified to just (see [7, Corollary 12])

$$a_1(n) = \frac{1}{2}\left(4^n - \binom{2n}{n}\right).$$

Table 1 gives the first few values of the sequence $a_s(n)$ for $s = 1, 2, 3, 4$. Notice that the sequence $a_s(n)$ was studied by Merlini et al. [13] in the context of the Tennis Ball Problem.

$s\backslash n$	1	2	3	4	5	6	7	8	9
$s=1$	1	5	22	93	386	1586	6476	26333	106762
$s=2$	1	9	69	502	3564	24960	173325	1196748	8229849
$s=3$	1	14	156	1622	16347	161970	1588176	15465222	149866020
$s=4$	1	20	295	4000	52290	670316	8491720	106740640	1334461075

Table 1: Values of the total area.

4 The Semiperimeter of the Polyominoes

By setting $q = 1$ in (1) we obtain the functional equation

$$A^{(s)}(x;p,1;v) = p^2x + \frac{px}{1-v}A^{(s)}(x;p,1;1) + \left(x\sum_{k=1}^{s} p^{k+1}v^k - \frac{pxv}{1-v}\right)A^{(s)}(x;p,1;v). \tag{9}$$

Then

$$\left(1 - x\sum_{k=1}^{s} p^{k+1}v^k + \frac{pxv}{1-v}\right)A^{(s)}(x;p,1;v) = p^2x + \frac{px}{1-v}A^{(s)}(x;p,1;1). \tag{10}$$

Define the function

$$K(v) = 1 - x\sum_{k=1}^{s} p^{k+1}v^k + \frac{pxv}{1-v} = 1 - \frac{p^2xv(1-(pv)^s)}{1-pv} + \frac{pxv}{1-v}.$$

Let $v_0 = v_0(x,p)$ be a root of $K(v) = 0$. This functional equation can be solved again by the kernel method. In this case, if we assume that $v = v_0$, where v_0 satisfies $K(v_0) = 0$, we obtain

$$A^{(s)}(x;p,1;1) = p(v_0 - 1).$$

Note that the equation $K(v_0) = 0$ can be written as

$$w_0 = px(w_0+1)\frac{1-p-p^{s+1}w_0(w_0+1)^s}{(1-p)(1-\frac{pw_0}{1-p})},$$

where $w_0 = v_0 - 1$. Using the Lagrange inversion formula we obtain that the coefficient of x^n in w_0 (here, we assume that $|p|< 1$) is given by

$$[x^n]w_0 = \frac{1}{n}\sum_{0\le i+j\le n-1} \frac{(-1)^j p^{n+1+i+(s+1)j}}{(1-p)^{i+j}}\binom{n-1+i}{i}\binom{n}{j}\binom{n+sj}{n-1-i-j}.$$

Hence, we can state the following result.

Theorem 4.1. *The coefficient of x^n, $n \ge 1$, in $A^{(s)}(x;p,1;1)$ is given by*

$$\mathsf{Per}_n^{(s)}(p) := \frac{1}{n}\sum_{0\le i+j\le n-1} \frac{(-1)^j p^{n+1+i+(s+1)j}}{(1-p)^{i+j}}\binom{n-1+i}{i}\binom{n}{j}\binom{n+sj}{n-1-i-j}.$$

For example, $\mathsf{Per}_3^{(2)}(p) = p^4 + 3p^5 + 5p^6 + 2p^7 + p^8$. Figure 6 shows the weights of the polyominoes corresponding to this term.

Corollary 4.2. *The total semiperimeter over the polyominoes associated with the members of $\mathcal{C}_n^{(s)}$ is given by*

$$\left.\frac{\partial \mathsf{Per}_n^{(s)}(p)}{\partial p}\right|_{p=1}$$

Table 2 gives the first few values of the total semiperimeter sequence for $s = 1, 2, 3, 4$.

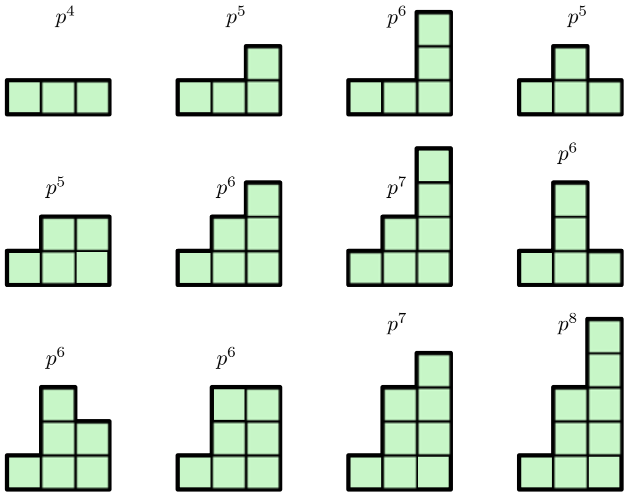

Figure 6: Weights for the polyominoes associated with the words in $\mathcal{C}_3^{(2)}$.

$s\backslash n$	1	2	3	4	5	6	7	8	9
$s=1$	2	7	25	91	336	1254	4719	17875	68068
$s=2$	2	12	71	430	2652	16576	104652	665874	4263050
$s=3$	2	18	150	1275	11033	96768	857440	7658001	68827440
$s=4$	2	33	439	5900	80535	1113273	15541258	218637585	3094921424

Table 2: Values of the total semiperimeter.

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