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On the switch-length of two connected graphs with the same degree sequence

Rosário Fernandes*

CMA and Departamento de Matemática
da Faculdade de Ciências e Tecnologia
Universidade Nova de Lisboa
2829-516 Caparica, Portugal
`mrff@fct.unl.pt`

Abstract

Let G be a simple graph containing distinct vertices x, y, z, w such that the edges $\{x, y\},\ \{z, w\} \in G$ and $\{x, z\},\ \{y, w\} \notin G$. The process of deleting the edges $\{x, y\},\ \{z, w\}$ from G and adding $\{x, z\},\ \{y, w\}$ to G is referred to as a switch (or 2-switch) in G. Let G_1 and G_2 be two connected simple graphs with the same vertex set V such that for all $v \in V$, the degree of v in G_1 is the same as in G_2. It is well known that G_2 can be obtained from G_1 by a sequence of switches. Moreover, there is one such sequences of switches with only connected graphs. For some classes of graphs, we study the problem of finding bounds for the minimum number of switches required for transforming G_1 into G_2 such that all graphs in the sequence are connected.

1 Introduction

Let $G = (V(G), E(G))$ be an undirected graph without loops and parallel edges; that is, a *simple graph* where $V(G)$ denotes the set of vertices of G and $E(G)$ denotes the set of its edges (set of unordered pairs of elements of $V(G)$). Let x, y, z, w be distinct vertices such that the edges $\{x, y\},\ \{z, w\} \in G$ and $\{x, z\},\ \{y, w\} \notin G$. The process of deleting the edges $\{x, y\},\ \{z, w\}$ from G and adding $\{x, z\},\ \{y, w\}$ to G is referred to as a *switch* (or 2-*switch*) in G.

Many researchers have studied this switch operation in simple graphs and papers have been written in many contexts, [2–4, 6–12].

Since the switch operation does not change the degree of each vertex, it preserves the degree sequence of the graph. Hakimi [14] proved that two graphs have the same

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degree sequence if and only if there is a finite sequence of switches that transforms one into another (see also [5, 13]).

Let G_1 and G_2 be two simple graphs on the same set of vertices V such that $d_{G_1}(v) = d_{G_2}(v)$, for all $v \in V$. In these conditions, we say that G_1 and G_2 have the same *V-degree sequence*. Since there is a finite sequence of switches that transforms G_1 into G_2, Will [17] and, separately Bereg and Ito [1], studied the minimum number of switches required for this transformation and obtained similar results. Moreover, Will showed that the computation of this number is NP-complete. In this paper, we use the notation of [1]. They defined a new graph

$$F(G_1, G_2) = (V, E(F)),$$

where $E(F) = (E(G_1) \cup E(G_2)) \setminus (E(G_1) \cap E(G_2))$ and its edges are colored with red if the edge belongs to $E(G_1) \setminus E(G_2)$ and with blue if the edge belongs to $E(G_2) \setminus E(G_1)$. They proved that there is *a disjoint decomposition of the edges of* $F(G_1, G_2)$ into red-blue alternating circuits and defined the number $\rho(G_1, G_2)$ as *the maximum number of circuits* for which there is one such edge disjoint decomposition of $F(G_1, G_2)$.

If $C_1, \ldots, C_{\rho(G_1,G_2)}$ is an edge disjoint decomposition of the edges of $F(G_1, G_2)$ into disjoint red-blue alternating circuits, then we say that $C_1, \ldots, C_{\rho(G_1,G_2)}$ is a $\rho(G_1, G_2)$-*decomposition* of $F(G_1, G_2)$.

More recently, Jaume et al. [15] defined the concept of *t-switch* as a switch that transforms a tree to another tree. Generalizing the notion of t-switch we define a *c-switch* in a connected graph G as a switch that transforms G into a connected graph. Note that if G is a tree, then a t-switch is the same as a c-switch.

Let G_1 and G_2 be two simple connected graphs with the same V-degree sequence. Taylor [16] proved that there is a sequence of c-switches for transforming G_1 into G_2.

We denote the *minimum number of c-switches that transform* G_1 *into* G_2 by $\psi_c(G_1, G_2)$.

The main focus of this paper is to study, for some classes of graphs, bounds for the minimum number of c-switches required for transforming one graph of this class into another one of the same class.

Through this paper we consider G_1 and G_2 to be two simple connected graphs with the same V-degree sequence. For every $v \in V(G_1)$ we denote by $N_{G_1}(v)$ the set of all vertices adjacent to v in G_1. The degree of v in G_1 is denoted by $d_{G_1}(v)$. If v and w are two distinct vertices of G_1, then we denote a path between them by $v - w$. We denote by $|V|$ (respectively, $|E(G_1)|$) the number of elements of V (respectively, $E(G_1)$). We will also consider the set

$$\Lambda(G_1, G_2) = \{\ell \in V : \ d_{G_1}(\ell) = 1 \text{ and } N_{G_1}(\ell) = N_{G_2}(\ell)\}.$$

It is well known that if G_1 and G_2 are two simple connected graphs with n vertices, then they have at least $n-1$ edges. Moreover, if they have $n-1$ edges, then they are trees. On the other hand, if they have n edges, then they are unicycle graphs (they have one cycle) and if they have $n+1$ edges, then they are bicycle graphs (they have at least two cycles).

The main results of this paper are the next theorems.

Theorem 1.1 *Let G_1 and G_2 be two connected unicycle graphs with the same V-degree sequence. If G_1 has at least one pendant vertex, then there is a unicycle graph Q obtained from G_1 by at most one c-switch such that $\Lambda(Q, G_2) \neq \emptyset$.*

Theorem 1.2 *Let G_1 and G_2 be two connected graphs with the same V-degree sequence such that $|V| = n$, $|E(G_1)| = n+1$. If G_1 has at least one pendant vertex, then there is a bicycle graph Q obtained from G_1 by at most one c-switch such that $\Lambda(Q, G_2) \neq \emptyset$.*

Theorem 1.3 *Let G_1 and G_2 be two connected unicycle graphs with the same V-degree sequence. Then $\psi_c(G_1, G_2) \leq |E(G_1)| - 2$.*

Theorem 1.4 *Let G_1 and G_2 be two bicycle graphs with the same V-degree sequence. Then $\psi_c(G_1, G_2) \leq |E(G_1)|$.*

The paper is organized as follows. In Section 2, we state known results in the literature related with the problem we study, which we also describe in detail. In Section 3 we prove Theorem 1.1. In Section 4, we consider bicycle graphs G_1 and G_2 and prove Theorem 1.2. In Section 5, we give an upper bound for $\psi_c(G_1, G_2)$ when G_1 is a tree or a unicycle graph or a bicycle graph, generalizing Theorems 1.3 and 1.4. Section 6 is dedicated to the circuits of a $\rho(G_1, G_2)$-decomposition of $F(G_1, G_2)$ and their consequences. Finally, in Section 7, we give some concluding remarks and open problems.

2 Background

Let G_1 and G_2 be two simple connected graphs with the same V-degree sequence. Since the number of red edges in $F(G_1, G_2)$ is equal to the number of its blue edges, Bereg and Ito [1] defined the number $r(G_1, G_2)$ as $|E(G_1) \setminus E(G_2)|$ and proved the following result.

Theorem 2.1 [1] *Let G_1 and G_2 be two simple connected graphs with the same V-degree sequence. Then the minimum number of switches for transforming G_1 into G_2 is equal to*

$$r(G_1, G_2) - \rho(G_1, G_2).$$

The proof of Theorem 2.1 presented a process for transforming G_1 into G_2 by $r(G_1, G_2) - \rho(G_1, G_2)$ switches, as follows.

Process 1 [1]

Step 1. *Between all $\rho(G_1, G_2)$-decompositions of $F(G_1, G_2)$, choose one with a circuit C having the minimum number of edges. Let a, b, c, d be a path of C such that $\{a, b\}$ and $\{c, d\}$ are red edges (elements of $E(G_1) \setminus E(G_2)$).*

Step 2.

- *If C has four edges, then R is the graph obtained from G_1 by deleting the edges $\{a, b\}$ and $\{c, d\}$ and adding the edges $\{b, c\}$ and $\{a, d\}$. Go to Step 3.*
- *Else if C has more than four edges and $\{a, d\} \notin E(G_1) \cup E(G_2)$, then R is the graph obtained from G_1 by deleting the edges $\{a, b\}$ and $\{c, d\}$ and adding the edges $\{b, c\}$ and $\{a, d\}$. Go to Step 3.*
- *Else if C has more than four edges and $\{a, d\} \in E(G_1) \cap E(G_2)$, then R' is the graph obtained from G_2 by deleting the edges $\{b, c\}$ and $\{a, d\}$ and adding the edges $\{a, b\}$ and $\{c, d\}$. Go to Step 3.*

Step 3.

- *If R is different from G_2, then $J = R$. Go to Step 1 with J and G_2.*
- *Else if R' is different from G_1, then $K = R'$. Go to Step 1 with G_1 and K.*
- *Otherwise, stop.*

Let T_1 and T_2 be two trees with the same V-degree sequence. By Theorem 2.1, the minimum number of switches for transforming T_1 into T_2 is equal to $r(T_1, T_2) - \rho(T_1, T_2)$. However, Theorem 2.1 did not mention which graphs appeared in the sequence of $r(T_1, T_2) - \rho(T_1, T_2)$ switches for transforming T_1 into T_2. Jaume et al. [15] considered the following process for t-switches.

Process 2 [15]

Step 1.

- *If $\Lambda(T_1, T_2) = \emptyset$, then consider $Q_1 = T_1$ and $Q_2 = T_2$ and go to Step 3.*
- *Otherwise, consider $P_1 = T_1$, $P_2 = T_2$ and go to Step 2.*

Step 2.

- *If $E(P_1) = E(P_2)$, then stop.*
- *Otherwise, remove from P_1 and P_2 the vertices of $\Lambda(P_1, P_2)$. We obtain two trees, K_1 from P_1 and K_2 from P_2. Go to Step 1 with K_1 and K_2.*

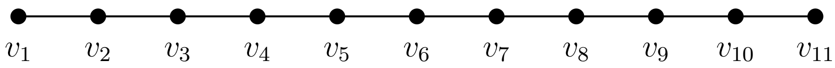
Step 3. *Let x be a pendant vertex of Q_1, y be its adjacent vertex in Q_1 and z be its adjacent vertex in Q_2. Let w be a vertex adjacent to z in Q_1 such that w does not belong to the path $x - z$, in Q_1. Let K be the tree obtained from Q_1 by deleting the edges $\{x, y\}$ and $\{z, w\}$ and adding the edges $\{x, z\}$ and $\{y, w\}$. Consider $R_1 = K$ and $R_2 = Q_2$ and go to Step 1 with R_1 and R_2.*

Since in Process 1 we can use edges not incident to pendant vertices and in Process 2 we must have edges incident to a pendant vertex (one in T_1 and another in T_2), a natural question arises: "Can we say that the minimum number of t-switches for transforming T_1 into T_2 is equal to $r(T_1, T_2) - \rho(T_1, T_2)$?"

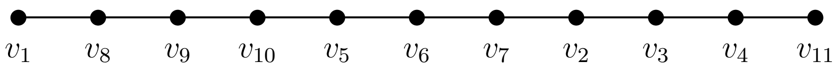
As we can see in the next example, sometimes this happens and sometimes it does not. Note that Process 2 does not give the minimum number of t-switches.

Example 2.2 *Consider the trees T_1 and T_2*

T_1

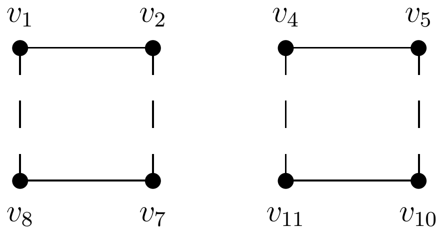


T_2



Drawing the edges of $E(T_1) \setminus E(T_2)$ by continuous lines and the edges of $E(T_2) \setminus E(T_1)$ by discontinuous lines, the graph $F(T_1, T_2)$ is

$F(T_1, T_2)$

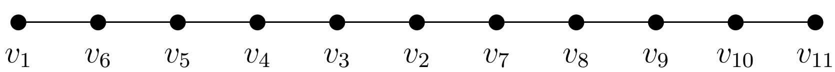


Hence the minimum number of switches for transforming T_1 into T_2 is

$$r(T_1, T_2) - \rho(T_1, T_2) = 4 - 2 = 2.$$

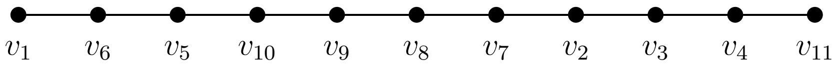
However, if we make the switch represented by the first circuit of $F(T_1, T_2)$, then we get a disconnected graph with a cycle. On the other hand, if we make the switch represented by the second circuit of $F(T_1, T_2)$, then we get a disconnected graph with a cycle. Consequently, we need at least three t-switches for transforming T_1 into T_2. For example, the tree T_3

T_3



is obtained from T_1 by deleting the edges $\{v_1, v_2\}$ and $\{v_6, v_7\}$ and adding the edges $\{v_1, v_6\}$ and $\{v_2, v_7\}$. Let T_4 be the tree

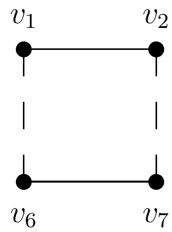
T_4



obtained from T_3 by deleting the edges $\{v_4, v_5\}$ and $\{v_{10}, v_{11}\}$ and adding the edges $\{v_5, v_{10}\}$ and $\{v_4, v_{11}\}$. The tree T_2 is obtained from the tree T_4 by deleting the edges $\{v_1, v_6\}$ and $\{v_8, v_7\}$ and adding the edges $\{v_1, v_8\}$ and $\{v_6, v_7\}$. Thus, $\psi_c(T_1, T_2) = 3$.

In the same way, drawing the edges of $E(T_1) \setminus E(T_3)$ by continuous lines and the edges of $E(T_3) \setminus E(T_1)$ by discontinuous lines, the graph $F(T_1, T_3)$ is

$F(T_1, T_3)$



Therefore the minimum number of switches for transforming T_1 into T_3 is $r(T_1, T_3) - \rho(T_1, T_3) = 2 - 1 = 1$. In fact, as we saw, T_3 is obtained from T_1 by one t-switch.

3 Connected unicycle graphs

Let G_1 and G_2 be two unicycle graphs. The main important result of this section is Theorem 1.1.

Let
$$\Omega(G_1, G_2) = \{\ell \in V : \ d_{G_1}(\ell) \neq 1 \text{ and } N_{G_1}(\ell) \cap N_{G_2}(\ell) \neq \emptyset\}.$$
Note that if $c \in \Omega(G_1, G_2)$ and $b \in N_{G_1}(c) \cap N_{G_2}(c)$ has degree at least 2, then $b \in \Omega(G_1, G_2)$.

First, we will consider the case when G_1 and G_2 are cycles.

Proposition 3.1 *Let G_1 and G_2 be two cycles with the same V-degree sequence. Then there is a cycle P obtained from G_1 by at most one c-switch such that $E(P) \cap E(G_2) \neq \emptyset$.*

Proof. If $\Omega(G_1, G_2) \neq \emptyset$, then there exist $\ell, p \in \Omega(G_1, G_2)$ such that $\{\ell, p\} \in E(G_1) \cap E(G_2)$.

Now, suppose that $\Omega(G_1, G_2) = \emptyset$. Let $\{a, b\} \in E(G_1)$ and $c \in V$ be adjacent to a in G_2. Thus, $c \neq b$, $c \neq a$ and $\{a, c\} \notin E(G_1)$. Let $d \in V$ such that $\{c, d\} \in E(G_1)$ and d does not belong to the path $a - c$ that contains b, in G_1. If we remove the edges $\{a, b\}$ and $\{c, d\}$ from G_1 and add the edges $\{a, c\}$ and $\{b, d\}$, then we get a cycle P obtained from G_1 by one c-switch. Moreover, $\{a, c\} \in E(P) \cap E(G_2)$. □

Now, we can prove Theorem 1.1.

Proof of Theorem 1.1. If $\Lambda(G_1, G_2) \neq \emptyset$ then the result follows.

Suppose that $\Lambda(G_1, G_2) = \emptyset$. Let a be a pendant vertex in G_1 and b, c be its adjacent vertices in G_1 and G_2, respectively. Since $d_{G_2}(c) \geq 2$, let d be a vertex of G_1 such that $\{c, d\} \in E(G_1)$ and there is a path $a - c$ in G_1, without d. If we remove the edges $\{a, b\}$ and $\{c, d\}$ from G_1 and add the edges $\{a, c\}$ and $\{b, d\}$, then we get a connected graph P obtained from G_1 by one c-switch. This implies that P is a unicycle graph and $\Lambda(P, G_2) \neq \emptyset$. □

4 Connected bicycle graphs

Let G_1 and G_2 be two connected bicycle graphs. Let

$$\Upsilon(G_1, G_2) = \begin{array}{l} \{\{\ell, p\} \in E(G_1) \cap E(G_2) : \ d_{G_1}(\ell) = 2 = d_{G_1}(p) \text{ and} \\ G_1 - \{\ell, p\} \text{ and } G_2 - \{\ell, p\} \text{ are connected graphs}\}. \end{array}$$

There are three types of connected bicycle graphs without any pendant vertices:

- *The tight handcuff* which is formed by two cycles sharing a vertex.
- *The loose handcuff* which is formed by two disjoint cycles connected by a path.
- *The theta* which has two vertices with degree three that are connected by three disjoint paths and the other vertices have degree two.

We first show that if G_1 and G_2 are connected bicycle graphs without any pendant vertices, then there are bicycle graphs Q_1 and Q_2 obtained, respectively, from G_1 and G_2 by c-switches such that $\Upsilon(Q_1, Q_2) \neq \emptyset$.

Lemma 4.1 *Let G_1 and G_2 be two connected graphs with the same V-degree sequence and without any pendant vertices. If G_1 is a tight handcuff graph, then there is a bicycle graph Q obtained from G_1 by at most one c-switch such that $\Upsilon(Q, G_2) \neq \emptyset$.*

Proof. Since G_1, G_2 are connected graphs with the same V-degree sequence and G_1 is a tight handcuff graph, G_2 is also a tight handcuff graph. If $\Upsilon(G_1, G_2) \neq \emptyset$, then we get the result.

Suppose that $\Upsilon(G_1, G_2) = \emptyset$. Since G_1 has two cycles, C_1 and C_2, shared by one vertex and each cycle has at least three vertices, there exist $a, b \in V$ such that a, b are vertices of C_1, $\{a, b\} \in E(G_1)$ and $d_{G_1}(a) = d_{G_1}(b) = 2$. Let $c \in V$ adjacent to a in G_2 such that $d_{G_2}(c) = 2$. As G_2 is a tight handcuff graph, $\{a, c\}$ belongs to a cycle in G_2. Moreover, $c \neq b$, $c \neq a$ and $\{a, c\} \notin E(G_1)$.

If c belongs to C_1, then there is a vertex $d \in V$ such that $\{c, d\} \in E(G_1)$ and d does not belong to any path $a - c$ that contains b, in G_1. If we remove the edges $\{a, b\}$ and $\{c, d\}$ from G_1 and add the edges $\{a, c\}$ and $\{b, d\}$, then we get a tight handcuff graph P obtained from G_1 by one c-switch and $\{a, c\}$ belongs to a cycle in P. Since $\{a, c\} \in E(P) \cap E(G_2)$, we have $\Upsilon(P, G_2) \neq \emptyset$.

If c belongs to C_2, then there is a vertex $g \in V$ such that $\{c, g\} \in E(G_1)$ and $d_{G_1}(g) = 2$. If we remove the edges $\{a, b\}$ and $\{c, g\}$ from G_1 and add the edges $\{a, c\}$ and $\{b, g\}$, then we get a tight handcuff graph J obtained from G_1 by one c-switch and $\{a, c\}$ belongs to a cycle in J. Since $\{a, c\} \in E(J) \cap E(G_2)$, we get $\Upsilon(J, G_2) \neq \emptyset$. □

Lemma 4.2 *Let G_1 and G_2 be two theta graphs with the same V-degree sequence. Then there is a bicycle graph Q obtained from G_1 by at most one c-switch such that $\Upsilon(Q, G_2) \neq \emptyset$.*

Proof. Since G_1 is a simple graph, G_1 has at least two vertices of degree two. If G_1 has at most three vertices of degree two, then it is easy to conclude that G_1 and G_2 are the same graph or G_2 is obtained from G_1 by one c-switch. Consequently, we get the result. Thus, we can suppose that G_1 has at least four vertices of degree two.

If $\Upsilon(G_1, G_2) \neq \emptyset$, then we get the result.

Suppose that $\Upsilon(G_1, G_2) = \emptyset$. Let d and f be the two vertices of G_1 with degree three and let P_1, P_2 and P_3 be the three disjoint paths $d - f$ in G_1.

Case 1. Suppose that there are $a, b, c \in V$ such that $\{a, b\} \in E(G_1)$, $\{a, c\} \in E(G_2)$ and $d_{G_1}(a) = d_{G_1}(b) = d_{G_1}(c) = 2$. This implies that $\{a, c\}$ belongs to a cycle in G_2 Moreover, $c \neq b$, $c \neq a$ and $\{a, c\} \notin E(G_1)$. Suppose that a, b belong to P_1. We have two possibilities; $c \in P_1$ or not.

First, assume that c belongs to P_1. Let g be the vertex in P_1 such that $\{c, g\} \in E(G_1)$ and, one and only one of the vertices g and b belong to the path $a - c$, in P_1. If we remove the edges $\{a, b\}$ and $\{c, g\}$ from G_1 and add the edges $\{a, c\}$ and $\{b, g\}$, then we get a theta graph S obtained from G_1 by one c-switch. Since $\{a, c\} \in E(S) \cap E(G_2)$ and $\{a, c\}$ belongs to a cycle in S, we get $\Upsilon(S, G_2) \neq \emptyset$.

Now, assume that c does not belong to P_1 and b belongs to the path $d - a$, in P_1. Suppose that c belongs to P_2. Let $z \in V$ adjacent to c in G_1 and such that z does not belong to the path $d - c$, in P_2. If we remove the edges $\{a, b\}$ and $\{c, z\}$ from G_1 and add the edges $\{a, c\}$ and $\{b, z\}$, then we get a theta graph R obtained from G_1 by one c-switch. Since $\{a, c\} \in E(R) \cap E(G_2)$ and $\{a, c\}$ belongs to a cycle in R, we get $\Upsilon(R, G_2) \neq \emptyset$.

Case 2. Suppose that there are not $a, b, c \in V$ such that $\{a, b\} \in E(G_1)$, $\{a, c\} \in E(G_2)$ and $d_{G_1}(a) = d_{G_1}(b) = d_{G_1}(c) = 2$. Then G_1 has four vertices of degree 2. Let x, y, z, w be these four vertices. Therefore, if we suppose that in G_1, x, y are vertices of P_1, then z is a vertex of P_2 and w is a vertex of P_3. Moreover, G_2 is a theta graph with three paths between d and f, R_1, R_2 and R_3. Then, in G_2, x is a vertex of R_1, y is a vertex of R_2 and z, w are vertices of R_3. It is easy to conclude that the result follows. □

Lemma 4.3 *Let G be a loose handcuff graph. Then there is a theta graph K obtained from G by one c-switch.*

Proof. Let C_1 and C_2 be the cycles of G. Let $a, b, c, d \in V(G)$ such that $d_G(a) = d_G(b) = d_G(c) = d_G(d) = 2$, $\{a, b\} \in C_1$ and $\{c, d\} \in C_2$. If we remove the edges

$\{a, b\}$ and $\{c, d\}$ from G and add the edges $\{a, c\}$ and $\{b, d\}$, then we get a theta graph K obtained from G by one c-switch. □

Lemma 4.4 *Let G_1 and G_2 be connected graphs with the same V-degree sequence and without any pendant vertices. If G_1 is a loose handcuff graph, then there are bicycle graphs Q_1 and Q_2 obtained, respectively, from G_1 and G_2 by at most three c-switches such that $\Upsilon(Q_1, Q_2) \neq \emptyset$.*

Proof. By Lemma 4.3, there is a theta graph K_1 obtained from G_1 by one c-switch and there is a theta graph K_2 obtained from G_2 by at most one c-switch. By Lemma 4.2, we get the result. □

Now we can prove Theorem 1.2.

Proof of Theorem 1.2.
If $\Lambda(G_1, G_2) \neq \emptyset$, then the result follows.

Suppose that $\Lambda(G_1, G_2) = \emptyset$. Let a be a pendant vertex of G_1, b be its adjacent vertex in G_1 and c be its adjacent vertex in G_2. Since $d_{G_2}(c) \geq 2$, let d be a vertex of G_1 such that $\{c, d\} \in E(G_1)$ and so that there is a path $a - c$ in G_1, without d. If we remove the edges $\{a, b\}$ and $\{c, d\}$ from G_1 and add the edges $\{a, c\}$ and $\{b, d\}$, then we get a connected graph P obtained from G_1 by one c-switch, and $\Lambda(P, G_2) \neq \emptyset$. □

5 Upper bound for the number of c-switches

In the Introduction we mentioned the number $\psi_c(G_1, G_2)$ as the minimum number of c-switches for transforming G_1 into G_2, when G_1 and G_2 are connected graphs. The focus of this section is to describe an upper bound for this number.

Theorem 5.1 *Let T_1 and T_2 be two trees with at least three vertices and the same V-degree sequence. Then $\psi_c(T_1, T_2) \leq |E(T_1)| - 2$.*

Proof. As T_1 has at least three vertices, $|E(T_1)| - 2 \geq 0$. Assume that T_1 and T_2 are two trees with $\Lambda(T_1, T_2) = \emptyset$. In each round of Process 2 we get at least one pendant vertex that is adjacent to a same vertex in the two trees considered and, consequently, we get a new common edge in the two trees. On the other hand, if each one of our two trees only have three edges in a path, then applying Process 2 once, we get two equal trees. Therefore the result follows. □

By Theorem 5.1 and the results of Sections 3 and 4 we can prove Theorems 1.3 and 1.4.

Proof of Theorem 1.3. Suppose that G_1 is a cycle. By Proposition 3.1, there is a cycle P obtained from G_1 by at most one c-switch such that $E(P) \cap E(G_2) \neq \emptyset$. Removing a common edge from P and G_2 we obtain two trees, T_1 and T_2. By Theorem 5.1, $\psi_c(T_1, T_2) \leq |E(T_1)| - 2$. Consequently,

$$\psi_c(G_1, G_2) \leq \psi_c(T_1, T_2) + 1 \leq |E(T_1)| - 1 = |E(G_1)| - 2.$$

Suppose that G_1 has a pendant vertex. By Theorem 1.1, there is an unicycle graph Q obtained from G_1 by at most one c-switch such that $\Lambda(Q, G_2) \neq \emptyset$. Removing a common pendant vertex from Q and G_2 we have two unicycle graphs. Repeating this argument we obtain two cycles. By the above arguments we conclude the result. □

Proof of Theorem 1.4. Suppose that G_1 has no pendant vertices. By Lemmas 4.1, 4.2 and 4.4, there are bicycle graphs Q_1, Q_2 obtained, respectively, from G_1 and G_2 by at most three c-switches such that $\Upsilon(Q_1, Q_2) \neq \emptyset$. Removing an element of $\Upsilon(Q_1, Q_2)$ from Q_1 and Q_2 we obtain two connected unicycle graphs, H_1 and H_2. By Theorem 1.3, $\psi_c(H_1, H_2) \leq |E(H_1)| - 2$. Consequently,

$$\psi_c(G_1, G_2) \leq \psi_c(H_1, H_2) + 3 \leq |E(H_1)| + 1 = |E(G_1)|.$$

Suppose that G_1 has a pendant vertex. By Theorem 1.2, there is a bicycle graph Q obtained from G_1 by at most one c-switch such that $\Lambda(Q, G_2) \neq \emptyset$. Removing an element of $\Lambda(Q, G_2)$ from Q and G_2 we have two bicycle graphs. Repeating this argument we obtain two bicycle graphs with no pendant vertices. By the above arguments we conclude the result. □

Proposition 5.2 *Let G_1 and G_2 be two trees or two connected unicycle graphs with at least three vertices and the same V-degree sequence. Then $r(G_1, G_2) - \rho(G_1, G_2) \leq |E(G_1)| - 2$.*

Proof. By Theorem 2.1, the minimum number of switches for transforming G_1 into G_2 is equal to

$$r(G_1, G_2) - \rho(G_1, G_2).$$

Hence $r(G_1, G_2) - \rho(G_1, G_2) \leq \psi_c(G_1, G_2)$. By Theorems 5.1 and 1.3, the result follows. □

Using Theorems 5.1 and 1.3, and Proposition 5.2, we get the following result.

Corollary 5.3 *There do not exist two trees, nor two connected unicycle graphs G_1 and G_2, with the same V-degree sequence such that $\rho(G_1, G_2) = 1$, and having no common edge.*

Proof. Suppose there are graphs G_1 and G_2 satisfying the conditions of this corollary such that $\rho(G_1, G_2) = 1$ and having no common edge. Consequently,

$$|E(G_1)| - 1 = |E(G_1) \setminus E(G_2)| - 1 = r(G_1, G_2) - \rho(G_1, G_2).$$

This is impossible by Proposition 5.2. □

6 The $\rho(G_1, G_2)$-decomposition

In this section we obtain some results about the circuits of a $\rho(G_1, G_2)$-decomposition of $F(G_1, G_2)$, where G_1 and G_2 are two simple connected graphs with the same V-degree sequence.

Proposition 6.1 *Let G_1 and G_2 be two simple connected graphs with the same V-degree sequence. Let a, b, c be three distinct elements of V such that*

$$\{a, b\} \in E(G_1) \setminus E(G_2) \text{ and } \{b, c\} \in E(G_2) \setminus E(G_1).$$

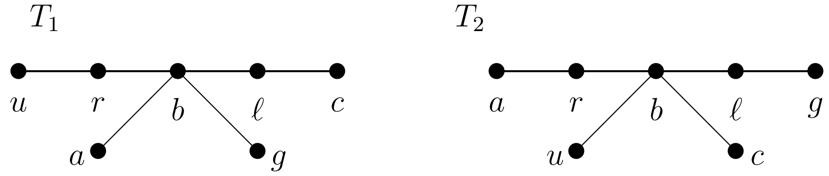
Let C be a circuit of a $\rho(G_1, G_2)$-decomposition of $F(G_1, G_2)$, to which $\{a, b\}$ and $\{b, c\}$ belong. Then there is a disjoint red-blue alternating circuit C' of $F(G_1, G_2)$ with the same edges as C and the path a, b, c.

Moreover, C' is a circuit of a $\rho(G_1, G_2)$-decomposition of $F(G_1, G_2)$.

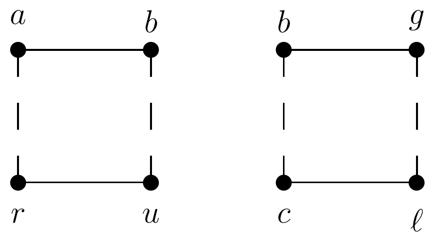
Proof. If a, b, c is a path in C, then the result follows. Suppose that a, b, c is not a path in C. Let f and g be the vertices such that a, b, f and g, b, c are paths in C. Suppose that $a, b, f, \ldots, g, b, c$ is the walk in C, without repeated edges, from a to c where the two referred paths belong. Denote by b_f and b_g the vertex b next to f and g in this walk, respectively. If W is obtained from C by deleting the edges $\{b_f, f\}$ and $\{b_g, c\}$ and adding the edges $\{b_f, c\}$ and $\{b_g, f\}$, then W is the union of two red-blue alternating circuits of the edges of $F(G_1, G_2)$. This contradicts the hypothesis. Thus, $a, b, f, \ldots, c, b, g$ is the walk in C, without repeated edges, from a to g where the two referred paths belong. If C' is the circuit obtained from C by deleting the edges $\{b_f, f\}$ and $\{b_g, c\}$ and adding the edges $\{b_f, c\}$ and $\{b_g, f\}$, then a, b, c is a path in C' and we get the result. □

As we can see in the next example, Proposition 6.1 does not hold when there is not a circuit in a $\rho(G_1, G_2)$-decomposition of $F(G_1, G_2)$ where $\{a, b\}$ and $\{b, c\}$ belong.

Example 6.2 *Consider the trees T_1 and T_2*



Drawing the edges of $E(T_1) \setminus E(T_2)$ with continuous lines and the edges of $E(T_2) \setminus E(T_1)$ with discontinuous lines, the unique $\rho(T_1, T_2)$-decomposition of $F(T_1, T_2)$ is



Therefore, there is no $\rho(T_1, T_2)$-decomposition having a circuit with the edges $\{a, b\}$ and $\{b, c\}$.

Proposition 6.3 *Let G_1 and G_2 be two simple connected graphs with the same V-degree sequence. Let a, b, c, d be four distinct elements of V such that*

$$\{a, b\}, \{c, d\} \in E(G_1) \setminus E(G_2) \text{ and } \{b, c\} \in E(G_2) \setminus E(G_1).$$

Let C be a circuit of a $\rho(G_1, G_2)$-decomposition of $F(G_1, G_2)$ to which $\{a, b\}$, $\{b, c\}$ and $\{c, d\}$ belong. Then there is a disjoint red-blue alternating circuit C'' of $F(G_1, G_2)$ with the same edges as C and the path a, b, c, d.

Moreover, C'' is a circuit of a $\rho(G_1, G_2)$-decomposition of $F(G_1, G_2)$.

Proof. By Proposition 6.1, let C' be a disjoint red-blue alternating circuit of $F(G_1, G_2)$ with the same edges as C and the path a, b, c. If a, b, c, d is a path in C', then the result follows. Suppose that a, b, c, d is not a path in C'. Let f be the vertex such that a, b, c, f is a path of C'. Let r, c, d be the path in C' where the edge $\{c, d\}$ belongs. Suppose that $d, c, r, \ldots, a, b, c, f$ is the walk in C', without repeated edges, from d to f where the two referred paths belong. Denote by c_r and c_f the vertex c next to r and f in this walk, respectively. If W is obtained from C' by deleting the edges $\{c_r, r\}$ and $\{c_f, f\}$ and adding the edges $\{c_f, r\}$ and $\{c_r, f\}$, then W is the union of two red-blue alternating circuits of the edges of $F(G_1, G_2)$. This contradicts the hypothesis. Hence $r, c, d, \ldots, a, b, c, f$ is the walk in C', without repeated edges, from d to f where the two referred paths belong. Let C'' be the circuit obtained from C' by deleting the edges $\{c_r, r\}$ and $\{c_f, f\}$ and adding the edges $\{c_f, r\}$ and $\{c_r, f\}$. Then a, b, c, d is a path in C'' and we get the result. □

Proposition 6.4 *Let G_1 and G_2 be two connected graphs, with the same V-degree sequence, such that $\rho(G_1, G_2) = 1$ and there is a sequence of c-switches for transforming G_1 into G_2. Let a, b, c, d be four distinct elements of V such that*

$$\{a, b\}, \ \{c, d\} \in E(G_1) \setminus E(G_2), \ \{b, c\} \in E(G_2) \setminus E(G_1), \ \{a, d\} \notin E(G_1)$$

and there is a path $a - d$ in G_1, to which b belongs and c does not.
Let H be the graph obtained from G_1 by removing the edges $\{a, b\}$ and $\{c, d\}$ and

adding the edges $\{b, c\}$ and $\{a, d\}$. Then H is a connected graph. Moreover, if there is a sequence of c-switches for transforming H into G_2, then

$$\psi_c(G_1, G_2) \leq \psi_c(H, G_2) + 1.$$

Proof. Since $\rho(G_1, G_2) = 1$ by Proposition 6.3 we can assume that a, b, c, d is a path in the circuit C of a $\rho(G_1, G_2)$ decomposition of $F(G_1, G_2)$. Let $x, y \in V$. As G_1 is connected, there is a path $x - y$ in G_1. If $\{a, b\}$ and $\{c, d\}$ do not belong to this path, then the mentioned path is a path $x - y$ in H. Suppose that $\{a, b\}$ or $\{c, d\}$ belong to the path $x - y$ in G_1. As there is a path $a - d$ in G_1, to which b belongs and c does not, there is a path $b - d$ in G_1, without a and c. Hence, if we remove the edge $\{a, b\}$ from the path $x - y$ in G_1 and add the path $a, d - b$, then we get a walk $x - y$ without the edge $\{a, b\}$. If we remove the edge $\{c, d\}$ from this last walk $x - y$ in G_1 and add the path $c, b - d$, then we get a walk $x - y$ without the edges $\{a, b\}$ and $\{c, d\}$. Consequently, there is a path $x - y$ in H and H is a connected graph obtained from G_1 by one c-switch.

If there is a sequence of c-switches for transforming H into G_2, then there is a sequence of c-switches for transforming G_1 into G_2 and the graph H is obtained from G_1 by the first c-switch. Therefore, $\psi_c(G_1, G_2) \leq \psi_c(H, G_2) + 1$. □

7 Conclusions

We have defined graphs with the same V-degree sequence as graphs on the same set of vertices V such that the degree of each $v \in V$ is the same in all these graphs. We also have defined a c-switch in a connected graph as a switch that transforms a connected graph into another connected graph.

Let G_1 and G_2 be two connected graphs with n vertices, at most $n+1$ edges and the same V-degree sequence. We have studied bounds for the minimum number of switches required for transforming G_1 into G_2.

We conclude with some open questions.

1. Describe all pairs, (G_1, G_2), of connected graphs with n vertices and at most $n + 1$ edges such that $\psi_c(G_1, G_2) = r(G_1, G_2) - \rho(G_1, G_2)$.

2. Describe all pairs, (G_1, G_2), of connected graphs with n vertices and at most $n + 1$ edges such that $\psi_c(G_1, G_2) = |E(G_1)| - 2$.

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