AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **83(1)** (2022), Pages 163–172

Graphs with disjoint 2-dominating sets

Michael A. Henning*

Department of Mathematics and Applied Mathematics
University of Johannesburg
Auckland Park 2006, South Africa
mahenning@uj.ac.za

Jerzy Topp

Institute of Applied Informatics
University of Applied Sciences
82-300 Elbląg, Poland
j.topp@pwsz.elblag.pl

Abstract

A subset $D \subseteq V(G)$ is a dominating set of a multigraph G if every vertex in $V(G) \setminus D$ has a neighbor in D, while D is a 2-dominating set of G if every vertex belonging to $V(G) \setminus D$ is joined by at least two edges with a vertex or vertices in D. A graph G is a $(2,2)$-dominated graph if it has a pair (D, D') of disjoint 2-dominating sets of vertices of G. In this paper we present two characterizations of minimal $(2,2)$-dominated graphs.

1 Introduction

For notation and graph theory terminology we generally follow [7]. Specifically, let $G = (V(G), E(G))$ be a graph with possible multi-edges and multi-loops, and with vertex set $V(G)$ and edge set $E(G)$. For a vertex v of G, its *neighborhood*, denoted by $N_G(v)$, is the set of vertices adjacent to v. The *closed neighborhood* of v, denoted by $N_G[v]$, is the set $N_G(v) \cup \{v\}$. In general, for a subset $X \subseteq V(G)$, the *neighborhood* of X, denoted by $N_G(X)$, is defined to be $\bigcup_{v \in X} N_G(v)$, and the *closed* neighborhood of X, denoted by $N_G[X]$, is the set $N_G(X) \cup X$. The 2-*neighborhood* of v, denoted by $N_G^2(v)$, is the set of vertices at distance 2 from v in G, that is, $N_G^2(v) = \{u \in V(G) \colon d_G(u,v) = 2\}$. The *closed* 2-*neighborhood* of v, denoted by $N_G^2[v]$, is the set of vertices within distance 2 from v in G, and so $N_G^2[v] = N_G[v] \cup N_G^2(v)$.

* Research supported in part by the University of Johannesburg

ISSN: 2202-3518

If A and B are disjoint sets of vertices of G, then we denote by $E_G(A, B)$ the set of edges in G joining a vertex in A with a vertex in B. For one-element sets we write $E_G(v, B)$, $E_G(A, u)$, and $E_G(u, v)$ instead of $E_G(\{v\}, B)$, $E_G(A, \{u\})$, and $E_G(\{u\}, \{v\})$, respectively. If v is a vertex of G, then by $E_G(v)$ we denote the set of edges incident with v in G. The *degree* of a vertex v in G, denoted by $d_G(v)$, is the number of non-loop edges incident with v plus twice the number of loops incident with v. A vertex of degree one is called a *leaf.* A vertex is *isolated* if its degree equals zero. The smallest and largest degrees in a graph G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. For an integer $k \geq 1$, we let $[k] = \{1, \ldots, k\}$.

A set of vertices $D \subseteq V(G)$ of G is a *dominating set* if every vertex in $V(G) \setminus D$ has a neighbor in D, while D is a k*-dominating set*, where k is a positive integer, if every vertex belonging to $V(G) \setminus D$ is joined by at least k edges with a vertex or vertices in D. If G is a graph without multiple-edges, then a subset $D \subseteq V(G)$ is a k*-dominating set* of G if $|N_G(v) \cap D| \geq k$ for every $v \in V(G) \setminus D$.

If k and ℓ are positive integers, then a pair (D_1, D_2) of proper and disjoint subsets of the vertex set $V(G)$ of a graph G is a (k, ℓ)*-pair* in G if D_1 is a k-dominating set of G, and D_2 is an ℓ-dominating set of G. A graph G is said to be a (k, ℓ)*-dominated graph* if it contains a (k, ℓ)-pair. It is obvious from the above definition, that if a graph G is a (k, ℓ)-dominated graph, then necessarily $\max\{k, \ell\} \leq \Delta(G)$, and $1 \leq \min\{k, \ell\} \leq \delta(G)$. Trivially, if G is a (k, ℓ)-dominated graph, then G is a (k', ℓ')-dominated graph, where $1 \leq k' \leq k$ and $1 \leq \ell' \leq \ell$. In addition, if G is a (k, ℓ)-dominated graph, then G is an (ℓ, k)-dominated graph. Thus we may suppose that if G is a (k, ℓ)-dominated graph, then $k \leq \ell$.

We observe that a complete graph K_n is a (k, ℓ)-dominated graph (for positive integers k and ℓ) if and only if $k + \ell \leq n$. Moreover, we observe that a complete bipartite graph $K_{m,n}$ is a (m, n)-dominated graph. A cycle C_n is a $(2, 2)$-dominated graph if and only if n is an even positive integer, while every cycle of odd length is a $(1, 2)$-dominated graph but not a $(2, 2)$-dominated graph.

Of the four graphs in Fig. 1, the graphs F, H, and the Cartesian product $K_2 \,\Box\, C_5$ are examples of $(2, 2)$-dominated graphs, while the Cartesian product $K_2 \,\Box\, K_4$ is an example of a $(3, 3)$-dominated graph (and a $(1, 4)$-dominated graph). The appropriate $(2, 2)$- and $(3, 3)$-pairs in these graphs are determined by the sets of black and white vertices, respectively, illustrated in Fig. 1. More generally, we show in Corollary 2.3 that if G and H are graphs without isolated vertices, then the Cartesian product $G \,\Box\, H$ is a $(2, 2)$-dominated graph.

Ore [11] was the first to observe that a graph without isolated vertices contains two disjoint dominating sets. That is, Ore observed that every such graph is a $(1, 1)$-dominated graph. Subsequently, various properties of graphs having disjoint dominating sets of different types have been extensively studied, for example, in papers [1]–[10] and [12], to mention just a few. All $(1, 2)$-dominated graphs were characterized in [6, 9, 10]. In this paper, we study $(2, 2)$-dominated graphs, and, in particular, we present two characterizations of minimal $(2, 2)$-dominated graphs. It is worth mentioning here that it follows from [1, Theorem 12] that in the general case

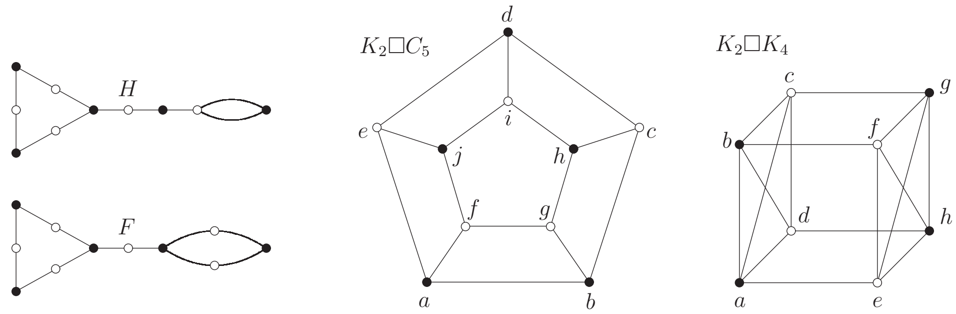

Figure 1: Graphs F, H, $K_2\Box C_5$, and $K_2\Box K_4$

it is $\mathcal{NP}$-complete to decide whether a given graph G is a $(2,2)$-dominated graph if $\delta(G) \geq 2$.

2 Elementary properties of $(2,2)$-dominated graphs

In this section, we present properties of $(2,2)$-dominated graphs that we need in order to prove our main results.

Definition 1. A connected graph G is said to be a *minimal* $(2,2)$-*dominated graph*, if G is a $(2,2)$-dominated graph and no proper spanning subgraph of G is a $(2,2)$-dominated graph.

From this definition, we immediately have the following observations.

Observation 2.1. *Every spanning supergraph of a* $(2,2)$*-dominated graph is a* $(2,2)$*-dominated graph, and every* $(2,2)$*-dominated graph is a spanning supergraph of some minimal* $(2,2)$*-dominated graph.*

Observation 2.2. *Every bipartite graph of minimum degree at least* 2 (*and every spanning supergraph of such a graph*) *is a* $(2,2)$*-dominated graph.*

As a consequence of Observation 2.2, we have the following result.

Corollary 2.3. *If G and H are graphs without isolated vertices, then their Cartesian product $G \Box H$ is a* $(2,2)$*-dominated graph.*

Proof. We may assume that G and H are connected graphs each of order at least 2. Let T_G and T_H be spanning trees of G and H, respectively. Then $T_G \Box T_H$ is a bipartite spanning subgraph of $G \Box H$ and $\delta(G \Box H) = 2$. Thus, by Observations 2.2 and 2.1, $T_G \Box T_H$ and $G \Box H$ are $(2,2)$-dominated graphs. $\Box$

In view of Observation 2.1, minimal $(2,2)$-dominated graphs can be viewed as skeletons of $(2,2)$-dominated graphs, skeletons which can be extended to any $(2,2)$-dominated spanning supergraph.

The next theorem presents general properties of minimal $(2,2)$-dominated graphs.

Theorem 2.4. *A graph G is a minimal $(2,2)$-dominated graph if and only if G has the following three properties.*

(a) $\delta(G) \geq 2$.
(b) *G is a bipartite graph.*
(c) *Every edge of G is incident with a vertex of degree 2 in G.*

Proof. Assume first that G is a minimal $(2,2)$-dominated graph, and let (D_1, D_2) be a $(2,2)$-pair in G. Since D_1 and D_2 are disjoint, every vertex v of G belongs to $V(G) \setminus D_1$ or to $V(G) \setminus D_2$, and thus v is joined by at least two edges with a vertex or vertices in D_1 or D_2 (since D_1 and D_2 are 2-dominating sets), implying that $\delta(G) \geq 2$. We now claim that D_1 and D_2 form a partition of $V(G)$. Suppose, to the contrary, that $V(G) \setminus (D_1 \cup D_2) \neq \emptyset$. Then, for every $v \in V(G) \setminus (D_1 \cup D_2)$, the pair $(D_1 \cup \{v\}, D_2)$ is a $(2,2)$-pair in $G - E_G(v, D_1)$ (and $(D_1, D_2 \cup \{v\})$ is a $(2,2)$-pair in $G - E_G(v, D_2)$), a contradiction to the minimality of G. From the minimality of G it also follows that G is a bipartite graph in which the sets D_1 and D_2 form a bipartition, for if two vertices x and y belonging to D_1 (or D_2) were adjacent in G, then (D_1, D_2) would be a $(2,2)$-pair in $G - xy$, a contradiction. Finally, no two vertices of degree at least 3 are adjacent in G, for if a vertex $x \in D_1$ of degree at least 3 were adjacent to a vertex $y \in D_2$ of degree at least 3, then (D_1, D_2) would be a $(2,2)$-pair in $G - xy$. From this and from the fact that $\delta(G) \geq 2$, it follows that every edge of G is incident with a vertex of degree 2 in G (and, therefore, $\delta(G) = 2$).

Assume now that G is a bipartite graph with partite sets A and B, in which $\delta(G) \geq 2$ and every edge of G is incident with a vertex of degree 2 in G. Then (A, B) is a $(2,2)$-pair in G and therefore G is a $(2,2)$-dominated graph. Now, if e is an edge of G, then $G - e$ has a vertex of degree 1 (since e is incident with a vertex of degree 2) and therefore $G - e$ is not a $(2,2)$-dominated graph. Consequently, G is a minimal $(2,2)$-dominated graph. □

If H is a graph (with possible multi-edges or multi-loops), then the *subdivision graph* of H, denoted by $S(H)$, is the graph obtained from H by inserting a new vertex into each edge and each loop of H. We remark that the graphs F in Fig. 1, G in Fig. 2, and G in Fig. 3 are examples of subdivision graphs. We note that the subdivision graph $S(H)$ of H is a bipartite graph. On the other hand, we have the following useful observation.

Observation 2.5. *A connected graph G is a subdivision graph if and only if G is a connected bipartite graph with partite sets A and B such that at least one of them consists only of vertices of degree 2. Furthermore, a connected bipartite graph G with partite sets A and B such that $\delta(G) \geq 2$ and $|B| \geq |A|$ is a subdivision graph if and only if $d_G(x) = 2$ for every $x \in B$.*

We state the next two important corollaries of Theorem 2.4 that will prove very helpful to us. This corollary states that every minimal $(2,2)$-dominated graph (and therefore every $(2,2)$-dominated graph) can be constructed from a subdivision graph.

Corollary 2.6. *If a minimal* $(2,2)$*-dominated graph has multi-edges, then at least one of the vertices incident with them is of degree* 2.

Corollary 2.7. *If* H *is a graph with* $\delta(H) \geq 2$ *and with possible multi-edges or multi-loops, then its subdivision graph* $S(H)$ *is a minimal* $(2,2)$*-dominated graph.*

3 Constructive characterization of minimal $(2,2)$-dominated graphs

We remark that both graphs F and H in Fig. 1 are minimal $(2,2)$-dominated graphs. But only F is a subdivision graph. Thus, not every minimal $(2,2)$-dominated graph is a subdivision graph. Surprisingly, there are interesting connections between minimal $(2,2)$-dominated graphs and subdivision graphs. To prepare the ground for our explanation, let us begin with the following definition of a $\mathcal{P}$-contraction, which will play an important role in our considerations.

Let G be a bipartite graph. We define a vertex v as a *contractible vertex* of G if v is not incident with a multi-edge. Let v be a contractible vertex in G, and let $\mathcal{P}(v)$ be a partition of the neighborhood $N_G(v)$ of v. Recall that $N_G^2(v)$ is the set of vertices at distance 2 from v in G, while $N_G^2[v]$ is the set of vertices within distance 2 from v in G. Let $G' = G(\mathcal{P}(v))$ denote a graph in which

$$V(G') = (V(G) \setminus N_G(v)) \cup (\{v\} \times \mathcal{P}(v)),$$

and where

$$N_{G'}(u) = N_G(u) \ \text{ if } \ u \in V(G') \setminus N_G^2[v],$$

$$N_{G'}\big((v,S)\big) = N_G(S) \ \text{ if } \ (v,S) \in \{v\} \times \mathcal{P}(v),$$

$$N_{G'}(v) = \{v\} \times \mathcal{P}(v) \ \text{ and } \ |E_{G'}(v,(v,S))| = 1 \ \text{ for each } \ S \in \mathcal{P}(v),$$

and

$$N_{G'}(u) = \{(v,S)\colon S \in \mathcal{P}(v) \text{ and } N_G(u) \cap S \neq \emptyset\} \cup (N_G(u) \setminus N_G(v))$$

for every vertex $u \in N_G^2(v)$. Moreover, in this case when $u \in N_G^2(v)$ and $(v,S) \in N_{G'}(u)$, then $|E_{G'}(u,(v,S))| = |E_G(u,S)|$.

The graph $G(\mathcal{P}(v))$ is called a $\mathcal{P}$*-contraction* of G with respect to the partition $\mathcal{P}(v)$. To illustrate this construction, we present on the left side of Fig. 2 a graph G with a specified vertex v and a partition $\mathcal{P}(v) = \{S_1, S_2, S_3, S_4\}$ of the neighborhood $N_G(v)$ of v into four subsets indicated by ellipses. The graph on the right side of Fig. 2 is the associated $\mathcal{P}$-contraction $G(\mathcal{P}(v))$ of G with respect to the partition $\mathcal{P}(v)$.

The following observation follows readily from the definition of the $\mathcal{P}$-contraction of a graph.

Observation 3.1. *If* G *is a bipartite graph and* $\mathcal{P}(v) = \{S_1, \ldots, S_k\}$ *is a partition of the neighborhood* $N_G(v)$ *of a contractible vertex* v *of* G*, then the following properties hold in the* $\mathcal{P}$*-contraction* $G' = G(\mathcal{P}(v))$ *of* G.

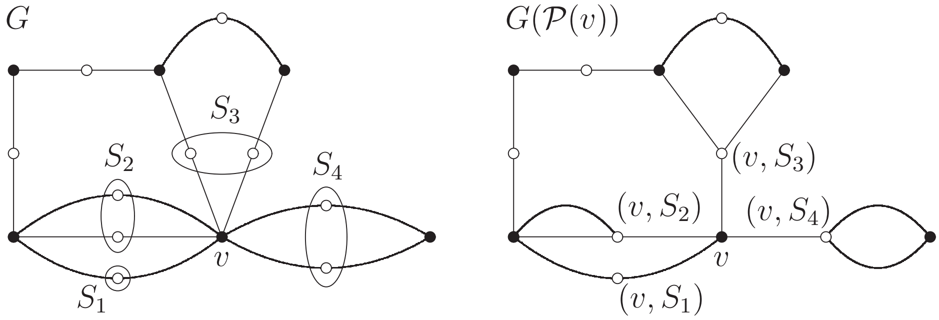

Figure 2: Graph G and its $\mathcal{P}$-contraction $G(\mathcal{P}(v))$

(a) *G' is a bipartite graph.*
(b) $d_{G'}(v) = |\mathcal{P}(v)| = k$.
(c) $d_{G'}((v, S_i)) = 1 + \sum_{u \in S_i} \big(d_G(u) - 1\big)$ *for every* $S_i \in \mathcal{P}(v)$.
(d) $d_{G'}(x) = d_G(x)$ *for every* $x \in V(G') \setminus N_G^2[v]$.
(e) *G' is isomorphic to G if $|\mathcal{P}(v)| = d_G(v)$, that is, if $\mathcal{P}(v)$ consists of singletons.*

We are interested in determining partitions $\mathcal{P}(v)$ of $N_G(v)$ which transform a minimal $(2, 2)$-dominated graph G into a minimal $(2, 2)$-dominated graph $G(\mathcal{P}(v))$. We begin with the following lemma.

Lemma 3.2. *Let G be a minimal $(2, 2)$-dominated graph, and let $\mathcal{P}(v)$ be a partition of $N_G(v)$ for some contractible vertex v of G, say $\mathcal{P}(v) = \{S_1, \dots, S_k\}$, where $1 \le |S_1| \le \cdots \le |S_k|$. Then the $\mathcal{P}$-contraction $G(\mathcal{P}(v))$ of G is a minimal $(2, 2)$-dominated graph if and only if at least one of the following two statements holds.*

(a) $k = |N_G(v)|$.
(b) $k = 2$ *and* $d_G(x) = 2$ *for every* $x \in N_G(S_i) \setminus \{v\}$ *if* $|S_i| \ge 2$ $(i \in \{1, 2\})$.

Proof. From the fact that G is a minimal $(2, 2)$-dominated graph and from Theorem 2.4 it follows that G is a bipartite graph, $\delta(G) = 2$, and every edge of G is incident with a vertex of degree 2. Let G' denote the $\mathcal{P}$-contraction $G(\mathcal{P}(v))$ of G, where v is a contractible vertex of G, $\mathcal{P}(v) = \{S_1, \dots, S_k\}$ is a partition of $N_G(v)$ and $1 \le |S_1| \le \cdots \le |S_k|$.

We shall show that G' is a minimal $(2, 2)$-dominated graph if and only if at least one of the statements (a) and (b) holds. Since the result is obvious if $k = |N_G(v)|$ (as in this case $\mathcal{P}(v) = \{x \colon x \in N_G(v)\}$ and G' is isomorphic to G, see Observation 3.1(e)), we may assume that $k < |N_G(v)|$. Then $k < |N_G(v)| = |S_1| + \ldots + |S_k| \le k|S_k|$ and therefore $|S_k| \ge 2$. In addition, it follows from Theorem 2.4 that neither the case $k = 1 < |N_G(v)|$ nor the case $3 \le k < |N_G(v)|$ is possible as otherwise either v is of degree 1 in G' or v and (v, S_k) are adjacent vertices of degree at least 3 in G'. Thus it remains to consider the case $k = 2 < |N_G(v)|$.

It follows from Observation 3.1 (a)–(d) that G' is a bipartite graph and $\delta(G') = 2$, because G is bipartite, $\delta(G) = 2$, and $k = 2$. Consequently, by Theorem 2.4, to prove that G' is a minimal $(2, 2)$-dominated graph, it suffices to show that every edge of G' is incident with a vertex of degree 2. Since the edges $v(v, S_1)$ and $v(v, S_2)$ are incident with v, which is of degree 2 in G', and every edge of G', which is not incident with (v, S_1) or (v, S_2), has inherited this property from the graph G, the graph G' is a minimal $(2, 2)$-dominated graph if and only if every edge of G' incident with (v, S_1) or (v, S_2) (and different from $v(v, S_1)$ and $v(v, S_2)$) is incident a vertex of degree 2 in G' (and in G). This property holds if and only if $|S_1| = 1$ and $d_G(x) = 2$ for every $x \in N_G(S_2) \setminus \{v\}$ or $2 \leq |S_1| \leq |S_2|$ and $d_G(x) = 2$ for every $x \in N_G(S_1 \cup S_2) \setminus \{v\}$, that is, if and only if $d_G(x) = 2$ for every $x \in N_G(S_i) \setminus \{v\}$ if $|S_i| \geq 2$ where $i \in \{1, 2\}$. This completes the proof. □

A 2-$\mathcal{P}$-contraction of a graph G is a $\mathcal{P}$-contraction $G(\mathcal{P}(v))$ of G with respect a partition $\mathcal{P}(v) = \{S_1, S_2\}$ of $N_G(v)$, where v is a contractible vertex of degree at least 3, and $d_G(x) = 2$ for every $x \in N_G(S_i) \setminus \{v\}$ if $|S_i| \geq 2$ ($i \in \{1, 2\}$). It follows from Lemma 3.2 that every 2-$\mathcal{P}$-contraction transforms a minimal $(2, 2)$-dominated graph into a minimal $(2, 2)$-dominated graph.

By $\mathcal{M}$ we denote the family of all connected minimal $(2, 2)$-dominated graphs. We are now in a position to present a constructive characterization of the family $\mathcal{M}$. For this purpose, let $\mathcal{F}$ be the family of graphs that:

(1) contains the subdivision graph $S(H)$ for every connected graph H with $\delta(H) \geq 2$ (and possibly with multi-edges and multi-loops); and

(2) is closed under 2-$\mathcal{P}$-contractions.

Examples of graphs $G = S(H)$, and 2-$\mathcal{P}$-contractions $F = G(\mathcal{P}(v))$, $S = F(\mathcal{P}(u))$, and $T = S(\mathcal{P}(w))$ belonging to the family $\mathcal{F}$ are given in Fig. 3.

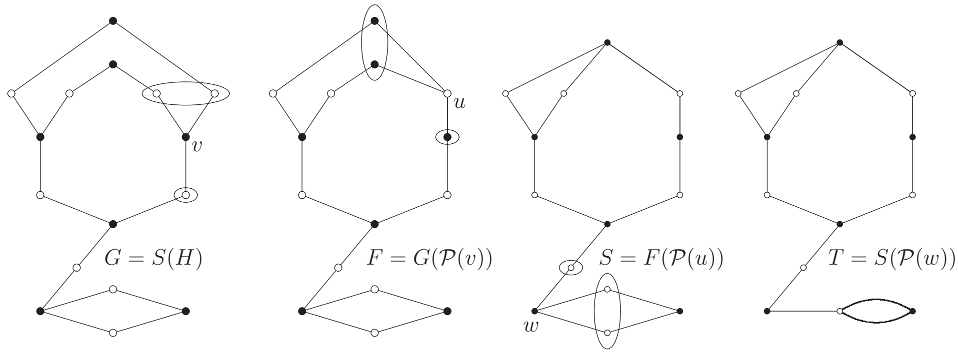

Figure 3: Graphs G, F, S, and T belonging to the family $\mathcal{F}$

The following theorem provides a constructive characterization of minimal $(2, 2)$-dominated graphs.

Theorem 3.3. *A connected graph G belongs to the family $\mathcal{M}$ if and only if G belongs to the family $\mathcal{F}$.*

Proof. It follows from Corollary 2.7 and Lemma 3.2 that $\mathcal{F} \subseteq \mathcal{M}$. Thus it remains to prove that $\mathcal{M} \subseteq \mathcal{F}$. Assume that G is a connected graph belonging to $\mathcal{M}$. By Theorem 2.4, G is a bipartite graph with $\delta(G) = 2$ such that every edge of G is incident with a vertex of degree 2 in G. Let A and B be the partite sets of G. Let

$$A_G^2 = \{x \in A \colon d_G(x) = 2\} \text{ and } A_G^3 = A \setminus A_G^2,$$

and

$$B_G^2 = \{x \in B \colon d_G(x) = 2\} \text{ and } B_G^3 = B \setminus B_G^2.$$

By induction on $k = \min\{|A_G^3|, |B_G^3|\}$ we will prove that $G \in \mathcal{F}$. If $k = 0$, then at least one of the sets A and B consists of vertices of degree 2, implying that G is a subdivision graph (by Observation 2.5) and proving that G belongs to $\mathcal{F}$. Thus, let k be a positive integer, and assume that $|A_G^3| \geq |B_G^3| = k$.

Among all vertices $u \in A_G^3$ and $v \in B_G^3$, let u and v be chosen to be at minimum distance apart in G, that is, $d_G(u, v) = \min\{d_G(x, y) \colon x \in A_G^3 \text{ and } y \in B_G^3\}$. Let $P \colon v = v_0, v_1, \ldots, v_\ell = u$ be a shortest (u, v)-path in G. Since u and v belong to different partite sets of G, we note that ℓ is odd. Further since G is a minimal $(2, 2)$-dominated graph, the set $A_G^3 \cup B_G^3$ is an independent set, implying that $\ell \geq 3$. By the choice of the path P, every internal vertex of the path P has degree 2 in G, while the vertices u and v are both of degree at least 3. Further from the minimality of G and by Theorem 2.4, every neighbor of v has degree 2 in G.

Without loss of generality we assume that the subset $N_G(v) \setminus \{v_1\}$ of $N_G(v)$ is the union of two disjoint sets $\{p_1, \ldots, p_m\}$ and $\{s_1, \ldots, s_n\}$, where each vertex p_i has degree 2 in G and is joined by a pair of parallel edges with v (say by edges e_i and e_i'), while each vertex s_j has degree 2 in G and is adjacent to a vertex, say s_j', different from v. We remark that possibly $s_i' = s_j'$ if $i \neq j$, and possibly one of the sets $\{p_1, \ldots, p_m\}$ and $\{s_1, \ldots, s_n\}$ is empty. Now let G' be a graph with vertex set $V(G') = (V(G) \setminus \{v\}) \cup V^*$, where

$$V^* = \{(v, e_1), (v, e_1'), \ldots, (v, e_m), (v, e_m')\} \cup \{(v, s_1), \ldots, (v, s_n)\},$$

and with edge set $E(G')$ obtained from $E(G)$ as follows:

- deleting all edges incident with v in G,
- adding an edge from v_1 to every vertex in the set V^*,
- adding an edge from p_i to both the vertices (v, e_i) and (v, e_i') for all $i \in [m]$, and
- adding an edge from s_i to the vertex (v, s_i) for all $i \in [n]$.

That is, defining

$$\begin{aligned} E_1 &= E(G) \setminus E_G(v), \\ E_2 &= \{v_1x \colon x \in V^*\}, \\ E_3 &= \{(v,e_i)p_i, (v,e_i')p_i \colon i \in [m]\}, \\ E_4 &= \{(v,s_i)s_i \colon i \in [n]\}, \end{aligned}$$

we have $E(G') = E_1 \cup E_2 \cup E_3 \cup E_4$. An illustration of the construction of the graph G' from the graph G is given in Fig. 4.

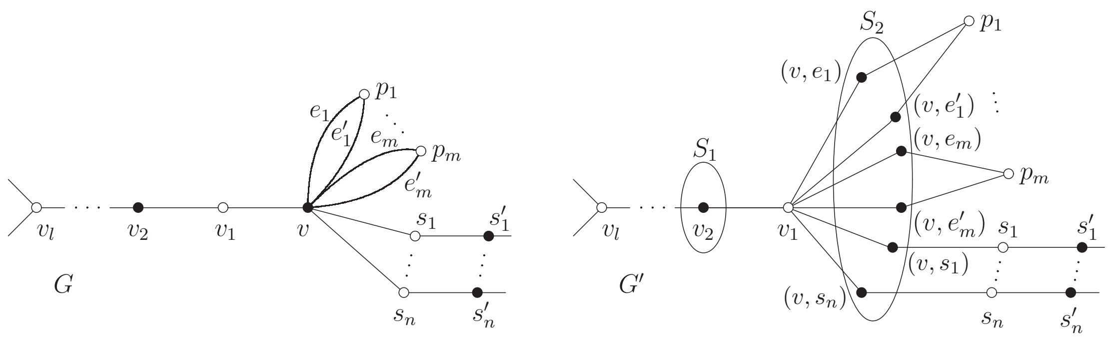

Figure 4: Graphs G and G' such that $G'(\mathcal{P}(v_1))$ is isomorphic to G

We note that every new vertex added to G when constructing G' has degree 2 in G'. Further, the degrees of all vertices in G different from v remain unchanged in G', except for the vertex v_1 whose degree changes from 2 to $1 + 2m + n$. It follows from the fact that G is a minimal $(2, 2)$-dominated graph and from Theorem 2.4 that G' is a minimal $(2, 2)$-dominated graph, that is, from the fact that $G \in \mathcal{M}$ it follows that $G' \in \mathcal{M}$. Recall that by assumption, we have $|A^3_G| \ge |B^3_G| = k$. Since $B^3_{G'} = B^3_G \setminus \{v\}$ and $A^3_{G'} = A^3_G \cup \{v_1\}$, we therefore have

$$\min\{|A^3_{G'}|, |B^3_{G'}|\} = \min\{|A^3_G| + 1, |B^3_G| - 1\} = |B^3_G| - 1 = k - 1 < k.$$

Applying the induction hypothesis to the graph G', we have that $G' \in \mathcal{F}$. Finally, if $\mathcal{P}(v_1) = \{S_1, S_2\}$ is a partition of $N_{G'}(v_1)$, where $S_1 = \{v_2\}$ and $S_2 = N_{G'}(v_1) \setminus \{v_2\} = V^*$, then the 2-$\mathcal{P}$-contraction $G'(\mathcal{P}(v_1))$ belongs to the family $\mathcal{F}$ (by Lemma 3.2). Consequently, the graph G belongs to $\mathcal{F}$ as G is isomorphic to $G'(\mathcal{P}(v_1))$. This completes the proof of Theorem 3.3. □

Acknowledgements

The authors wish to thank Dieter Rautenbach and the anonymous referees whose valuable comments improved the clarity of the paper.

References

[1] J. Bang-Jensen, S. Bessy, F. Havet and A. Yeo, Bipartite spanning sub(di)graphs induced by 2-partition, *J. Graph Theory* **92** (2019), 130–151.

[2] M. A. Henning, C. Löwenstein and D. Rautenbach, Remarks about disjoint dominating sets, *Discrete Math.* **309** (2009), 6451–6458.

[3] M. A. Henning, C. Löwenstein and D. Rautenbach, Partitioning a graph into a dominating set, a total dominating set, and something else, *Discuss. Math. Graph Theory* **30** (2010), 563–574.

[4] M. A. Henning, C. Löwenstein, D. Rautenbach and J. Southey, Disjoint dominating and total dominating sets in graphs, *Discrete Appl. Math.* **158** (2010), 1615–1623.

[5] M. A. Henning and I. Peterin, A characterization of graphs with disjoint total dominating sets, *Ars Math. Contemp.* **16** (2019), 359–375.

[6] M. A. Henning and D. F. Rall, On graphs with disjoint dominating and 2-dominating sets, *Discuss. Math. Graph Theory* **33** (2013), 139–146.

[7] M. A. Henning and A. Yeo, *Total Domination in Graphs*, Springer Monographs in Mathematics, Springer, 2013.

[8] C. Löwenstein and D. Rautenbach, Pairs of disjoint dominating sets and the minimum degree of graphs, *Graphs Combin.* **26** (2010), 407–424.

[9] M. Miotk, J. Topp and P. Żyliński, Disjoint dominating and 2-dominating sets in graphs, *Discrete Optim.* **35** (2020), 100553.

[10] M. Miotk and P. Żyliński, Spanning trees with disjoint dominating and 2-dominating sets, *Discuss. Math. Graph Theory* **42** (2022), 299–308.

[11] O. Ore, *Theory of Graphs*, Amer. Math. Soc. Colloq. Publ. **38**, Amer. Math. Soc., Providence, RI, 1962.

[12] J. Southey and M. A. Henning, A characterization of graphs with disjoint dominating and paired-dominating sets, *J. Comb. Optim.* **22** (2011), 217–234.

(Received 5 Nov 2021; revised 25 Mar 2022)