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Most generalized Petersen graphs of girth 8 have cop number 4*

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Abstract

A generalized Petersen graph $GP(n,k)$ is a regular cubic graph on $2n$ vertices (the parameter k is used to define some of the edges). It was previously shown (Ball et al., 2015) that the cop number of $GP(n,k)$ is at most 4, for all permissible values of n and k. In this paper we prove that the cop number of "most" generalized Petersen graphs is exactly 4. More precisely, we show that unless n and k fall into certain specified categories, then the cop number of $GP(n,k)$ is 4. The graphs to which our result applies all have girth 8.

In fact, our argument is slightly more general: we show that in any cubic graph of girth at least 8, unless there exist two cycles of length 8 whose intersection is a path of length 2, then the cop number of the graph is at least 4. Even more generally, in a graph of girth at least 9 and minimum valency δ, the cop number is at least $\delta + 1$.

1 Introduction

A robber is on the loose and you need to determine how many cops are needed to ensure his capture. Cops and robbers is a pursuit-evasion game played on graphs [14, 15] by two players on a simple graph G. The game starts with the cop player placing up to her allowed number of cops on her choice of vertices in G, followed by the robber placing his single (in the original version of the game, which is what we will be considering) pawn on his choice of vertex in G. Both players are fully aware of the structure of the graph and the positions of all the pawns, and take turns with the cop player having the first play. On their turn, the players may elect to have

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their pawn(s) remain static at the current position by passing play, or to move to one of its neighbouring vertices. The cop player may move all of her pawns in a single turn. Cops may also congregate at the same vertex, but this may not have any advantage during the course of the game. The game ends when a cop occupies the same vertex as the robber—the robber is thus captured, and the cop player wins. If the robber has a strategy that will enable him to evade the cops forever, then he wins. The question to consider is this: how many cops must be engaged in play to guarantee that the cops can win?

Definition 1.1. The *cop number* of a graph G, denoted $c(G)$, is the smallest positive integer k such that k cops suffice to capture the robber in a finite number of moves played on the graph G.

An interesting family of graphs to play the game on, and the family of graphs that we will be discussing throughout this paper, is the generalized Petersen graph family.

Definition 1.2. A *generalized Petersen graph* $GP(n, k)$ is a graph with vertex set

$$\{a_0, a_1, \ldots, a_{n-1}, b_0, b_1, \ldots, b_{n-1}\}$$

and edge set

$$\{a_i a_{i+1}, a_i b_i, b_i b_{i+k} : 0 \leq i \leq n-1\}$$

where subscripts are read modulo n, $n \geq 5$, and $k < n/2$.

The assumption that $k \neq n/2$ ensures that generalized Petersen graphs are always cubic. The fact that the same graph would be produced as $GP(n, k)$ and $GP(n, n-k)$ allows the assumption that $k < n/2$. Note that the Petersen graph is $GP(5, 2)$.

Definition 1.3. The *girth* of a graph is the length of a shortest cycle contained in the graph.

In this paper we will be focusing on generalized Peterson graphs with a girth of 8; we will explore the relationship between generalized Peterson graphs of girth 8 and their cop numbers.

For the purpose of this paper, we will separate the vertex set into two separate sets, A and B, which are the vertices labelled a_i in our definition (generally drawn as the outer cycle) and the vertices labelled b_i (generally drawn as the inner cycle), respectively.

In [2], Ball et al. showed that for any generalized Petersen graph $GP(n, k)$, the cop number is at most 4. The goal of this paper is to show that many generalized Petersen graphs of girth 8 have a cop number of exactly 4, and classify possible exceptions.

Since the parameters of generalized Petersen graphs of girth 8 are understood (see Table 1), our concluding result will be a reasonably short list of families of parameters that includes all generalized Petersen graphs (up to isomorphism) that do *not* have a cop number of 4. This list includes all generalized Petersen graphs that do not have a girth of 8. This result is presented in Corollary 5.3. What is known about the cop number of generalized Petersen graphs from their parameters is summarized in Table 3.

2 Previous research

In this study, we have applied a classic version of cops and robbers, whereby moves are limited and the location of the players are visible to their opponents with perfect information (as if there were two helicopters perched over the neighbourhood reporting relative locations to the escaped robber and the pursuing cops). Cops and robbers, however, is a game with many different versions all with different rules. Some variations include games played without perfect information [11], "lazy" cops and robbers [5], or traps [8].

In addition to varying rules of play, there are also varying types of graphs upon which the game can be played. We focus on generalized Peterson graphs in this paper, although our main result applies to any cubic graph of girth 8. In [2] the analysis was extended to I graphs. They proved that the cop number of a connected I-graph $I(n, k, j)$ is less than or equal to 5.

For generalized Petersen graphs, most research on the cop number has focussed on its relation to the parameters n and k, rather than focussing on the girth of the graph. However, the girth of a generalized Petersen graph is straightforward to determine from n and k. Most generalized Petersen graphs have girth 8, and therefore fall within the scope of this paper, but there are infinite families whose girth is g for each $3 \leq g \leq 7$. We summarize the information about this relationship in Table 1. In understanding this table, it is important to be aware of the isomorphism classes of generalized Petersen graphs.

Proposition 2.1 (Steimle and Staton, [16]). *Two generalized Petersen graphs* $GP(n, k)$ *and* $GP(n, \ell)$ *are isomorphic if and only if* $k = \ell$ *or* $k\ell \equiv \pm 1 \pmod{n}$*.*

In Table 1, only the smallest value of k in any given isomorphism class is listed. For example, when $n = 2k + 1$ the graph $GP(n, k)$ has girth 5, but this relation is not included in the table because the corresponding graph is isomorphic to $GP(n, 2)$. (Taking $\ell = 2$ and $n = 2k + 1$ gives $k\ell \equiv -1 \pmod{n}$.)

For some families that do not have girth 8, the cop number is already known, or tighter bounds have been found. For example, when $k = 1$, $GP(n, 1)$ has a girth of 4, and its cop number is known to be 2 [2]. Likewise, when $k = 3$ the girth is 6 (except for some small values of n) and the cop number is known to be at most 3 [2].

It is well-known that a graph has a cop number of 1 if and only if it has a *pitfall*, also known as a *corner* (see for example [6, pp. 30–33]). (A *pitfall* is a vertex whose neighbourhood is dominated by a neighbouring vertex.) No generalized Petersen graph contains a pitfall, so the cop number of any generalized Petersen graph is at least 2. In order for a generalized Petersen graph to have a cop number of 2, it must

Table 1: The graph $GP(n,k)$ has girth between 3 and 8. This table from [4, Theorem 5] gives the girth for $GP(n,k)$ as long as k has the smallest possible value in that isomorphism class (see Proposition 2.1 to determine the isomorphism class).

Girth 3	**Girth 4**	**Girth 5**	**Girth 6**	**Girth 7**	**Girth 8**
$n = 3k$	$n = 4k$	$n = 5k$	$n = 6k$	$n = 7k$	otherwise
	$k = 1$	$k = 2$	$k = 3$	$k = 4$	
		$n = 5k/2$	$n = 2k + 2$	$n = 7k/2$	
				$n = 7k/3$	
				$n = 2k + 3$	
				$n = 3k \pm 2$	

have girth 3 or 4, since for a cubic graph having girth 5 or more while having cop number 2 would contravene a bound of Aigner and Fromme [1, Theorem 3], which states that if $\delta(G) \geq d$ and the girth of G is at least 5 then $c(G) \geq d$. For the sake of completeness, we will give a brief explicit proof of this in our situation in Section 3. This means that when it is proved in [2] that the cop number is at most 3, it will actually be 3 unless the girth is 4 or less. So in fact, when $k = 3$ the cop number of $GP(n,k)$ is 3 unless $n \in \{9, 12\}$.

Although they do not mention this, the argument given in [2] for $k = 3$ also serves to show that the cop number is at most 3 when $k = 2$ (in which case the girth is 5 except for some small values of n). We present this result here.

Proposition 2.2. *The cop number of* $GP(n,2)$ *is at most* 3. *In fact, as long as the girth of this graph is* 5 *(that is, unless* $n \in \{6,8\}$*), the cop number of* $GP(n,2)$ *is exactly* 3.

Proof. This is very similar to the proof of [2, Theorem 5.1]. As it is not hard to prove following their strategy but involves very different techniques from the rest of this paper, we omit the details. ☐

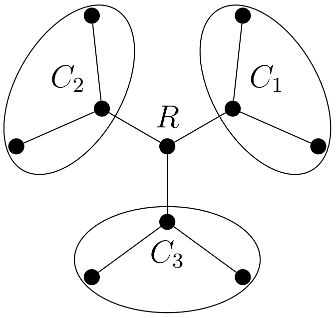
In Table 2, we list all generalized Petersen graphs with $n \leq 40$ whose cop number achieves the upper bound of 4 given in [2]. (Our table is based on the table in [2].) There are 60 such graphs, 57 of which have girth 8. Note that of these 60 graphs, only $GP(28,8)$, $GP(35,10)$, and $GP(35,15)$ have girth 7. We remark that two of the sets of parameters that appear in Table 2 did not appear in [2]; however, the source code created and used by the authors of [2] is freely available [3] and we used it to verify that these two sets of parameters do indeed produce graphs whose cop number is 4. The parameters missing from their paper are $n = 25$ with $k = 7$ and $n = 40$ with $k = 17$. A list of the cop number of every generalized Petersen graph with $n \leq 30$ is also given in [7], and their list also shows the cop number of $GP(25,7)$ being 4.

We have presented here only a very limited description of some of the extensive research relating to the game of cops and robbers, and to generalized Peterson graphs.

Table 2: $GP(n, k)$ with cop number 4, for $n \leq 40$ [2] [7]

n	k	girth	n	k	girth
25	7	8	34	6, 10, 13, 14	8, 8, 8, 8
26	10	8	35	6, 8, 10, 13, 15	8, 8, 7, 8, 7
27	6	8	36	8, 10, 14, 15	8, 8, 8, 8
28	6, 8	8, 7	37	6, 7, 8, 10, 11, 14, 16	8, 8, 8, 8, 8, 8, 8
29	8, 11, 12	8, 8, 8	38	6, 7, 8, 11, 14, 16	8, 8, 8, 8, 8, 8
31	7, 9, 12, 13	8, 8, 8, 8	39	6, 7, 9, 11, 15, 16, 17	8, 8, 8, 8, 8, 8, 8
32	6, 7, 9, 12	8, 8, 8, 8	40	6, 7, 9, 11, 12, 15, 17	8, 8, 8, 8, 8, 8, 8
33	6, 7, 9, 14	8, 8, 8, 8			

Figure 1: These cops have a mate in 2.



3 Cop number 2

Before we begin stating our results, we must define some terminology:

Definition 3.1. The cops have achieved a *mate in* 2 if for each edge incident to the robber there is a cop at distance no more than 2 away from the robber along a path that uses that edge.

A visual representation of this situation is shown in Figure 1, using the distance-2 subgraph of the robber's vertex. (In most of this paper, since we are assuming that the girth is 8, this subgraph is a tree.) As illustrated by the figure, if one cop is in each of the three branches, occupying any one of the three vertices within distance 2 of the robber, the robber will lose in at most two moves. (We are assuming that the cops play optimally. It will take two moves only if all cops all start at distance 2 and the robber passes).

So if the cops have not achieved a mate in 2, the robber has a legal move (other than passing) that does not result in him being caught on the cops' turn. Thus, if there is no configuration in which the cops can achieve a mate in 2, then the cops cannot win, since the robber can on each turn take the legal move that does not result in his being caught.

In order for two cops to be able to achieve a mate in 2, some of the vertices in Figure 1 must be identified. This means that the girth must be 4 or less. Putting this (and our previous observation that generalized Petersen graphs do not have pitfalls) together with the information from Table 1, we obtain the following result.

Proposition 3.2. *A generalized Petersen graph cannot have cop number* 1. *A generalized Petersen graph can have cop number* 2 *only if* $k = 1$, $n = 3k$, *or* $n = 4k$.

As previously mentioned, it has been observed that when $k = 1$ the cop number is indeed 2, and this is also true when $n = 3k$ or $n = 4k$ and $k \in \{2, 3\}$. However, when $k = 4$ and $n = 3k$ or $n = 4k$ the cop number is 3. All of these results are found in [2].

4 Achieving a mate in 2 with three cops

We can extend the definition of a mate in 2 to apply to the situation that occurs immediately before the mate in 2:

Definition 4.1. The cops have achieved a *mate in* 3 if the robber's best possible move (aside from passing) allows the cops to achieve a mate in 2 with their move. This can be generalized to a mate in n, for $n > 3$: the cops have achieved a mate in n if the robber's best possible move (aside from passing) allows the cops to achieve a mate in $n - 1$ with their move.

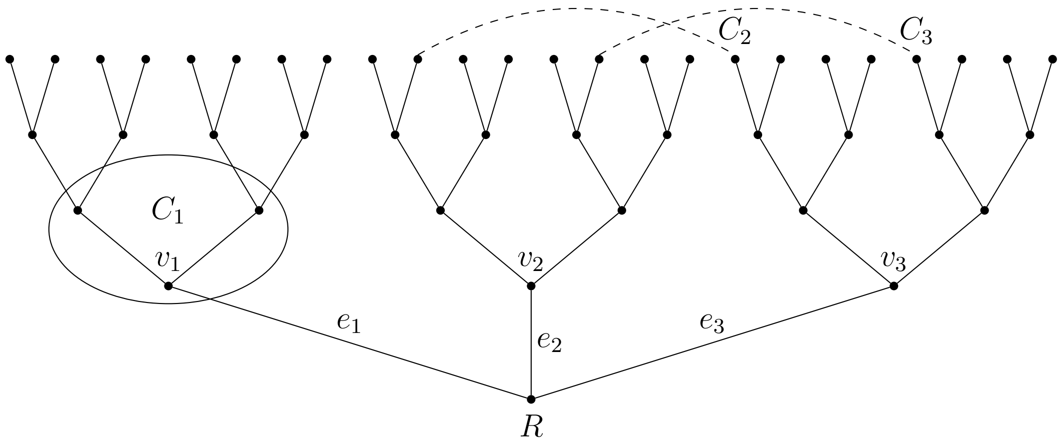
We ignore passing as one of the robber's options in this definition only because it complicates descriptions without affecting the underlying situation. As in our definition of a mate in 2, it may be possible for the robber (by passing) to remain in a mate in n position for one additional turn, before entering a mate in $n-1$ position.

With these definitions in hand, we state our key lemma. This will be the essential ingredient that we use to prove that many generalized Petersen graphs of girth 8 have cop number 4. We will analyse the positions the cops must be in relative to the robber, in order for three cops to achieve a mate in 3. Our strategy is to show that many generalized Petersen graphs have no subgraphs on which three cops can achieve a mate in 3; of course, this implies that three cops cannot win in such graphs. We note that the fact that some generalized Petersen graphs do contain subgraphs that theoretically allow three cops to achieve a mate in 3 does not necessarily imply that the cop number of these graphs is 3 (or less); it may not be possible for the cops to actually force the robber into a mate in 3 position. Full understanding of the cop numbers for graphs where such configurations exist is beyond the scope of this paper.

Lemma 4.2. *Suppose that we play cops and robbers on a cubic graph* G *of girth at least* 8. *Further suppose that we are playing with three cops.*

The only configuration in which the cops have achieved a mate in 3 *but have not achieved a mate in* 2 *can be described as follows (note that the graph must have girth* 8 *for this to be possible):*

Figure 2: These cops have achieved a mate in 3. (Vertices joined by dashed edges are identified, and C_1 is on any one of the encircled vertices)



- *two of the cops are sitting at points antipodal to the robber on cycles of length* 8*;*
- *these two cycles have as their intersection a path of length two consisting of two of the three edges incident with the robber's vertex; and*
- *the third cop is at distance* 1 *or* 2 *from the robber, on a path that includes the third edge incident with the robber's vertex.*

Before proving this lemma, we provide some more commentary.

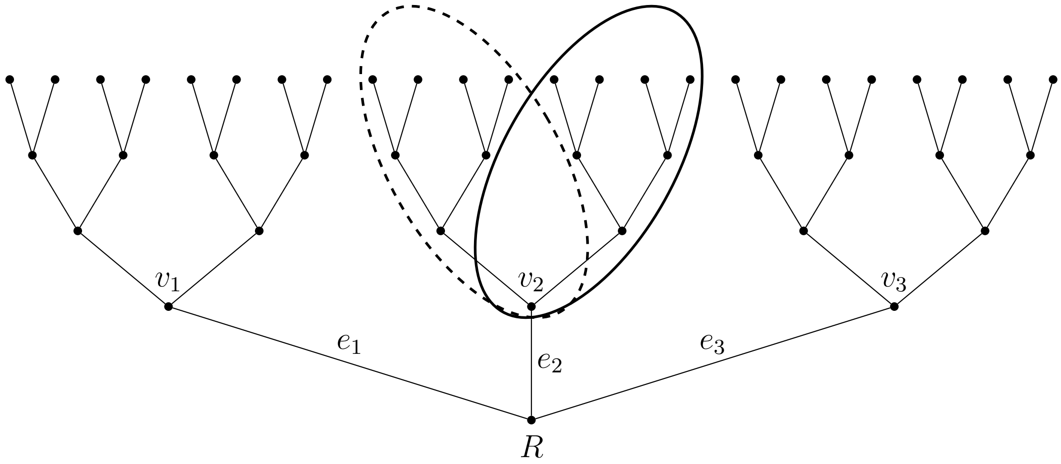
Figure 2 shows a configuration in which the cops have achieved a mate in 3. (Note that the dashed lines do not represent edges. Instead they signify that the two vertices on each end are actually the same vertex.) Cops C_2 and C_3 are each at the antipodal point of a cycle of length 8 from the robber, and the intersection of these two cycles is the path of length 2 consisting of the edges e_2 and e_3. (We do not rule out the possibility that some of the other vertices at distance 4 from the robber are also in fact identified.)

As you can see, the cops in Figure 2 have achieved a mate in 3. The robber cannot move to v_1 without being caught by cop C_1. His options if he moves to v_2 or v_3 are analogous to each other, so we will only discuss the case where he moves to v_2. The cop C_1 follows the robber, maintaining its current distance of 1 or 2. Cops C_2 and C_3 come down the middle branch of the tree toward v_2. Since they started at a distance of 4 from the robber and both are moving toward him (and he also moved toward them), at the end of their turn they are each at distance 2, one on the left branch from v_2 and the other on the right branch (while C_1 is on the third branch from v_2: the one that includes edge e_2), so the cops have achieved a mate in 2. (The robber can only afford to pass without being caught if cop C_1 begins at distance 2 from him. In this case, cop C_1 moves to v_1 and the other two cops remain in their current positions. On the robber's next turn, he must move to v_2 or v_3 to avoid being caught by cop C_1, and our previous analysis is unchanged.)

Proof of Lemma 4.2. The proof of this lemma is by case analysis of the relative "current" positions of the robber and the cops at any given point in the game when it is the robber's turn to move. We start by assuming that we are playing on a cubic graph of girth at least 8. We further assume that we are playing with three cops, and that in the "current" position they have not achieved a mate in 2. We will show that under these assumptions, the cops have also not achieved a mate in 3 unless their configuration is as described in the statement of this lemma.

Label the edges incident with the robber's current vertex as e_1, e_2, and e_3, and use v_1, v_2, and v_3 to denote the vertices at the opposite end of e_1, e_2, and e_3 (respectively) from the robber's current position. When we speak of a "branch from v_i" ($i \in \{1,2,3\}$) we mean the set of 8 vertices that include v_i itself, along with all of the vertices that are at distance 4 or less from the robber along a path that includes a fixed one of the two other neighbours of v_i. Figure 3 has each of the two branches from v_2 circled. Note that by this definition, v_i itself is on both of the branches from v_i.

Figure 3: Branches from v_2 (the left branch is in a dashed ellipse).



Recall that since the girth is at least 8, if a cop is at distance 4 or more from the robber, more than one of the vertices v_1, v_2, and v_3 may lie on a shortest path between the cop and the robber. Since the cops have not achieved a mate in 2, there is at least one vertex (v_1, v_2, or v_3) that does not lie on a shortest path of length 2 or less between the robber and some cop. The cases are as follows:

1. there is no cop within distance 2 of the robber's current position.
2. there is at least one vertex v_i ($i \in \{1,2,3\}$) that has at least one cop-free branch, and any cop on a branch from v_i is not at distance 2 or less from the robber.
3. every vertex v_i ($i \in \{1,2,3\}$) either has a cop at distance 2 or less from the robber on one of its branches (and this is the case for some v_i), or has no cop-free branch.

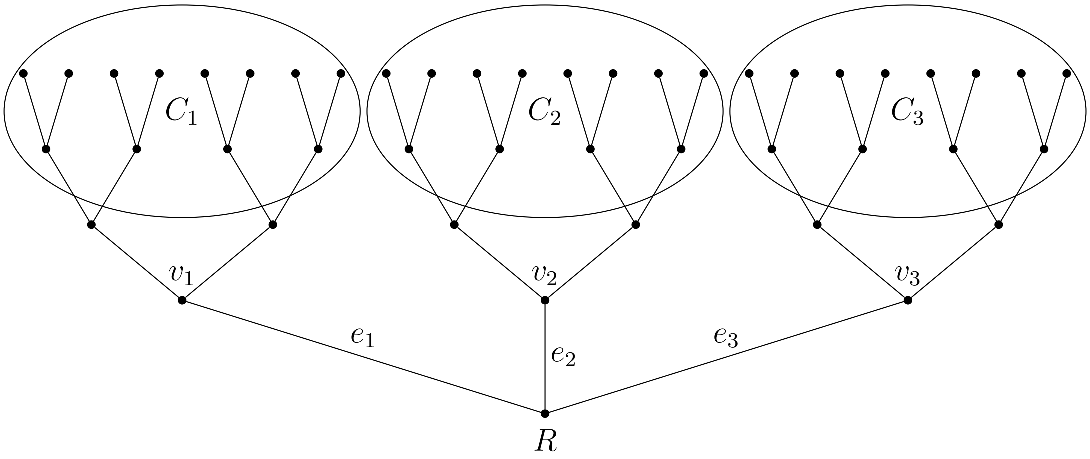
To begin with, let us explain briefly why these cases are exhaustive. If Case 1 does not apply, then there is at least one cop within distance 2 of the robber. Without loss of generality, let us suppose that this cop is on v_1 or one of its neighbours. If Case 3 does not apply, then there is some vertex v_j that has no cop at distance 2 or less from the robber and has a cop-free branch. So Case 2 applies to v_j.

This case analysis will be illustrated with figures. Keep in mind that if the girth is 8, some of the vertices at distance 4 from the robber in our figures may be the same as each other, but there are no other duplicated vertices in these illustrations.

Case 1. There is no cop within distance 2 of the robber's current position.

This situation is illustrated in Figure 4. This case does cover configurations that are not exactly like the illustration. For example, one cop might be sitting on a vertex that has paths of length 4 to the robber's position through both v_1 and v_2, or both of v_1's branches might be cop-free. However, no cop is on any uncircled vertex of the illustrated subgraph.

Figure 4: There is no cop within distance 2 of the robber's position.

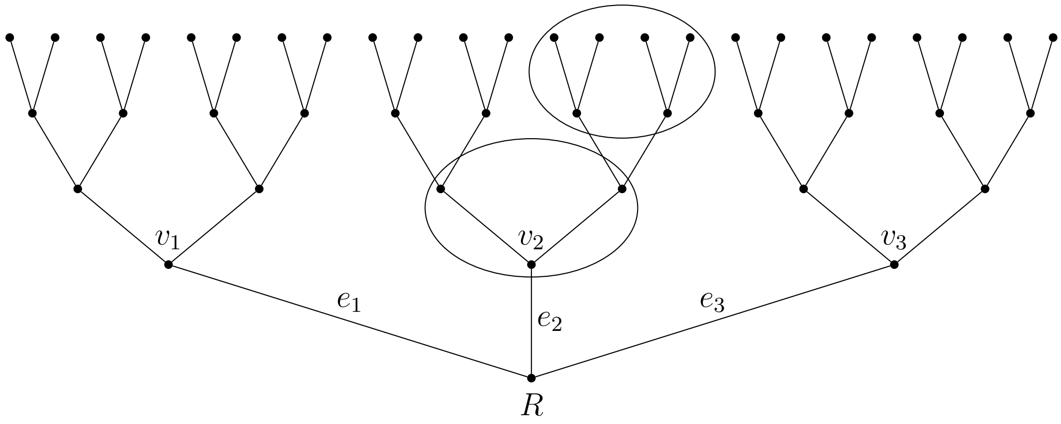


In Case 1, the robber may move to any of its adjacent vertices (v_1, v_2, or v_3). Without loss of generality, suppose that the robber chooses to move to vertex v_1. The cops' initial positions mean that none of them can use their move to reach the robber's initial vertex, or vertices v_2 or v_3. Therefore, the cops cannot achieve a mate in 2. This means that the cops had not achieved a mate in 3 previously.

Case 2. There is at least one vertex v_i ($i \in \{1, 2, 3\}$) that has at least one cop-free branch, and any cop on a branch from v_i is not at distance 2 or less from the robber.

In Figure 5 we have illustrated this case. Without loss of generality, suppose (as in the illustration) that vertex v_2 satisfies the hypothesis. Since there is no cop at distance 2 or less there is no cop on v_2 or its neighbours; since one of the branches from v_2 is cop-free, we may assume without loss of generality that if any cops are here at all, they are on the left-hand branch.

Figure 5: There is a cop-free branch from v_2, and any cops on a branch from v_2 are not at distance 2 or less (from the robber). So (assuming the right branch is cop-free) there are no cops in encircled areas.



In this case, the robber should move to vertex v_2. Since no cop was within distance 2 of the robber on either branch from v_2, no cop is in a position to capture the robber immediately on v_2. Since the right-hand branch from v_2 was cop-free, after the cops' move no cop can be within distance 2 of v_2 on this branch. Therefore the cops will not have achieved a mate in 2 after their move.

Once again, this means that the cops had not achieved a mate in 3 in the initial configuration.

CASE 3. EVERY VERTEX v_i ($i \in \{1, 2, 3\}$) EITHER HAS A COP AT DISTANCE 2 OR LESS FROM THE ROBBER ON ONE OF ITS BRANCHES (AND THIS IS THE CASE FOR SOME v_i), OR HAS NO COP-FREE BRANCH.

Without loss of generality, we may assume that cop C_1 is at distance 2 or less from the robber, being on vertex v_1 or one of its neighbours (other than the robber's vertex).

Suppose momentarily that there is a second cop at distance 2 or less from the robber on one of the branches from one of the other neighbours of the robber's vertex. Without loss of generality, we may assume that this neighbour is v_2. Since neither of these cops is on a branch of v_3, there cannot be a cop on each of the branches from v_3 unless that cop is also at distance 2 or less from the robber. This contradicts our hypothesis that the cops have not already achieved a mate in 2.

Thus, v_2 cannot have a cop-free branch, so there must be one cop on each of its branches (and neither of these cops is within distance 2 of the robber) and (similarly) one cop on each of the branches from v_3 (not within distance 2 of the robber). Due to our girth hypothesis, a single cop cannot be on both branches from a vertex v_i unless she is on vertex v_i (and thus within 2 of the robber). Since C_1 is not on either branch from v_2 or either branch from v_3, these four branches must be covered by cops C_2 and C_3. The only way a cop can be on a branch from v_2 and a branch from v_3 is for it to be sitting at distance 4 from the robber, on a vertex that is simultaneously on a branch from v_2 and a branch from v_3.

This means that each of C_2 and C_3 is at the antipodal vertex of a cycle of length 8 from the robber, and (since all four branches from v_2 and v_3 are involved in these 8-cycles) the intersection of these two cycles consists precisely of the edges e_2 and e_3. This is the configuration described in the statement of our lemma as achieving a mate in 3 and illustrated in Figure 2. ☐

This lemma allows us to prove our main result.

Theorem 4.3. *Let G be a cubic graph of girth at least 8. Unless G contains two cycles of length 8 whose intersection is a path of length 2 (with two edges and three vertices), we conclude that $c(G) \geq 4$: in particular, if G is a generalized Petersen graph, then this means $c(G) = 4$.*

Proof. Suppose that we play the game on G with 3 cops.

Choose any initial locations for the cops, and consider the set of vertices that are at distance 3 from C_1. Since the girth of G is at least 8 and the valency is 3, there are 12 such vertices, all distinct. At most two of these vertices can hold the other two cops. Therefore, there are at least 10 vertices the robber can choose for his initial position that are not occupied by any of the cops, and are not within distance 2 of cop C_1. Therefore the robber has choices for his initial move that do not leave him in a position where the cops have already achieved a mate in 2.

Using Lemma 4.2 inductively, we see that since the cops have not achieved a mate in 2, the robber can always prevent them from achieving a mate in 2, and can therefore win the game. (Our hypothesis about the structure of G together with Lemma 4.2 implies that the cops can never achieve a mate in 3, and can therefore never achieve a mate in 2.)

Therefore, $c(G) > 3$, so $c(G) \geq 4$.

By [2], if G is a generalized Petersen graph, then $c(G) \leq 4$, so $c(G) = 4$. ☐

Notice that this result does not necessarily imply that cubic graphs of girth 8 that do have cycles of length 8 whose intersection is a path of length 2 do actually have cop number 3 or less. All we have shown so far about such graphs is that they do admit placements for 3 cops and a robber in which the cops have achieved a mate in 3.

Just as three cops could never actually achieve a mate in 2 in other cubic graphs of girth 8, it is possible that three cops cannot actually achieve a mate in 3 in some (or all) families of cubic graphs in which a mate in 3 configuration exists. To determine this will require further understanding of the structure of these graphs.

In the next section, we undertake an analysis of which generalized Petersen graphs of girth 8 admit a mate in 3 configuration. Before doing so, we provide a generalization of Theorem 4.3 that may be of broader interest but uses very similar arguments. It is possible that this result is known, but we did not find it in our research into the literature on this problem. The most closely-related result seems to be Frankl's bound [9], showing that if $\delta(G) > d$ and the girth of G is at least $8t-3$ then $c(G) > d^t$.

However, for girth 9 we would only have $t = 1$, so Frankl's bound would only imply that $c(G) \geq \delta(G)$, not (as we will show) that $c(G) > \delta(G)$.

Theorem 4.4. *If G is a connected graph of minimum valency $\delta \geq 3$ and girth at least 9, then $c(G) > \delta$.*

Proof. We want to show that δ cops cannot capture the robber with optimal play on both sides, because the robber can always avoid entering a situation where the cops have achieved a mate in 2. So assume that we are playing the game with δ cops.

As in the proof of Theorem 4.3, since every vertex has valency at least δ and the girth is at least 9, there are at least $\delta(\delta-1)^2 \geq 4\delta$ vertices at distance 3 from the initial position of cop C_1, and the robber can choose any one of these that is not occupied by or adjacent to one of the other $\delta - 1$ cops. So the robber has numerous options for an initial position v that avoid a mate in 2, since it is not within distance 2 of cop C_1. (Every one of the δ or more neighbours of v must have a cop on either itself or one of its neighbours in order to achieve a mate in 2, which requires all δ cops, so C_1 needs to be involved in any mate in 2 configuration.)

Because we are assuming the girth is at least 9 (rather than 8), we can employ a simplified version of the proof of Lemma 4.2. Inductively assume that the cops have not yet achieved a mate in 2, and call the robber's current vertex v. Then there is some neighbour u of v such that no cop is on u or any of the neighbours of u. If it is possible to choose u so that no cop is at distance 4 or less from v on a path that passes through u, then we do so.

With the goal of contradicting our choice of u, suppose that there is more than one cop at distance 4 or less from v whose shortest path uses u. Since there are at least δ neighbours of v and only δ cops, and because the girth of G is at least 9, the fact that the shortest paths from two of the cops to v have distance 4 or less and pass through u means that there must be some other neighbour x of v such that none of the shortest paths from a cop to v that have distance 4 or less pass through x. But in this circumstance, our criteria for choosing should have led us to choose x rather than u, a contradiction.

So we may assume that at most one cop is at distance 4 or less from v along a shortest path that uses u. The robber should move to u. Since there was no cop on u or any of its neighbours, the cops cannot immediately capture the robber.

Since u has at least two neighbours other than v ($\delta \geq 3$), one of these neighbours (say y) is not used in the shortest path from any cop who was within distance 4 of v, to v. But this means that after the robber has moved to u, there will be no cop on y or any of its neighbours, so the cops have not achieved a mate in 2.

Thus, the robber always has a move that does not allow the cops to achieve a mate in 2 if they had not already achieved a mate in 2, so the cops cannot win. ☐

Figure 6: If v is in A, the labelling is as follows (where v is now labelled a_i).

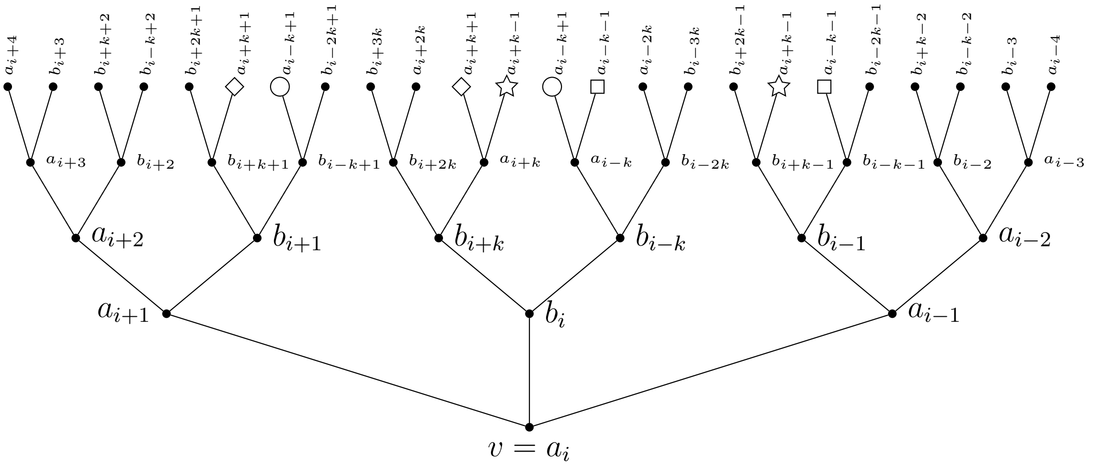
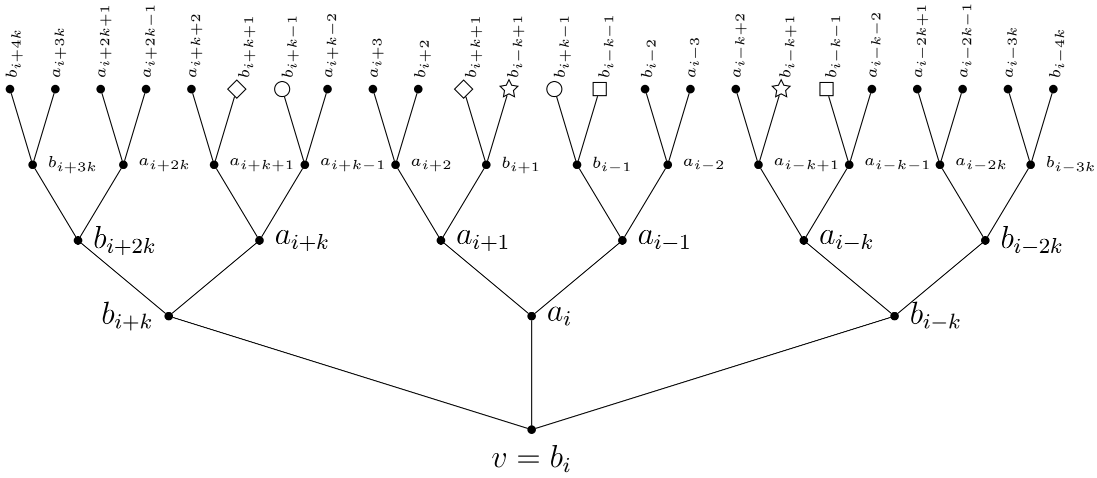


Figure 7: If v is in B, the labelling is as follows (where v is now labelled b_i).



5 Graphs of girth 8 in which three cops can achieve a mate in 3

In this section, we determine for which values of n and k the graph $GP(n,k)$ admits two cycles of length 8 whose intersection is a path of length 2. In other words, we are finding all the families of generalized Petersen graphs that admit a configuration in which it is theoretically possible for three cops to achieve a mate in 3.

Assuming the path of length 2 has a fixed vertex v as its central vertex, we see that we have two possible labellings for the collection of vertices that are within distance 4 of v, as v can be in either A (shown in Figure 6) or in B (shown in Figure 7).

Notice that some vertices at distance 4 from v are already the same as others. The nodes of identified vertices are illustrated by four pairs of matching white shapes in Figures 6 and 7. In any generalized Petersen graph, these yield 4 cycles of length 8 that include the given vertex v. In fact, if $v \in \{a_i, b_i\}$ then each of these cycles includes both a_i and b_i. However, no pair of these cycles has for their intersection a path of length 2 with v at its center (their pairwise intersections all have either 1 or 3 edges). So in order for a mate in 3 configuration to be possible, some of the other vertices at distance 4 from v must be identified.

We make a list of all possible values of n and k that *may* give us the structure we are looking for. These values were found by solving equations modulo n for each possible pair of vertices at distance 4 from v (without pairing vertices in A with vertices in B, as they could not possibly be the same). For example, we may have $b_{i-2k+1} = b_{i+2k-1}$, which occurs if $4k - 2 \equiv 0 \pmod{n}$. We ignore any solutions that do not meet the definition of a generalized Petersen graph (which requires that $n \geq 5$ and $k < n/2$).

The relationships between n and k that result in additional cycles of length 8 or less are as follows, when $v \in A$:

- $k = 1, 2, 3, 5$;
- $k = n - 3$ (which can only arise if $n = 5$ and $k = 2$);
- $k = n - 5$ (which can only arise if $n - 5 < n/2$, so $n \leq 9$);
- $n = 6, 8$;
- $n = 2k + i$ where $i \in \{1, 2, 4\}$;
- $n = 3k + i$ where $i \in \{0, \pm 1, \pm 3\}$;
- $n = 4k + i$ where $i \in \{0, \pm 2\}$;
- $n = 5k \pm 1$ or $n = (5k \pm 1)/2$; and
- $n = 6k$.

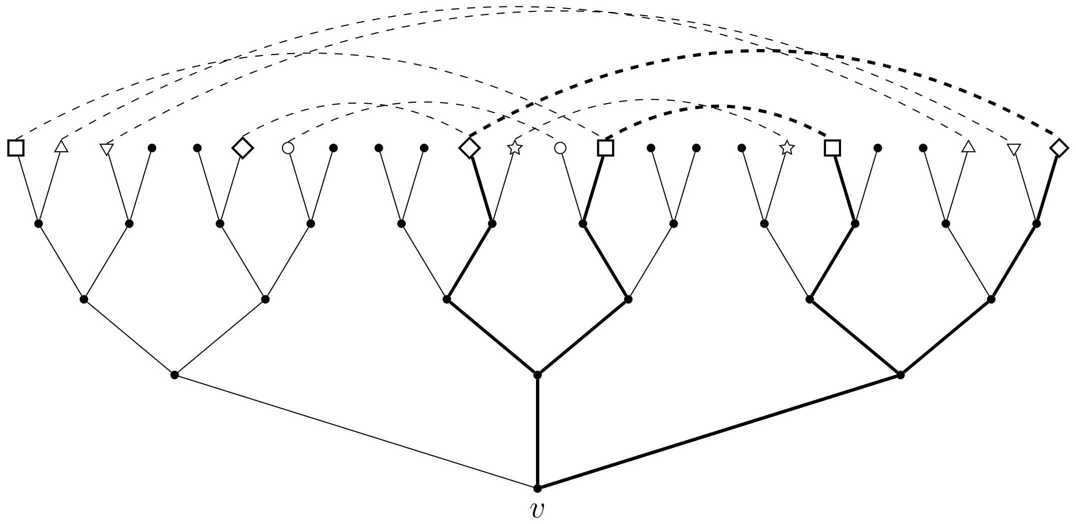
Comparing this to the values in Table 1 and using the smallest value of k from any isomorphism class (from the classes as given in Proposition 2.1), we see that of these relationships, only the following can arise in generalized Petersen graphs of girth 8:

- $k = 5$;
- $n = 2k + 4$;
- $n = 3k \pm 3$; and
- $n = 4k \pm 2$.

When $v \in B$, the exact same cases arise with the addition of one extra case where $n = 8k$.

Although $n = ak + b$ and $n = ak - b$ produce different graphs, the structures of their cycles of length 8 as shown in the distance 4 subgraph from any vertex are identical. Thus, we can illustrate the cycle structure of the cases $n = 3k \pm 3$ with a single figure, and likewise the cases $n = 4k \pm 2$. Additionally, the cycle structure of the $n = 3k \pm 3$ case is the same regardless of whether $v \in A$ or $v \in B$, and the cycle structure of $n = 2k + 4$ where $v \in A$ is the same as the cycle structure of $n = 4k \pm 2$ where $v \in B$ and vice versa, so these will also share figures in the following analysis.

Figure 8: $v \in A$ and $k = 5$



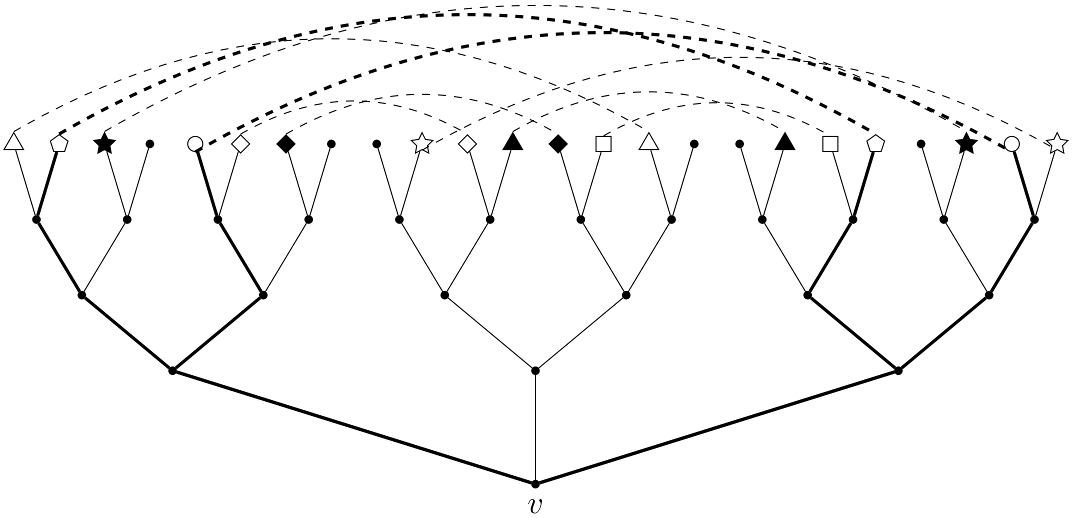
The first case we consider is $k = 5$ with $v \in A$. Figure 8 shows the structure of the subgraph of vertices at distance at most 4 from v. One pair of cycles of length 8 whose intersection is a path of length 2 have been thickened for clarity. In this graph, each of the three 2-paths centred at v serves as the intersection for two cycles of length 8. One of these cycles of length 8 can be found by extending the 2-path upwards from each end along a path of length 3, to the square-shaped vertex. The other can be found by extending the 2-path upwards from each end along a path of length 3, to the diamond-shaped vertex. Altogether this gives six cycles of length 8 (three involving the square vertex and three involving the diamond vertex) that come in pairs forming the structure that makes it possible for the cops to achieve a mate in 3.

When $k = 5$ and $v \in B$ we do not find two cycles of length 8 whose intersection is a path of length 2. However, this simply means that the cops cannot achieve a mate in 3 when the robber is on a B vertex in a graph with $k = 5$. Since we have just seen that the cops can achieve a mate in 3 when the robber is on an A vertex, this family of generalized Petersen graphs must be included as possible exceptions.

The next case to consider is $n = 2k + 4$ with $v \in A$, which is shown in Figure 9. As previously noted, this figure also covers the case $n = 4k \pm 2$ with $v \in B$. In this case, there are again three pairs of cycles of length 8 that intersect in paths of length 2 centred at v. One is to use the vertices indicated by the pentagon and the circle, another is to use the square and the white star. The third way is to use the vertices indicated by the white triangle and the white diamond. Again we have made the edges of the first pair of these 8-cycles bold.

When $n = 2k + 4$ and $v \in B$ or $n = 4k \pm 2$ and $v \in A$, the cycle structure is shown in Figure 10. There are two ways to make cycles of length 8 that intersect in paths of length 2. We can use the vertices indicated by the white triangle and the

Figure 9: Vertices within 4 of v when $v \in A$ and $n = 2k + 4$, and when $v \in B$ and $n = 4k \pm 2$



white diamond, or we can use the vertices indicated by the circle and the square. Again we have made the edges of the first pair of these 8-cycles bold.

Our next case is $n = 3k \pm 3$, shown in Figure 11. Whether v is in A or B makes no difference to the structure of the cycles. In this case there are three ways to make cycles of length 8 that intersect in paths of length 2. We can use the vertices indicated by the pentagon and the circle, or we can use the vertices indicated by the white triangle and diamond. Finally, we can use the vertices indicated by the white star and square. Again, only the first of these has been made bold.

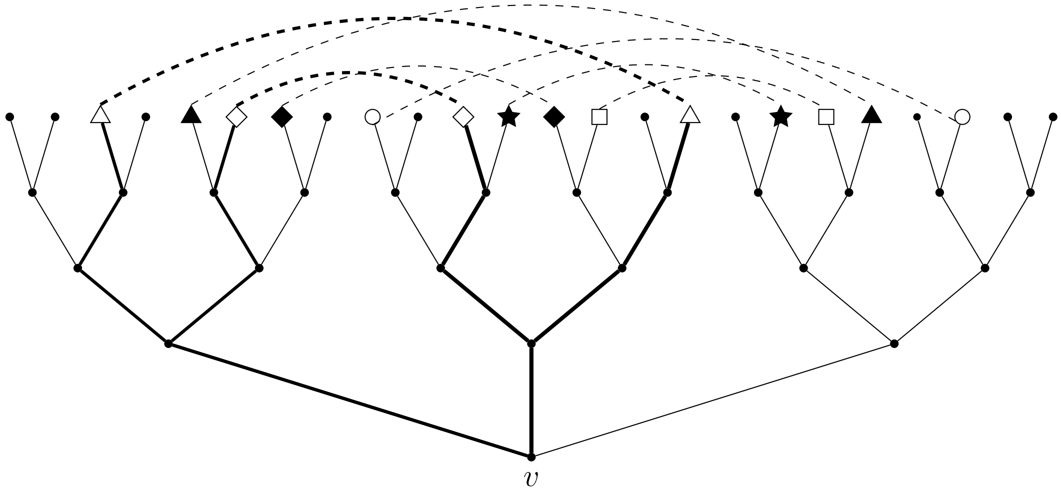
The final case to consider is the $n = 8k$ case, which only results in extra identified vertices at distance 4 from v if $v \in B$. However, this case does not result in any cycles of length 8 whose intersection is a path of length 2.

With the above analysis, we have proved the following lemma. Its corollaries follow immediately from combining this lemma with Theorem 4.3.

Lemma 5.1. *Suppose that $GP(n, k)$ has girth 8, and includes two cycles of length 8 whose intersection is a path of length 2. Then up to isomorphism, its parameters have one of the following forms:*

- $k = 5$;
- $n = 2k + 4$;
- $n = 3k \pm 3$; *or*
- $n = 4k \pm 2$.

Figure 10: Vertices within 4 of v when $v \in B$ and $n = 2k+4$, and when $v \in A$ and $n = 4k \pm 2$



Corollary 5.2. *Suppose that a generalized Petersen graph $GP(n,k)$ has girth 8 and is presented with k as small as possible in its isomorphism class. Then the cop number $c(GP(n,k)) = 4$ except possibly if one of the following is true:*

- $k = 5$;
- $n = 2k + 4$;
- $n = 3k \pm 3$; *or*
- $n = 4k \pm 2$.

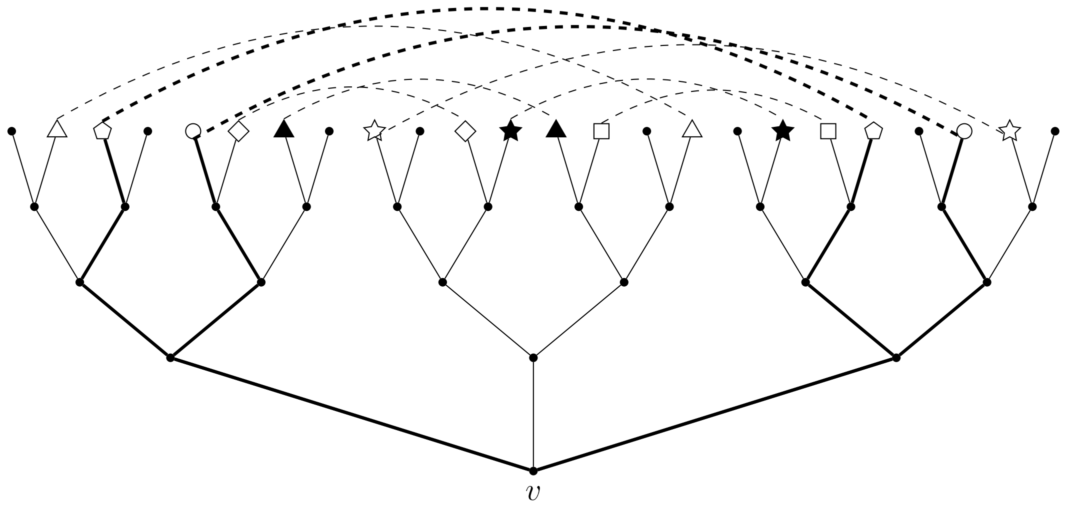
Using the information from Table 1, we can replace our hypothesis about the girth by adding additional parameters to the forbidden list.

Corollary 5.3. *Suppose that a generalized Petersen graph $GP(n,k)$ is presented with k as small as possible in its isomorphism class. Then the cop number $c(GP(n,k)) = 4$ except possibly if one of the following is true:*

- $k \in \{1,2,3,4,5\}$;
- $n = 2k + i$ *with* $i \in \{2,3,4\}$;
- $n = 3k + i$ *with* $i \in \{0,\pm 2,\pm 3\}$;
- $n = 4k + i$ *with* $i \in \{0,\pm 2\}$;
- $n = 5k/i$ *with* $i \in \{1,2\}$;
- $n = 6k$; *or*
- $n = 7k/i$ *with* $i \in \{1,2,3\}$.

Our results do not guarantee the cop number is 3 or less in any of these cases; indeed, from Table 2 we know that there are generalized Petersen graphs with girth 7 (which are therefore included in the above list) that also have cop number 4. All we really know (except where other results apply, such as the cop number when $k = 1$) about the cases that are listed as exceptional in Corollary 5.3 is that they either have

Figure 11: $n = 3k \pm 3$



girth less than 8, or they admit a configuration in which the cops could theoretically achieve a mate in 3.

For generalized Petersen graphs with $n \leq 40$, the cop numbers are known [2]. (Although a complete list does not appear in [2], the information can be completed using their algorithm [3].) We can also find the cop number for all generalized Petersen graphs with $n \leq 30$ in [7]. While it might be of interest to provide more information about all of these graphs, for the sake of brevity we mention only some highlights here. A table that includes all girths is included as an appendix in the arχiv version of this paper [13].

None of the graphs of girth 8 that appear in Table 2 have parameters that satisfy any of the conditions of Corollary 5.2. So it is possible (though the evidence is too limited to justify conjecturing it) that our result explains all graphs of girth 8 that have cop number 4.

The three graphs of girth 7 that have cop number 4 are precisely the graphs whose parameters have the form $n = 7k/i$ where $i \in \{2, 3\}$. It seems likely that this family of parameters results in different behaviour with respect to the game of cops and robbers, than the other families of parameters that give rise to graphs of girth 7. In fact, while this paper was in the refereeing process, it was proved that for $n \geq 42$ these graphs always have cop number 4 [12]. For completeness, these results have been added into Table 3.

As mentioned previously, a generalized Petersen graph cannot have cop number 1, and can only have cop number 2 if its girth is 3 or 4. Thus, if $n \notin \{3k, 4k\}$ and $k \neq 1$, then $c(GP(n, k)) \in \{3, 4\}$. All of the generalized Petersen graphs with $n \leq 40$ that have cop number 2 are listed in [2], although this is not clear from what they write. These graphs are every graph with $k = 1$ (this is a theoretical result proven in [2]), and the graphs with $n = 3k$ or $n = 4k$ for $k \in \{2, 3\}$. We conjecture that these are the only generalized Petersen graphs that have cop number 2.

In Table 3 we present what is known about the cop number of the generalized Petersen graph $GP(n,k)$ when $n > 40$ (we omit the small cases as these have been solved completely by computer and there are small exceptional situations that would make the table more complicated. All of these values were included in the appendix to the arχiv version of this paper, [13]). We let k be the smallest possible value such that $GP(n,k) \cong GP(n,\ell)$, so $k = \ell$ or $k\ell \equiv \pm 1 \pmod{n}$.

Table 3: The bounds on the cop number $c = c(GP(n,k))$ as long as $n > 40$ and k has the smallest possible value in that isomorphism class (see Proposition 2.1 to determine the isomorphism class). All of these bounds arise from results in [2, 12], or this paper.

$\boldsymbol{c = 2}$	$\boldsymbol{2 \le c \le 4}$	$\boldsymbol{c = 3}$	$\boldsymbol{3 \le c \le 4}$	$\boldsymbol{c = 4}$
$k = 1$	$n = 3k$	$k = 2$	$4 \le k \le 5$	otherwise
	$n = 4k$	$k = 3$	$n = 2k + i, i \in \{2, 3, 4\}$	
			$n = 3k + i, i \in \{\pm 2, \pm 3\}$	
			$n = 4k + i, i \in \{\pm 2\}$	
			$n = 5k/i, i \in \{1, 2\}$	
			$n = 6k$	

6 Conclusion

In this paper we focused on generalized Peterson graphs of girth 8. Given that such graphs can have a girth g with $3 \le g \le 8$, further research on cops and robbers played on generalized Peterson graphs of other girths is still needed.

In the introduction we looked at some of the previous work around cops and robbers on generalized Petersen graphs. Our paper attempts to fill the notable gap in research on the cop numbers of such graphs that have girth 8. More research, however, is indicated for girth 8 and for all of the other possible girths.

Problem 6.1. For each of the relationships between n and k listed in Corollary 5.3, what is $c(GP(n,k))$?

As previously stated, it has been shown that the specific relationships between n and k listed in Corollary 5.3 do not guarantee a cop number of 4. However, this in itself does not guarantee a cop number of 3, so any specific values for their cop numbers have not yet been proven.

In this study, we have applied a classic version of cops and robbers, whereby the players have perfect information. However, you could easily take away this perfect information and analyse how this changes the cop number. Different rules can significantly affect game play and outcomes.

Problem 6.2. How is the game of cops and robbers affected (in this context) if the robber doesn't have perfect information?

Our results rely heavily on all of the cops being able to move on any cop turn. In the *lazy cops* variant of the game, only one cop can move at a time. The "lazy cop number" is the number of cops required to guarantee that the cops can win with this rule variant.

Problem 6.3. What is the lazy cop number for various classes of generalized Petersen graphs? Which generalized Petersen graphs have lazy cop number 2?

In addition to varying rules of play, there are also varying types of graphs upon which the game can be played. We examined only generalized Petersen graphs in this paper; however, many different graph families could be analyzed for their cop numbers. While there has been a great deal of research along these lines, many families remain unexplored. For example, one could look at the cop numbers for a particular family of snark graphs.

Definition 6.4. A snark is a connected, bridgeless, simple, cubic graph whose edge-chromatic number is 4.

Since the Petersen graph is a snark, and our structural result applies to cubic graphs, such research would be closely related to this paper.

Definition 6.5. The flower snarks are an infinite family of snarks introduced by Rufus Isaacs [10].

Problem 6.6. What is the cop number of the flower snark J_n?

More research could also be done on cop numbers for I-graphs. The family of I-graphs is a generalization of the family of generalized Petersen graphs, in which the jumps on the "outer" cycle are also based on a parameter, rather than joining consecutive vertices. They are also cubic graphs that have girth at most 8 (although they are not necessarily connected). Our results certainly apply to some I-graphs, although we have not investigated this in any detail. However, the upper bound found in [2] for the cop number of these graphs was 5 rather than 4, and nothing in our arguments helps to distinguish whether a cop number is 4 or 5 in a cubic graph. The following question seems quite interesting:

Problem 6.7. When is the cop number of an I graph equal to 5?

Cops and robbers, in all its variations, is a fun and interesting game to play on different graphs. There is much more research to be done on this topic and many more facets to explore.

References

[1] M. Aigner and M. Fromme, A game of cops and robbers, *Discrete Appl. Math.* **8** (1984), 1–12.

[2] T. Ball, R. Bell, J. Guzman, M. Hanson-Colvin and N. Schonsheck, On the cop number of generalized Petersen graphs, *Discrete Math.* **340** (2017), 1381–1388.

[3] `http://users.math.msu.edu/users/robertbe/genPetersencode.html`.

[4] M. Boben, T. Pisanski and A. Žitnik, I-graphs and the corresponding configurations, *J. Combin. Des.* **13** (2005), 406–424.

[5] A. Bonato, Conjectures on Cops and Robbers, *Graph Theory. Problem Books in Mathematics* Springer, 2016, 31–42.

[6] A. Bonato and R. J. Nowakowski, *The Game of Cops and Robbers on Graphs* Amer. Math. Soc., Providence, RI, 1971.

[7] A. C. Burgess, R. A. Cameron, N. E. Clarke, P. Danziger, S. Finbow, C. W. Jones and D. A. Pike, Cops that Surround a Robber, *Discrete Appl. Math.* **285** (2020), 552–566.

[8] N. E. Clarke and R. J. Nowakowski, Cops, robber and traps, *Utilitas Math.* **60** (2001), 91–98.

[9] P. Frankl, Cops and robbers in graphs with large girth and Cayley graphs, *Discrete Appl. Math.* **17** (1987), 301–305.

[10] R. Isaacs, Infinite Families of Nontrivial Trivalent Graphs Which Are Not Tait Colorable, *Amer. Math. Monthly* **82** (1975), 221–239.

[11] V. Isler and N. Karnad, The role of information in the cop-robber game, *Theor. Comp. Sci.* **399.3** (2008), 179–190.

[12] H. Morris and J. Morris, On generalised Petersen graphs of girth 7 that have cop number 4, *Art of Discrete and Appl. Math.* **5** (2022), #P2.05.

[13] J. Morris, T. Runte and A. Skelton, Most generalized Petersen graphs of girth 8 have cop number 4, *preprint*, `http://arxiv.org/abs/2009.00693`.

[14] R. J. Nowakowski and P. Winkler, Vertex to vertex pursuit in a graph, *Discrete Math.* **43** (1983), 23–29.

[15] A. Quilliot, Thèse d'État, Université de Paris VI (1983).

[16] A. Steimle and W. Staton, The isomorphism classes of the generalized Petersen graphs, *Discrete Math.* **309** (2009), 231–237.

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