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Jump sizes for polygonal balancing numbers

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Abstract

Polygonal balancing numbers are generalizations of the ordinary triangular balancing numbers introduced by Behera and Panda [*Fib. Quart.* 37 (1999), 98–105]. An ordinary balancing number corresponds in a natural way to a solution in positive integers to the Pell Equation $x^2 - 2y^2 = 1$ and vice versa. Given a fundamental solution of the Pell Equation in positive integers, one can produce all other solutions by multiplication by a unit in $\mathbb{Z}[\sqrt{2}]$. A corresponding transformation on the sequence of ordinary balancing numbers produces all terms in the sequence.

For s-agonal balancing numbers, the companion equation is Pell-like. All solutions of the companion equation can be generated by multiplying by units of $\mathbb{Z}[\sqrt{2}]$, but it is no longer true that every solution in positive integers to the companion equation corresponds to an s-agonal balancing number. The general s-agonal balancing number must satisfy additional congruence conditions. The corresponding transformation on s-agonal balancing numbers must be applied a certain number $j(t)$ times, where $t = s - 2$. This integer $j(t)$ is the jump size. For an odd prime p we prove that if $p \equiv \pm 1$ modulo 8, then $j(p)$ divides $(p-1)/2$, and if $p \equiv \pm 3$ modulo 8, then $j(p)$ divides $p + 1$.

1 Introduction

Behera and Panda [4] defined a balancing number as an integer B such that

$$1 + 2 + \cdots + (B - 1) = (B + 1) + (B + 2) + \cdots + M,$$

for some M, which we call a counterbalancing number. For example, since $1+\cdots+5 = 7+8$, we see that $(B, M) = (6, 8)$ is a balancing, counterbalancing number pair. This is equivalent to the Diophantine equation involving triangular numbers given by

$$\frac{(B-1)B}{2} + \frac{B(B+1)}{2} = \frac{M(M+1)}{2}, \text{ or } B^2 = \frac{M(M+1)}{2}. \tag{1}$$

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Solving equation (1) for the positive counterbalancing number M, we have that

$$M = \frac{-1 + \sqrt{8B^2 + 1}}{2}. \tag{2}$$

Thus, $8B^2+1$ must be an odd, perfect square for B to be a balancing number. Panda and Behera [4] also found that a particular function and its inverse,

$$f(x) = 3x + \sqrt{8x^2 + 1} \text{ and } f^{-1}(x) = 3x - \sqrt{8x^2 + 1},$$

provide the next and previous balancing numbers from a given one:

$$B_{k+1} = f(B_k) = 3B_k + \sqrt{8B_k^2 + 1} \text{ and } B_{k-1} = f^{-1}(B_k) = 3B_k - \sqrt{8B_k^2 + 1}. \tag{3}$$

From equations (3), they deduced the fundamental recurrence relation

$$B_{k+1} = 6B_k - B_{k-1}, \text{ for } k \geq 2 \text{ with } B_0 = 0 \text{ and } B_1 = 1.$$

Panda [10] introduced the Lucas-balancing numbers, defined by

$$C_k = \sqrt{8B_k^2 + 1}, \text{ with } C_0 = 1 \text{ and } C_1 = 3, \tag{4}$$

as an analog of the Lucas numbers relative to the Fibonacci numbers. From equation (2), we see that $C_k = 2M_k + 1$. Thus, rather than basing an analysis of balancing numbers on B and C we may use M rather than C if it is to our advantage.

Using definition (4) and that of the function $f(x)$, we see that

$$B_{k+1} = 3B_k + C_k. \tag{5}$$

Although Liptai [8] does not explicitly refer to the Lucas-balancing numbers, he showed that the balancing numbers and Lucas-balancing numbers satisfy the Pell equation

$$C^2 - 8B^2 = 1. \tag{6}$$

The equations, functions, and recursions from (1) to (6) provide the foundations for subsequent studies of balancing numbers and Lucas-balancing numbers, and their properties.

Balancing numbers have been generalized in numerous ways [1–3, 6–14]. Especially in [3] the problem was generalized from balancing the triangular equation to balancing sequences for polygonal numbers of any number of sides $s \geq 3$. The ordinary balancing numbers correspond to the case $s = 3$.

The s-agonal balancing equation is

$$P(s, b-1) + P(s, b) = P(s, m), \tag{7}$$

where $P(s, n)$, the s-sided polygonal number of order n, is given by the well-known formula

$$P(s, n) = \sum_{i=1}^{n} \big((s-2)i - (s-3)\big) = \frac{1}{2}n\big((s-2)n - (s-4)\big). \tag{8}$$

Now we balance the sequence $\big((s-2)i-(s-3)\big)_{i\geq 1}$ at an index b to obtain the s-agonal-balancing equation (7). For b satisfying the s-agonal-balancing equation (7), we call b an s-agonal-balancing index with corresponding s-agonal-balancing number given by $B=(s-2)b-(s-3)$. Additionally, the s-agonal-counterbalancing index m corresponds to the s-agonal-counterbalancing number $M=(s-2)m-(s-3)$.

Substituting the expression for $P(s,n)$ from formula (8) into the s-agonal-balancing equation (7), we obtain

$$\frac{b-1}{2}\big((s-2)(b-1)-(s-4)\big)+\frac{b}{2}\big((s-2)b-(s-4)\big)=\frac{m}{2}\big((s-2)m-(s-4)\big).$$

Making the substitutions $b=(B+(s-3))/(s-2)$ and $b-1=(B-1)/(s-2)$ and simplifying, we have

$$\frac{1}{(s-2)}\left(2B^2+2(s-3)\right)=(s-2)m^2-(s-4)m.$$

Solving this quadratic equation for m, and selecting the positive root to ensure that m is positive, we obtain an explicit relationship between m and B given by

$$m=\frac{(s-4)+\sqrt{8B^2+s^2-8}}{2(s-2)}. \tag{9}$$

In order for m and B to be integers, equation (9) requires the following three conditions for B to be an s-agonal-balancing number:

$$\begin{aligned}&8B^2+s^2-8 \text{ is a perfect square,}\\ &(s-4)+\sqrt{8B^2+s^2-8} \text{ is divisible by } 2(s-2)\text{, and}\\ &B \text{ must be of the form } (s-2)b-(s-3).\end{aligned} \tag{10}$$

Since $C=\sqrt{8B^2+s^2-8}$, condition (10) may be expressed in the form

$$(s-4)+C \text{ is divisible by } 2(s-2).$$

For a given s-agonal-balancing number B and s-agonal-counterbalancing index m, we call C the corresponding Lucas-s-agonal-balancing number. Additionally, we see that there is a direct relationship between the s-agonal-counterbalancing number $M=(s-2)m-(s-3)$, its s-agonal-counterbalancing index m and the Lucas-s-agonal-balancing number C given by

$$m=\frac{(s-4)+C}{2(s-2)}=\frac{M+(s-3)}{(s-2)}. \tag{11}$$

Specifically, we have

$$C=2(s-2)m-(s-4) \text{ and } C=2M+(s-2).$$

We define extended s-agonal-balancing, Lucas-s-agonal-balancing and s-agonal-counterbalancing numbers by the three conditions.

Definition 1.1. For any $s \geq 3$, we say a triple (B, C, M) of integers, with $C > 0$ and $M > 0$ is an extended s-agonal-balancing triple if it satisfies the three conditions

$$C^2 - 8B^2 = s^2 - 8,$$

$$C \equiv -(s-4) \equiv s \pmod{2(s-2)}, \tag{12}$$

$$B \equiv -(s-3) \equiv 1 \pmod{(s-2)}. \tag{13}$$

We note that, as a consequence of equation (11), equation (12) may be rewritten in terms of M as

$$M \equiv 1 \pmod{(s-2)}.$$

Both reflect the requirement that m be an integer. Importantly, this allows both congruence conditions (12) and (13) to be expressed using the same modulus, $s-2$.

Remark 1.2. Notice that when $s = 3$ these conditions are automatically satisfied.

For a specific $s \geq 3$, we use $B^{[s]}$ to denote an s-agonal-balancing number, $C^{[s]}$ for the corresponding Lucas-s-agonal-balancing number and $M^{[s]}$ will denote the corresponding s-agonal-counterbalancing number. For completeness, we denote by $b^{[s]}$ and $m^{[s]}$ the corresponding s-agonal-balancing index and the s-agonal-counterbalancing index, respectively.

In the next section we develop matrix versions of the transformations between solutions of the s-agonal companion equation and the corresponding s-agonal numbers. In the third section we prove theorems on the jump sizes for prime powers. In the last section we include some possibilities for further research.

2 Transformations between solutions

We proceed by solving the family $y^2 - 8x^2 = s^2 - 8$ of s-agonal Pell equations. As was shown in [3] the solutions of an individual s-agonal Pell equation directly involve the (ordinary) balancing numbers and Lucas-balancing numbers.

For any $s \geq 3$ the *trivial solution* to the s-agonal Pell equation is given by

$$x_0^{[s]} = 1 \text{ and } y_0^{[s]} = s.$$

As with ordinary balancing numbers we may define functions that map a solution of the general s-agonal companion equation to another solution.

$$f(x;s) = 3x + \sqrt{8x^2 + (s^2 - 8)} \text{ and } g(y;s) = 3y + \sqrt{8y^2 - 8(s^2 - 8)}.$$

The inverses of f and g have corresponding forms [3]:

$$f^{-1}(x;s) = 3x - \sqrt{8x^2 + (s^2 - 8)} \text{ and } g^{-1}(y;s) = 3y - \sqrt{8y^2 - 8(s^2 - 8)}.$$

The forms for the k-fold compositions f^k and g^k directly involve the (ordinary) balancing numbers. By extending the recursions for the balancing numbers B_k and the Lucas-balancing numbers C_k to negative values of k the k-fold compositions of the inverses also have similar forms.

Theorem 2.1. *For $k \in \mathbb{N}$, the k-fold compositions of $f(x;s)$ and of $g(y;s)$ are given by*

$$f^k(x;s) = C_k x + B_k\sqrt{8x^2 + (s^2-8)} \text{ and } g^k(y;s) = C_k y + B_k\sqrt{8y^2 - 8(s^2-8)}.$$

Similarly, the k-fold compositions of $f^{-1}(x;s)$ and $g^{-1}(y;s)$ are

$$\begin{aligned} \left(f^{-1}\right)^k(x;s) &= C_k x - B_k\sqrt{8x^2 + (s^2-8)} \\ \text{and } \left(g^{-1}\right)^k(y;s) &= C_k y - B_k\sqrt{8y^2 - 8(s^2-8)}. \end{aligned}$$

The combined action of $f^k(x;s)$ and $g^k(y;s)$ may be represented by matrix multiplication.

Corollary 2.2. *For any solution (x,y) of the s-agonal Pell equation, and for any $k \in \mathbb{Z}$, we have*

$$\begin{aligned} f^k(x;s) &= \ C_k x + B_k y \\ g^k(y;s) &= 8B_k x + C_k y, \end{aligned}$$

where we have suppressed the apparent dependence of f^k on y, and of g^k on x. In matrix form, we see that

$$\begin{bmatrix} f^k(x;s) \\ g^k(y;s) \end{bmatrix} = \begin{bmatrix} C_k & B_k \\ 8B_k & C_k \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 8 & 3 \end{bmatrix}^k \begin{bmatrix} x \\ y \end{bmatrix}.$$

Since the matrix form for $[f^k(x;s), g^k(y;s)]^T$ is exactly the same as for the ordinary balancing generator functions $[f(x), g(y)]^T$, only the initial conditions distinguish solutions to the s-agonal Pell equations for different values of s.

Definition 2.3. We denote the *generator transformation* F defined by the pair of functions $f(x;s)$ and $g(y;s)$ by

$$F\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 3 & 1 \\ 8 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = F\begin{bmatrix} x \\ y \end{bmatrix}.$$

For s-agonal-balancing numbers, we are interested in solutions on the branch of the hyperbola $y^2 - 8x^2 = s^2 - 8$ with $y > 0$. However, because of the congruence restrictions, not every solution of the s-agonal Pell equation gives rise to an s-agonal balancing triple. There is however an integer j, the jump, which is a function of $t = s - 2$, so that the j-fold composition of F applied to a solution of the s-agonal Pell equation whose indices satisfy the congruence condition, gives another solution of the s-agonal Pell equation whose indices satisfy the congruence conditions.

Example 2.4. For the convenience of the reader we provide some tables of jump values for small powers of small primes. These are given in Tables 1 through 4.

Certainly the tables hint at patterns in the jump sizes for prime powers. However not every sequence of jump sizes for prime powers conforms. In particular $j(13) = j(169) = 14$ whereas one would expect $j(169) = 14 * 13^1$. Also $j(29) = 10$ although one would expect $j(29) = 30$.

s	4	6	10	18	34	66	130	258	514	1026
k	1	2	3	4	5	6	7	8	9	10
$t = 2^k$	2	4	8	16	32	64	128	256	512	2048
$j(t)$	1	2	4	8	16	32	64	128	256	512
$j(t)$	2^0	2^1	2^2	2^3	2^4	2^5	2^6	2^7	2^8	2^9

Table 1: Jump sizes for powers of 2

s	5	11	29	83	245	731	2189	6563	19685	59051
k	1	2	3	4	5	6	7	8	9	10
$t = 3^k$	3	9	27	81	243	729	2187	6561	19683	59049
$j(t)$	4	12	36	108	324	972	2916	8748	26244	78732
$j(t)$	4	$4 * 3^1$	$4 * 3^2$	$4 * 3^3$	$4 * 3^4$	$4 * 3^5$	$4 * 3^6$	$4 * 3^7$	$4 * 3^8$	$4 * 3^9$

Table 2: Jump sizes for powers of 3

s	7	27	127	627	3127	15627	78127	390627	1953127	9765627
k	1	2	3	4	5	6	7	8	9	10
$t = 5^k$	5	25	125	625	3125	15625	78125	390625	1953125	9765625
$j(t)$	6	30	150	750	3750	18750	93750	468750	2343750	11718750
$j(t)$	6	$6 * 5^1$	$6 * 5^2$	$6 * 5^3$	$6 * 5^4$	$6 * 5^5$	$6 * 5^6$	$6 * 5^7$	$6 * 5^8$	$6 * 5^9$

Table 3: Jump sizes for powers of 5

s	9	51	345	2403	16809	117651	823545	5764803	40353609	282475251
k	1	2	3	4	5	6	7	8	9	10
$t = 7^k$	7	49	343	2401	16807	117649	823543	5764801	40353607	282475249
$j(t)$	3	21	147	1029	7203	50421	352947	2470629	17294403	121060821
$j(t)$	3	$3 * 7^1$	$3 * 7^2$	$3 * 7^3$	$3 * 7^4$	$3 * 7^5$	$3 * 7^6$	$3 * 7^7$	$3 * 7^8$	$3 * 7^9$

Table 4: Jump sizes for powers of 7

3 Jump sizes for prime powers

Let (u, v) be a solution to the companion equation for s-agonal balancing numbers. Let $T : \mathbb{Z}^2 \longrightarrow \mathbb{Z}^2$ be the map sending $(a, b) \longrightarrow (3a+b, 8a+3b)$. We seek to find the jump size j which is the smallest positive integer so that $T^j(u, v) = (h, k)$ satisifies $u \equiv h \pmod{2(s-2)}$, and $v \equiv k \pmod{s-2}$. This value is a function of the prime factorization of t, where $t = s - 2$. Signify this as $j(t)$.

We will focus on the least integer $l(t)$ for which $T^{l(t)}(u, v) = (h, k)$ satisifies the congruences $u \equiv h \pmod{2(s-2)}$, and $v \equiv k \pmod{2(s-2)}$. As a price we can only deduce that $j(t)|l(t)$, since $v \equiv k \pmod{2(s-2)}$ implies $v \equiv k \pmod{s-2}$.

What we gain is that we can reframe our problem to compute the order of the matrix $A = \begin{bmatrix} 3 & 1 \\ 8 & 3 \end{bmatrix}$ which is an element of $SL(2, \mathbb{Z}_{2t})$ for any positive integer $t = (s-2)$.

Next write the unique prime factorization of t

$$t = 2^m \left(\prod_{p_i \equiv \pm 1 \pmod 8} p_i^{e_i} \right) \left(\prod_{q_j \equiv \pm 3 \pmod 8} q_j^{f_j} \right).$$

By the Chinese Remainder Theorem $l(t)$ is the least common multiple of the numbers $l(p^g)$ over all maximal prime power factors of $2t$.

3.1 The case $p = 2$

Theorem 3.1. *For all integers $n \geq 1$, $A^{2^n} \equiv \begin{bmatrix} 1 & 2^n \\ 0 & 1 \end{bmatrix} \pmod{2^{n+1}}$.*

Proof. We proceed by induction. When $n = 1$ we have $A^2 = \begin{bmatrix} 17 & 6 \\ 48 & 17 \end{bmatrix} \equiv \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ (mod 4).

Assume that $A^{2^n} \equiv \begin{bmatrix} 1 & 2^n \\ 0 & 1 \end{bmatrix} \pmod{2^{n+1}}$ for some $n \geq 1$, then

$A^{2^{n+1}} = (A^{2^n})^2 \equiv \begin{bmatrix} 1 & 2^n \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2^n \\ 0 & 1 \end{bmatrix} \pmod{2^{n+1}}$.

Thus modulo 2^{n+2}, we see that $A^{2^{n+1}}$ is equivalent to

$$\begin{bmatrix} 1 + a2^{n+1} & 2^n + c2^{n+1} \\ b2^{n+1} & 1 + d2^{n+1} \end{bmatrix} \begin{bmatrix} 1 + a2^{n+1} & 2^n + c2^{n+1} \\ b2^{n+1} & 1 + d2^{n+1} \end{bmatrix}$$

for some integers a, b, c, d. After matrix multiplication this is

$$\begin{bmatrix} 1 + 2^{n+2}[a + 2^{n-1}b + 2^n(a^2 + bc)] & 2^{n+1} + 2^{n+2}[2^{n-1}(a + d) + 2^n(ac + cd)] \\ 2^{n+2}[b + 2^n(ab + bd)] & 1 + 2^{n+2}[d + 2^{n-1}b + 2^n(bc + d^2)] \end{bmatrix}.$$

The result now follows when we apply 2^{n+2} as modulus. $\square$

$t = p$	$j(t)$	$(p-1)/2$	$t = p$	$j(p)$	$(p-1)/2$	$t = p$	$j(t)$	$(p-1)/2$
7	3	6	47	23	23	97	48	48
17	8	8	71	35	35	103	17	51
23	11	11	73	36	36	113	28	56
31	15	15	79	13	39	127	63	63
41	5	20	89	44	44	137	34	68

Table 5: Jump sizes for primes $p \equiv \pm 1 \pmod 8$

Corollary 3.2. *For all non-negative integers $l(2^n) = 2^n$.*

Proof. The preceeding theorem indicates that $l(2^n)|2^n$, and simultaneously that it cannot be a proper divisor. □

Corollary 3.3. *For all non-negative integers n, $j(2^n)|2^n$.*

Computationally the value of $l(2^n)$ is exactly twice the value of $j(2^n)$.

3.2 The case of a prime $p \equiv \pm 1 \pmod 8$

Theorem 3.4. *Let $p \equiv \pm 1 \pmod 8$ be a prime. Then $l(p)$ divides $(p-1)/2$.*

Proof. When p is an integral prime congruent to ± 1 modulo 8 the Legendre symbol $\left(\frac{2}{p}\right) = 1$. Thus there is an integer r so that $r^2 \equiv 2 \pmod p$. In this instance the matrix A is diagonalizable over the field $\mathbb{Z}_p$ with eigenvalues $\alpha = 3 + 2r$ and $\alpha^{-1} = 3 - 2r$ with corresponding eigenvectors the column vectors $\langle 1, 2r \rangle$ and $\langle 1, -2r \rangle$.

The diagonal matrix $B = \mathrm{diag}(\alpha, \alpha^{-1})$ has multiplicative order equal to $O_p(\alpha)$ which must be a divisor of $|\mathbb{Z}_p^*| = p - 1$. Since A is similar to B we have that $l(p)|(p-1)$. The small subtlety here is that $A \equiv I_2 \pmod 2$ so the value of $l(p)$ is determined by considering p as the only modulus.

Moreover, in $\mathbb{Z}_p$ we have $(1+r)^2 = 1 + 2r + r^2 \equiv 3 + 2r \pmod p$, so α is a square mod p and $l(p)$ divides $O_p(\alpha)$ which must divide $(p-1)/2$. □

Corollary 3.5. *For any integral prime p congruent to $\pm 1 \pmod 8$, $j(p)|(p-1)/2$.*

This agrees with numerical evidence as in Table 4 for values of $j(p)$ when $p \equiv \pm 1 \pmod 8$.

3.3 The case of a prime $q \equiv \pm 3 \pmod 8$

Theorem 3.6. *Let $q \equiv \pm 3 \pmod 8$ be a prime. Then $l(q)$ divides $(q + 1)$.*

Proof. When q is a prime congruent to ± 3 modulo 8 the Legendre symbol $\left(\frac{2}{q}\right) = -1$. Thus the polynomial $x^2 - 2$ is irreducible modulo q. Thus we can build $\mathbb{F} \cong GF(q^2) \cong$

$t = q$	$j(q)$	$q+1$	$t = q$	$j(q)$	$q+1$	$t = q$	$j(q)$	$q+1$
3	4	4	29	10	30	61	62	62
5	6	6	37	38	38	67	68	68
11	12	12	43	44	44	83	84	84
13	14	14	53	54	54	101	102	102
19	20	20	59	20	60	107	108	108

Table 6: Jump sizes for primes $q \equiv \pm 3 \pmod 8$

$\mathbb{Z}_q[x]/\langle x^2 - 2\rangle$. Now there is an element $r \in \mathbb{F}$ so that $r^2 \equiv 2 \pmod p$. In this instance the matrix A is diagonalizable over the field $\mathbb{F}$ with eigenvalues $\alpha = 3 + 2r$ and $\alpha^{-1} = 3 - 2r$ with corresponding eigenvectors the column vectors $\langle 1, 2r\rangle$ and $\langle 1, -2r\rangle$.

The diagonal matrix $B = \mathrm{diag}(\alpha, \alpha^{-1})$ has order in $SL(2, q^2)$ equal to $O(\alpha)$ which must be a divisor of $|\mathbb{F}^*| = (q^2 - 1)$. Let ϕ be the Frobenius automorphism for $\mathbb{F}$ where $\phi(z) = z^p$ for all $z \in \mathbb{F}$. We know that the fixed field of ϕ is the subfield of order q and that ϕ permutes the roots of $x^2 - 2$. Therefore

$$\phi(\alpha) = (3 + 2r)^q = 3^q + (2r)^q = 3 + 2^q r^q = 3 + 2(-r).$$

Consequently $\alpha^{q+1} = \alpha^q \alpha = (3 - 2r)(3 + 2r) = 1$. This means that the order of α divides $q + 1$. □

Corollary 3.7. *For any integral prime q congruent to $\pm 3 \pmod 8$, $j(q)|(q + 1)$.*

This agrees with numerical evidence in Table 5.

3.4 The case when t is an odd prime power

Theorem 3.8. *Let $p \equiv \pm 1 \pmod 8$ be a prime and let $t = p^e$ for some positive integer e. Then $l(t)$ divides $p^{e-1} l(p)$.*

Proof. If $l(p^e) = x$, then $A^{px} = (A^x)^p \equiv \begin{bmatrix} 1 + ap^e & bp^e \\ cp^e & 1 + dp^e \end{bmatrix}^p \pmod{p^{e+1}}$. But then

$$A^{px} \equiv \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + p^e \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right)^p \pmod{p^{e+1}}.$$

Since the identity matrix is central the binomial theorem applies and the Freshman's Dream (when a and b commute $(a + b)^n = a^n + b^n$) gives $l(p^{e+1})|px$.

By induction we have that $l(p^e)$ divides $p^{e-1} l(p)$. □

Corollary 3.9. *For any odd integer prime p and positive integer e, $j(p^e)|p^{e-1} l(p)$.*

4 Further research

In this short section we provide some directions for further research.

There are two generalizations of triangular balancing numbers which can be generalized to the polygonal case. First, in [2] the authors extended the definition of balancing number to upper k-gap balancing numbers. A second generalization is to add an integer weight w to the right hand side of the defining equation for balancing numbers. What are the jump sizes for the general upper k-gap polygonal balancing numbers, and/or the weighted polygonal balancing numbers?

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