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On sequences of P_n-line graphs

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Abstract

Let H be a connected graph of order at least 3 and G be a nonempty graph. The H-line graph of G, denoted by $HL(G)$, is that graph whose vertices are the edges of G and where two vertices of $HL(G)$ are adjacent if they are adjacent in G and lie in a common copy of H. For each nonnegative integer k, let $HL^k(G)$ denote the k-th iteration of the H-line graph of G. We say that the sequence $\{HL^k(G)\}$ converges if there exists a positive integer N such that $HL^k(G) \cong HL^{k+1}(G)$ for every $k \geq N$, and for $n \geq 3$ we set Λ_n as the set of all graphs G whose sequence $\{HL^k(G)\}$ converges when $H \cong P_n$. The sets Λ_3, Λ_4 and Λ_5 have been characterized. To progress towards the characterization of Λ_n in general, this paper defines and studies the following property: a graph G is minimally n-convergent if $G \in \Lambda_n$ but no proper subgraph of G is in Λ_n. In addition, we prove conditions that imply divergence, and use these results to develop some of the properties of minimally n-convergent graphs.

1 Introduction

In this paper all graphs are finite, simple, and undirected. If $S \subseteq V(G)$, then $G - S$ is used to denote the graph G where every vertex in S has been removed. Similarly, $G - v$ is used when $S = \{v\}$

Let H and G be graphs such that H is a connected graph of order at least 3, and G is a nonempty graph. Two edges e and f in a graph G are said to be H-adjacent if the edges are adjacent and lie in a common subgraph isomorphic to H. Define the H-line graph of G, or $HL(G)$, as that graph whose vertices are the edges of G and where two vertices of $HL(G)$ are adjacent if they are H-adjacent in G. Figure 1 shows an example of graphs G and $HL(G)$, where $H \cong P_5$. Notice that the edges e_1 and e_2 are adjacent and lie in a P_5 in G. By definition, it follows that e_1 and e_2, as vertices, are adjacent in $HL(G)$. On the other hand, edges e_2 and e_3 are adjacent in G but do not lie in any common P_5. This leads to e_2 and e_3, as vertices, not being adjacent in $HL(G)$.

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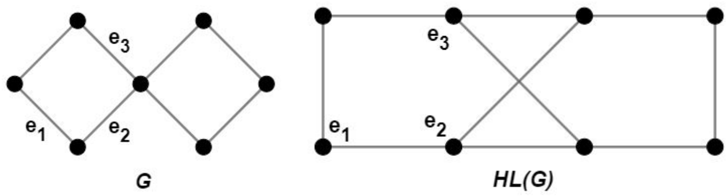

Figure 1: A graph and its H-line graph when $H \cong P_5$.

For $k \geq 0$, define $HL^{k+1}(G) = HL(HL^k(G))$ where $HL^0(G) = G$. The sequence $\{HL^k(G)\}$ is said to converge if there exists an integer N such that $HL^N(G) \cong HL^{N+1}(G)$. If the empty graph is part of the sequence, then the sequence is said to terminate. If the sequence does not converge nor terminate, then the sequence is said to diverge. Further, we call a graph F a limit graph if $F \cong HL(F)$. Figure 2 shows a graph G that is a limit graph when $H \cong P_{10}$. For convenience, if the sequence $\{HL^k(G)\}$ converges, we say that G is H-convergent, and similarly we say G is H-divergent when $\{HL^k(G)\}$ diverges.

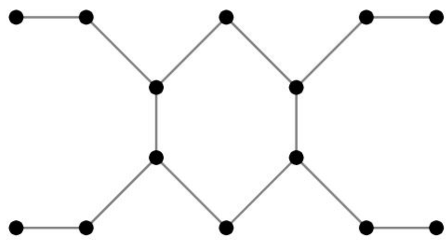

Figure 2: A limit graph with $H \cong P_{10}$.

Some special cases of H have been studied. The one studied the most by far is when $H \cong P_3$. If such is the case, then $HL(G) = L(G)$, the well-known line graph of G. With the exception of P_3, most of the research surrounding H-line graphs pertains to the characterization of H-convergent graphs. In [2], Chartrand et al. proved that no graph G is H-convergent when $H \cong K_{1,n}$ for $n \geq 3$ or when $H \cong K_n$ for $n \geq 4$. In [5], Jarrett proved that G is C_3-convergent if and only if C_3 is a subgraph of G. In [3], Chartrand et al. proved that if G has a subgraph isomorphic to C_4 but G contains no subgraph isomorphic to $K_1 + P_4$, $P_3 \times K_2$, $K_{2,3}$ or K_4, then G is C_4-convergent. However, a counterexample to the converse is also provided. Note that each of these four graphs have C_4 as a subgraph. This demonstrates that the result in [5] for C_3 does not generalize easily.

This paper will focus on the case when $H \cong P_n$ for $n \geq 4$ as the case in which $n = 3$ corresponds to the line graph. Define Λ_n as the set of all graphs G that are P_n-convergent. In [6], Chartrand proved that Λ_3 is composed of graphs whose components are cycles or $K_{1,3}$. In [2], Chartrand et al. proved that Λ_4 and Λ_5 are composed of graphs whose components are cycles of order at least 4 or 5, respectively, and the graphs in Figure 3. Characterizing Λ_n in general becomes harder as n

increases because new types of behaviour become possible. For example, Britto-Pacumio in [4] found and studied disconnected P_n-convergent graphs that had P_n-divergent components. See Figure 4 for an example found in [4]. If G' is a component of this graph, then $G' \not\cong HL(G')$ yet $G' \cong HL^2(G')$, so G' is P_n-divergent but $G = G' \cup HL(G')$ is P_n-convergent. This complex behaviour does not happen when $n = 4, 5$, and so the proofs that characterize Λ_4 and Λ_5 are difficult to replicate for a general n.

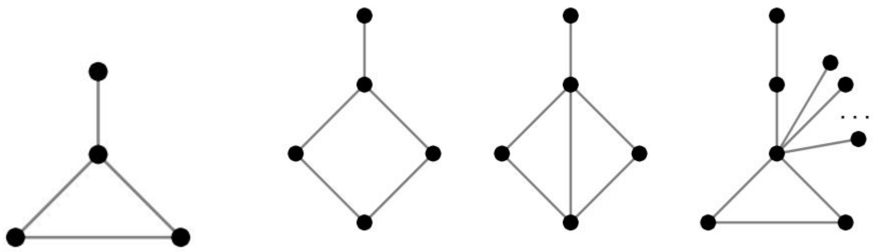

Figure 3: To the left, one graph in Λ_4, and to the right, three graphs in Λ_5.

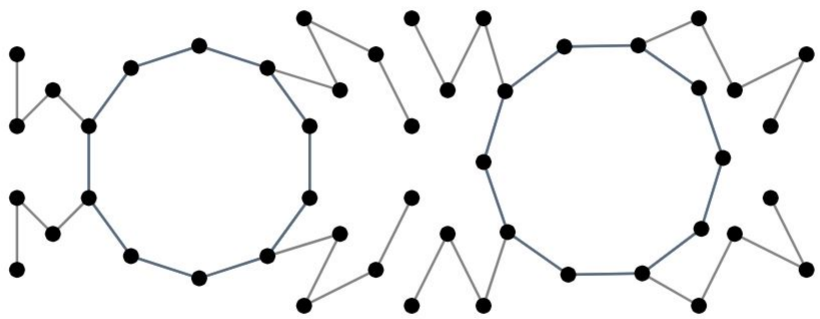

Figure 4: A disconnected limit graph when $H \cong P_{16}$.

To develop a route towards the characterization of Λ_n, we will study a subset of this set. We say that G is minimally n-convergent if $G \in \Lambda_n$ but no proper subgraph of G is in Λ_n. Further, let λ_n be the set of all graphs with this property. This definition is partially motivated by the graphs in Λ_5 shown in Figure 4 since requiring that a graph is minimally n-convergent eliminates the unnecessary structure. Minimal n-convergence, additionally, still captures the complex behavior shown in Figure 4 as this graph is minimally n-convergent.

The content of this paper is separated into two parts. The first one, Section 2, deals with conditions that imply divergence, along with a way to study this behaviour. The second part, Section 3, deals with the properties of minimal n-convergence with results proven by using the ones developed in Section 2. At the end of Section 3, we also provide a small summary of the ways in which the study of minimally n-convergent graphs can progress.

2 Conditions that imply divergence

Knowing the conditions that make a sequence diverge facilitates the study of sequences that do converge as they provide the properties that need to be avoided.

We start by categorizing divergence. Although not obvious at first, there are two kinds of divergence. The first is divergence by order. The sequence $\{HL^k(G)\}$ diverges by order if for every positive integer N, there exists an integer k such that $|V(HL^k(G))| \geq N$. Further, we say G is H-divergent by order if $\{HL^k(G)\}$ diverges by order.

The second kind of divergence is when the order is bounded yet the sequence of G does not converge. It is easy to generate graphs that are P_n-divergent by order. The other type of divergence is more difficult to obtain. In fact, the first paper on iterated H-line graph sequences, [2], conjectured that the second kind does not exist. Not much is known about the second kind of divergence, but we do know that it exists. As mentioned above, the connected graphs G such that $G \not\cong HL(G)$ but $G \cong HL^2(G)$ presented in [4] are an example of a graph with this kind of divergence. An important observation we will use in future proofs is that if G has a P_n-divergent by order subgraph, then G is P_n-divergent by order.

We start by covering two conditions that are known to cause divergence by order. For the first one we need to define a specific class of graphs. Let G_m^r for $r \geq 1$ and $m \geq 3$ be a unicyclic graph of order $m + r$ whose cycle has size m and where one of the vertices in the cycle is adjacent to a pendent vertex of a path of order r. See Figure 5 for an example. We have the following result due to Manjula in [1].

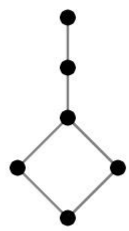

Figure 5: The graph G_4^2.

Theorem 2.1. *[1] If $n = m + r$, then the sequence $\{HL^k(G_m^r)\}$ converges to C_{m+r} in r iterations. Further, if $m + r > n$, then the sequence diverges by order.*

Since the class of graphs G_m^r where $m+r = n$ arises frequently, denote this family by δ_n, that is,

$$\delta_n = \{G_m^r : r + m = n\}.$$

Notice that if $G \in \delta_n$, then the sequence of any proper subgraph of G terminates. Thus, $\delta_n \subset \lambda_n$.

The second known condition that implies divergence that we will use also requires us to define another class of graphs. For $m \geq 4$, define F_m to be the graph of order m and size $m + 1$ consisting of a cycle of size m chorded by an edge that joins two vertices whose distance is 2. The following result is due to Chartrand et al. in [2].

Theorem 2.2. *[2] For $n \geq 4$ and $m \geq n$, the graph F_m is P_n-divergent by order.*

We use these two previous results to prove the following. Define the circumference of G, or $cr(G)$, as the size of the largest cycle in G, and where $cr(G) = 0$ if G is a forest.

Theorem 2.3. *Let $G' \subseteq G$ be a connected subgraph. If $cr(G') = m \geq n$ but $G' \ncong C_m$, then G is P_n-divergent by order.*

Proof. If a component of G is P_n-divergent by order, then G is P_n-divergent by order as well. Thus, we may assume that $G = G'$. Note that since $G \ncong C_m$, then there exists an edge $e = uv$ not in the cycle such that v is in the cycle. There are two cases.

The first case is when u is in the cycle. Let $G_0 \subseteq G$ contain the cycle and the edge e. Further, set $N(u) = \{v, u_1, u_2\}$ and $N(v) = \{u, v_1, v_2\}$ such that the vertices are labeled as in Figure 6. Since $m \geq n$, the sequence $v_1, ..., u_1, u, v, v_2, ..., u_2$ is a path of order at least n. Thus, e_1 and e_4 are P_n adjacent to e. By making a similar path, we notice that e_2 and e_3 are P_n-adjacent to e as well. Figure 7 gives a subgraph of $HL(G_0)$ which, as can be seen, is isomorphic to F_{m+1}. By Theorem 2.2, a subgraph of $HL(G_0)$ is P_n-divergent by order, and so G is P_n-divergent by order thus finishing this case.

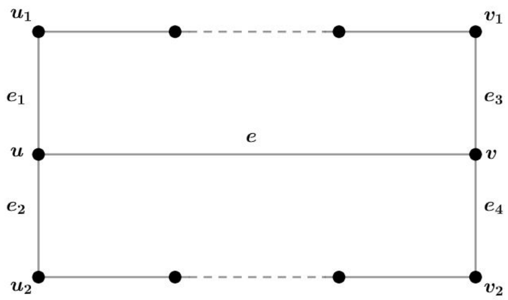

Figure 6: The subgraph G_0 of G.

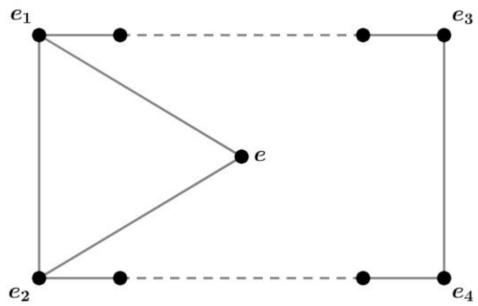

Figure 7: A subgraph of $HL(G_0)$.

The second case is when u is not in the cycle. This case, however, is a very simple case because then G has G_m^1 as a subgraph. Since $m \geq n$, it follows that $m + 1 > n$ so by Theorem 2.1 the sequence of G diverges by order. $\square$

Theorem 2.3 is important because it shows that if the sequence of a connected graph $G \in \Lambda_n$ ever reaches a point where $HL^k(G)$ has a subgraph isomorphic to C_m where $m \geq n$, then $HL^k(G)$ is in fact isomorphic to C_m. Although this condition is sufficient, we conjecture that it is also necessary.

Conjecture 2.4. A graph G is P_n-divergent by order if and only if there exists a k such that $HL^k(G)$ has a connected subgraph G' where $cr(G') = m \geq n$ but $G' \ncong C_m$.

Proving the above conjecture is just one step in understanding the structures that cause divergence. In particular, a characterization of the graphs G whose sequence ends up satisfying the condition of Theorem 2.3 would be beneficial.

We now provide another type of graph that is P_n-divergent by order. Let the graph $CL[x, y, z]$ be the graph with order $x + y + z + 1$ composed of three vertex disjoint paths of orders x, y and z respectively, and a vertex that is adjacent to a pendent vertex of each path. Observe that $CL[1, 1, 1]$ is the claw $K_{1,3}$.

Theorem 2.5. *If n and k are integers such that $k + 1 < n \leq 2k$, then $CL[k, k, n - k - 1]$ is P_n-divergent by order.*

Proof. Set $d = n-k-1$. We start by noticing that $HL(CL[k, k, d])$ is isomorphic to a unicyclic graph with C_3 as its cycle and where each vertex in the cycle is adjacent to a path of order $k - 1$ or $d - 1$. See Figure 8 for $HL(CL[k, k, d])$ and its indexation.

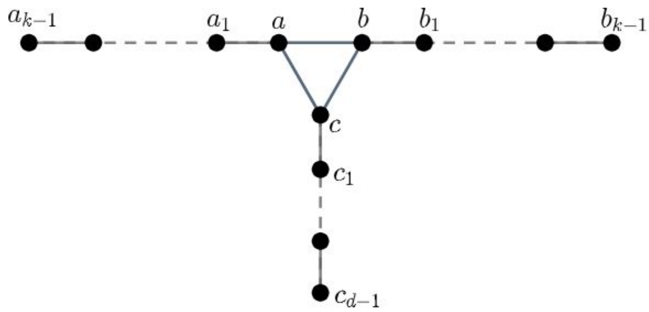

Figure 8: The graph $HL(CL[k, k, n - k - 1])$.

Notice that there is a path of order $(k-1)+3+(d-1)$ (which is equal to n) that includes the edges a_1a and ab. Similarly, there exists a similar path of order at least n that contains the edges a_1a and ac. In general, we notice that $HL^2(CL[k, k, d])$ has the graph of Figure 9 as a subgraph.

It is important to remark that the graph of Figure 9 is a subgraph of $HL^2(CL[k, k, d])$. In particular, the edges uv, uw and vw belong to $HL^2(CL[k, k, d])$. Nonetheless, this subgraph is enough to cause the sequence to diverge by order. Further, note that Figure 9 does not use the same labelings that were used in Figure 8. For example, the vertex labeled as a in Figure 9 corresponds to the edge a_1a in Figure 8. The case is similar for b and c. We do the labeling this way so that we can illustrate better what will happen in $HL^m(CL[k, k, d])$. Before going into these details, notice that in $HL^2(CL[k, k, d])$, there is a path of order $(k-2)+5+(d-2)$

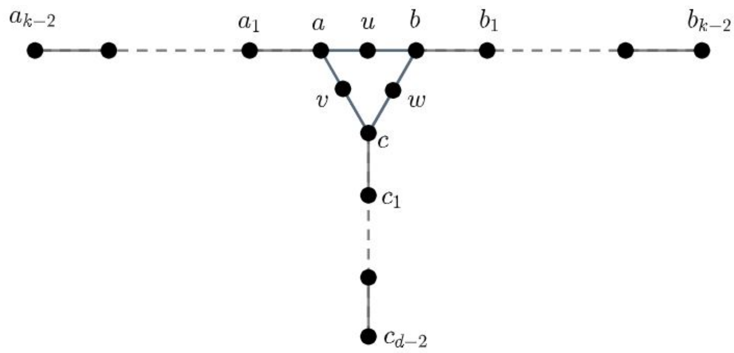

Figure 9: A subgraph of $HL^2(CL[k,k,n-k-1])$.

(which is equal to n) that includes the edges a_1a and au. Similarly, there is a path that contains a_1a and av. We can make similar statements for vertex b.

In general, assume that the m^{th} iteration of the sequence has a unicyclic subgraph with three vertices a, b, and c on the cycle, where each of a and b are adjacent to a path of order $k-m$, and c is adjacent to a path of order $d-m$. Further, assume that any two vertices in $\{a,b,c\}$ have distance m. If a_1 is the vertex adjacent to a in the path, then a_1a will be in a path of order $(k-m)+(m+1+m)+(d-m)$, which is equal to n. Through similar arguments for the other vertices adjacent with a, and by repeating this with b and c, we conclude that the $(m+1)^{\text{th}}$ iteration will have a subgraph with these same properties. Finally, since $HL(CL[k,k,d])$ has this property, then every graph in the sequence up to the d^{th} iteration has it.

This is enough to prove divergence by order because then $HL^d(CL[k,k,d])$ will have a subgraph isomorphic to G_{3d}^{k-d}. Since $k-d+3d>n$, Theorem 2.1 guarantees that this subgraph is P_n-divergent by order and thus $CL[k,k,n-k-1]$ is P_n-divergent by order too. □

Corollary 2.6. *Let v be the vertex with degree 3 in $CL[x,y,z]$, where x,y, and z are integers. If the edges incident to v are pairwise P_n-adjacent, then $CL[x,y,z]$ is P_n-divergent by order.*

Proof. Let P_1, P_2 and P_3 be the three paths joined by the vertex v where $|V(P_1)| = x, |V(P_2)| = y$, and $|V(P_3)| = z$. Notice that $x+y+1 \geq n$, so $y \geq n-1-x$. For now, assume that $y = n-x-1$. Since $y+z+1 \geq n$, it follows that $z \geq x$. As a consequence, the fact that $z+x+1 \geq n$ implies that $2x \geq n$. Thus, $CL[x,y,z]$ has as a subgraph isomorphic to $CL[x,n-x-1,x]$ where $2x \geq n$. By Theorem 2.5, $CL[x,n-x-1,x]$ is P_n-divergent by order, implying that $CL[x,y,z]$ is $P-n$-divergent by order. If it is the case that $y > n-x-1$, then $CL[x,n-x-1,z]$ is a subgraph of $CL[x,y,z]$ and the same proof applies. □

We will prove one more condition that implies divergence by order. For it, we need one more result due to Chartrand et al. in [2].

Theorem 2.7. *[2] If G is a connected graph, then $HL(G)$ contains at most one component that is not an isolated vertex.*

We remind the reader that two graphs G_1 and G_2 are not equal if and only if $V(G_1) \neq V(G_2)$ or $E(G_1) \neq E(G_2)$, and that this is possible even if $G_1 \cong G_2$. Our next result is that G is P_n-divergent by order if G contains two distinct subgraphs from the family $\delta_n = \{G_m^r : r + m = n\}$.

Theorem 2.8. *Let $G_1, G_2 \in \delta_n$ be subgraphs of the same component of G. If $G_1 \neq G_2$, then G is P_n-divergent by order.*

Proof. We may assume that G is connected. Set $G_1 \cong G_{m_1}^{r_1}$ and $G_2 \cong G_{m_2}^{r_2}$. We first consider the case where $G_1 \ncong G_2$. Without loss of generality, assume that $r_1 < r_2$. Theorem 2.1 implies that $HL^{r_1}(G_1) \cong C_n$. Since $r_1 < r_2$, we have that $HL^{r_1}(G_2) \ncong C_n$. Thus $HL^{r_1}(G)$ will have a subgraph isomorphic to C_n, but since the sequence of G_2 does not terminate, it follows that $HL^{r_1}(G) \ncong C_n$. By Theorem 2.3, G is P_n-divergent by order.

Assume then that $G_1 \cong G_2$. Set $m = m_1 = m_2$ and $r = r_1 = r_2$. Note that $HL^r(G_1) \cong HL^r(G_2) \cong C_n$, and the edges in both G_1 and G_2 will be the vertices of a cycle of size n in $HL^r(G)$. Further, observe that if $E(G_1) = E(G_2)$, then $V(G_1) = V(G_2)$ since $G_1 \cong G_2$, so it must be that $E(G_1) \neq E(G_2)$. Thus, $HL^r(G)$ will contain two different cycles of size n in the same component (we know that they are in the same component by Theorem 2.7). This satisfies the condition of Theorem 2.3, and so G is P_n-divergent by order. □

The natural generalization of this theorem is: if G has a component containing distinct subgraphs G_1 and G_2 such that $G_1, G_2 \in \Lambda_n$, then G has a sequence that diverges by order. Nonetheless, Figure 10 shows a counterexample to this. For a conjecture, we need to make one of these conditions stronger.

Conjecture 2.9. Let G be a connected graph, and let G_1 and G_2 be subgraphs of G. If $G_1 \ncong G_2$ and $G_1, G_2 \in \Lambda_n$, then G has a sequence that diverges by order.

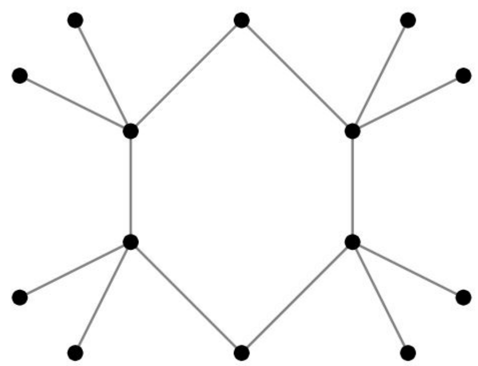

Figure 10: A graph whose sequence converges when $H \cong P_8$.

3 Properties of minimal n-convergence

3.1 Basic properties

We start by reminding the reader of the definition of minimally n-convergent graphs.

Definition 3.1. A graph G is said to be minimally n-convergent if $G \in \Lambda_n$ but no proper subgraph of G in Λ_n. Further, let λ_n be the set of all minimally n-convergent graphs.

Notice that every graph $G \in \Lambda_n$ has a subgraph $G' \subseteq G$ such that $G' \in \lambda_n$. This is why studying minimal n-convergence can be far reaching: obtaining properties about graphs in λ_n gives us properties about some subgraph of every graph in Λ_n. We will spend the rest of the paper developing results related to minimal n-convergence. Although the next result is easy to obtain, it provides a template for how to use the definition of minimal n-convergence in proofs.

Lemma 3.2. *If $G \in \lambda_n$, then every edge in G is in a copy of P_n.*

Proof. Suppose to the contrary that there exists $G \in \lambda_n$ with an edge e not belonging to a subgraph isomorphic to P_n. First, notice that e is an isolated vertex in $HL(G)$, as otherwise there exists an edge $f \in E(G)$ such that e is P_n-adjacent to f, meaning that there exists a path P_n containing both e and f which would contradict our assumption that e is not in a P_n. Thus, the edge e is an isolated vertex in $HL(G)$. From $G \in \Lambda_n$, we know that $HL(G) \in \Lambda_n$, and since e is an isolated vertex in $HL(G)$, it follows that $HL(G) - e \in \Lambda_n$ also.

Since e is not P_n-adjacent to f for every $f \in E(G)$, it follows that $E(HL(G - e)) = E(HL(G) - e)$. Also, from the definition of H-line graphs we also know that $V(HL(G - e)) = V(HL(G) - e)$, so we conclude that $HL(G - e) = HL(G) - e$. But then $HL(G - e) \in \Lambda_n$, so $G - e \in \Lambda_n$. This contradicts that G is minimally n-convergent thus finishing the proof. $\square$

Notice that Lemma 3.2 implies that if $G \in \lambda_n$, then $HL(G)$ has no isolated vertices. This contributes to the notion that every edge in G contributes to the structure of $HL(G)$ when G is minimally n-convergent.

Lemma 3.3. *Let $G \in \lambda_n$, and assume both that $G \notin \delta_n$ and that G is not a cycle. Further, let P be a non-extendable path in G that has order at least n. If p_1 and p_2 are the pendent vertices in P, then p_1 and p_2 are pendent vertices in G.*

Proof. Consider the case in which G is connected. If G is disconnected, the proof applies to one of its components. Assume, for a contradiction, that p_1 is not a pendant vertex in G. Since P cannot be extended, then p_1 must be adjacent to some vertex p of P. If $p = p_2$, then G has a cycle of size at least n. By hypothesis, G is not a cycle so $G \ncong C_n$, so by Theorem 2.3, it follows that G has a sequence that diverges by order, which is a contradiction. If $p \neq p_2$, then the subgraph of P with the edge p_1p is a subgraph G_0 isomorphic to G_m^r for some r and m. Since P has order at least n, it follows that $r + m \geq n$. However, it cannot be the case that

$r+m>n$ since G would have a sequence that diverges in order by Theorem 2.1. So $r+m=n$. However, Theorem 2.1 implies that $G_0 \in \Lambda_n$. And since $G \notin \delta_n$, then G_0 is a proper subgraph of G. This contradicts that G is minimally n-convergent thus finishing the proof. □

Lemma 3.4. *If $G \in \lambda_n$, then $HL(G)$ has the same number of components as G.*

Proof. Every component in G has every edge in a P_n by Lemma 3.2, so if G' is a component of G, it is not possible for $HL(G')$ to have an isolated vertex. Thus, by Theorem 2.7, $HL(G')$ is connected. In other words, every component of G generates exactly one component in $HL(G)$. □

3.2 Minimal n-convergence and unicyclicity

Knowing the structure of minimally n-convergent graphs can facilitate multiple proofs. We have the following conjecture about the structure of these graphs.

Conjecture 3.5. If $G \in \lambda_n$, then G has unicyclic components.

Establishing this conjecture can be very helpful when proving statements about minimally n-convergent graphs as it provides us with one and only one cycle to work with. Further, since every graph $G \in \Lambda_n$ has a subgraph in λ_n, Conjecture 3.5 would prove that no tree has a P_n-convergent sequence. We will use the rest of the paper to give results related to this conjecture. For this purpose, we need to study more carefully the relationship between H-line graphs and the property of unicyclicity.

Definition 3.6. Let C be the unique cycle in a unicyclic graph G.

- The subgraph A of G is called an arm if A is a component of $G - V(C)$.
- The armset of G, denoted by $\mathcal{A}(G)$, is the set

$$\mathcal{A}(G) := \{A : A \text{ is an arm of G}\}.$$

- The vertex $r \in V(C)$ is a called a root if r is adjacent to some vertex in an arm $A \in \mathcal{A}(G)$.
- The root identifier function, denoted by $\mathcal{A}_G : \mathcal{A}(G) \to V(C)$, is the function that takes A to the vertex of C that is adjacent to some vertex in A.

Note that $\mathcal{A}_G$ is well defined because if there were two roots r_1 and r_2 associated with an arm A, then G would not be unicyclic. Further, it is not necessary for $\mathcal{A}_G$ to be a one-to-one function. In particular, there exists unicyclic graphs $G \in \Lambda_n$ that have roots adjacent to multiple arms. For instance, the graph in Figure 10 has 4 roots but 8 arms. We need a result due to Britto-Pacumio in [4]. Remember that the circumference of G, or $cr(G)$, is the size of the largest cycle in G, and where $cr(G) = 0$ if G is a forest.

Theorem 3.7. *[4] If G is unicyclic and every edge of G is in a P_n, then $cr(HL(G)) \geq cr(G)$.*

Corollary 3.8. *Let $G \in \lambda_n$. If G is unicyclic, then $HL(G)$ is not a tree.*

The proof of the above corollary is immediate from Lemma 3.2 and Theorem 3.7. The rigid structures of unicyclic graphs allows for many proof techniques that make use of roots and arms.

Lemma 3.9. *Let $G \in \Lambda_n$ such that G is unicyclic. If e is an edge in an arm of G, then e cannot be in a cycle of $HL(G)$.*

Proof. Let C be the unique cycle of G. For a contradiction, assume that there exists an arm $A \in \mathcal{A}(G)$ and a cycle C' in $HL(G)$ such that $e \in E(A)$ and $e \in V(C')$. Set $C' : e_1, ..., e_p, e_1$ where $e_i \in E(G)$. Without loss of generality, assume that $e = e_1$, and let v be the vertex in G incident to both e_1 and e_2. Since both vertices incident to e are in A, we have that $v \in V(A)$. Let f be the edge incident to $\mathcal{A}_G(A)$, the root of A, and to some vertex u in A, the vertex in A adjacent to the root.

Notice that $e_i \neq e_j$ for $i \neq j$. If there exists an i such that $e_i \in E(C)$, then f would be in the sequence $e_1, e_2, ..., e_i$. However, f would also need to be in the sequence $e_i, e_{i+1}, ..., e_p, e_1$. Since f is not in the arm, we have that $f \neq e_1$ and $f \neq e_i$, so then we have a repeated element in the sequence $e_1, ..., e_p$, which is a contradiction. Thus, $e_i \notin E(C)$ for every i. In other words, every vertex of the cycle C' must be either in A or be f. Since A is a tree, we have that every edge in $V(C')$ must be incident to the same vertex as otherwise we can craft a similar argument to the case where there is an edge in the cycle of G. So every edge e_i is incident to v.

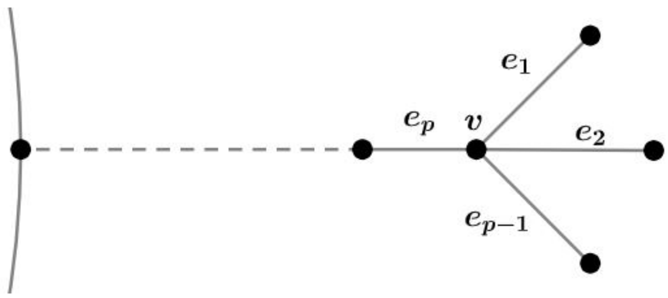

Figure 11: An illustration of a subgraph of G.

If $p = 3$, then the graph contains a claw where the edges incident to v, which would be $\{e_1, e_2, e_3\}$, are pairwise P_n-adjacent. By Corollary 2.6, the graph has a divergent sequence, which is a contradiction. Thus, $p > 3$. There exists a unique path between v and $\mathcal{A}_G(A)$. Without loss of generality, assume that this path contains the edge e_p. See Figure 11. Let $e_i = vv_i$ for every i. Let P_i denote the longest path that has as an endpoint v_i and which does not contain the edge e_i. For e_{p-1}, e_p, e_1 and e_2, denote the order of those paths by k_{p-1}, k_p, k_1 and k_2 respectively. Since e_1 is P_n-adjacent to e_2, it follows that $k_1 + k_2 + 1 \geq n$, or $k_1 + 1 \geq n - k_2$. Furthermore, it is not the case that e_1 is P_n-adjacent to e_{p-1} because then $\{e_1, e_p, e_{p-1}\}$ would be a set of pairwise P_n-adjacent edges in a claw, giving a contradiction by Corollary 2.6. Thus, $k_1 + k_{p-1} + 1 < n$. Combining both inequalities, we get that

$$n - k_2 + k_{p-1} \leq k_{p-1} + k_1 + 1 < n,$$

so $k_{p-1} < k_2$. By applying the same arguments, we also get that $k_p + 1 \geq n - k_{p-1}$ and $k_p + k_2 + 1 < n$. Combining both inequalities, we get that

$$k_2 + n - k_{p-1} \leq k_2 + k_p + 1 < n,$$

so $k_2 < k_{p-1}$, a contradiction. □

Corollary 3.10. *Let $G \in \Lambda_n$ such that G is unicyclic. If r is a root, then the edges incident to r cannot induce, as vertices, a graph with a cycle of order 4 or more in $HL(G)$.*

The corollary follows from the same proof technique used for Lemma 3.9: if G is a counterexample, then the cycle C' in $HL(G)$ contradicting Corollary 3.10 leads to contradictory inequalities. We now have all the tools needed for our last result. Remember that the girth of a graph G, denoted by $g(G)$, is the size of the smallest cycle in G.

Theorem 3.11. *Let $G \in \lambda_n$ such that $g(HL(G)) > 4$. If G has unicyclic components, then $HL(G)$ has unicyclic components.*

Proof. We may, again, assume G is connected as the proof applies to each component of G if G is disconnected. Since G is unicyclic and is in λ_n, it follows that $HL(G)$ must have a cycle.

For a contradiction, assume that there exists two cycles C_1 and C_2 in $HL(G)$ such that $C_1 \neq C_2$. Corollary 3.10 and $g(HL(G)) > 4$ imply that no root is incident to every edge of C_1 or C_2. Since no edge in the arms can be in a cycle of $HL(G)$, it must be that every edge in the cycle of G is in a cycle of $HL(G)$. Set C as the cycle of G, so $E(C) \subseteq V(C_1)$ and $E(C) \subseteq V(C_2)$. Since $C_1 \neq C_2$, there must exists an edge $e \in C_1$ that is not in C_2. This edge cannot be in C so it is incident to a root r. Set $E(r)$ as the set of edges incident to r, and let $E(r) \cap E(C) = \{f_1, f_2\}$. But f_1 and f_2 are both in $V(C_1)$ and $V(C_2)$, so

$$(E(r) \cap V(C_1)) \cup (E(r) \cap V(C_2))$$

induces at least one cycle in $HL(G)$. Every edge in this cycle is incident to r, which contradicts Corollary 3.10. □

Requiring that $g(HL(G)) > 4$ is most likely not necessary for the previous statement to remain true. Finding a proof that avoids using this assumption is desirable, but probably hard. To continue the study of minimally n-convergent graphs, we propose two directions. The first direction is working more towards the proof of Conjecture 3.5. One roadmap to such a proof is the following. First, verify that if $G \in \lambda_n$, then $HL(G) \in \lambda_n$ (such a statement sounds easy to prove but we could not do it). Second, confirm that limit graphs have unicyclic components. Lastly, improve Theorem 3.11 by removing the assumption that $g(HL(G)) > 4$. This, in turn, proves Conjecture 3.5.

A second direction to study λ_n, which has not been discussed in detail in this paper, is establishing the veracity of the following conjecture.

Conjecture 3.12. If $G \in \Lambda_n$ and G is not the disconnected union of two graphs in Λ_n, then there exists a unique graph G' in λ_n such that $G' \subseteq G$.

The existence of G' is already known. The conjecture adds that this graph is unique. This would imply that P_n-convergent graphs are actually just variations of graphs in λ_n. In other words, characterizing Λ_n would heavily depend on characterizing λ_n. Proving Conjecture 3.12, however, needs a more thorough development of the theory of minimal n-convergence (for instance, proving Conjecture 3.5). A good starting point for this direction is to prove or disprove Conjecture 2.9, as it deals with the P_n-convergence of graphs that have distinct P_n-convergent subgraphs.

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References

[1] K. Manjula, Some results on H-line graphs, *Indian J. Pure. Appl. Math.* 34(9) (2003), 1307–1314.

[2] G. Chartrand, H. Gavlas and M. Schultz, Convergent sequences of iterated H-line graphs, *Discrete Math.* 147 (1995), 73–86.

[3] G. Chartrand, M. Henning, E. Jarrett and C. Wall, Quadrilateral line graphs, *Utilitas Math.* 50 (1996), 3–17.

[4] R. Britto-Pacumio, Limits of iterated H-line graphs, *Discrete Math.* 199 (1-3) (1999), 1–10.

[5] E. Jarrett, On iterated triangular line graphs, Proc. Seventh Quadrennial Int. Conf. on the Theory and Applic'ns of Graphs, (1995), 589–599.

[6] G. Chartrand, Graphs and their associated line graphs, Ph.D. Thesis, Michigan State University, 1964.

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