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C_4-face-magic labelings on odd order projective grid graphs

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Abstract

For a graph $G = (V, E)$ embedded in the projective plane, let $\mathcal{F}(G)$ denote the set of faces of G. Then G is called a C_n-face-magic projective graph if there exists a bijection $f : V(G) \to \{1, 2, \ldots, |V(G)|\}$ such that for any $F \in \mathcal{F}(G)$ with $F \cong C_n$, the sum of all the vertex labels around C_n is a constant S. We consider the $m \times n$ grid graph, denoted by $\mathcal{P}_{m,n}$, embedded in the projective plane in the natural way. We show that for $m, n \geqslant 2$, $\mathcal{P}_{m,n}$ admits a C_4-face-magic labeling if and only if m and n have the same parity.

Let $m \geqslant 3$ and $n \geqslant 3$ be odd integers. We show that the C_4-face-magic value of a C_4-face-magic labeling on $\mathcal{P}_{m,n}$ is either $2mn + 1$, $2mn + 2$, or $2mn + 3$. In this paper, we characterize the C_4-face-magic labelings on $\mathcal{P}_{m,n}$ with C_4-face-magic value $2mn + 2$.

1 Introduction

Graph labelings were formally introduced in the 1970s by Kotzig and Rosa [15]. Graph labelings have been applied to graph decomposition problems, radar pulse code designs, X-ray crystallography and communication network models. The interested reader should consult Gallian's comprehensive dynamic survey on graph labelings [11] for further information.

We refer the reader to Chartrand, Lesniak and Zhang [5] for concepts and notation not explicitly defined in this paper. The graphs in this paper are connected multigraphs. The concept of a C_4-face-magic labeling was first applied to planar graphs. For a planar or projective graph $G = (V, E)$ embedded in the plane or projective plane, let $\mathcal{F}(G)$ denote the set of faces of G. Then, G is called a C_n-*face-magic planar* or *projective* graph if there exists a bijection $f : V(G) \to \{1, 2, \ldots, |V(G)|\}$ such that for any $F \in \mathcal{F}(G)$ with $F \cong C_n$, the sum of all the vertex labels around C_n is a constant S. Here, the constant S is called a C_n-*face-magic value* of G. More

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generally, C_4-face-magic planar graph labelings are a special case of (a, b, c)-magic labeling introduced by Lih [16]. For assorted values of a, b and c, Bača and others [1, 2, 3, 12, 13, 14, 16] have analyzed the problem for various classes of graphs. Wang [17] showed that the toroidal grid graph $C_m \times C_n$ has an antimagic labeling for all integers $m, n \geqslant 3$. Recall that a graph with q edges is called *antimagic* if its edges can be labeled with $1, 2, \ldots, q$ without repetition such that the sums of the labels of the edges incident to each vertex are distinct. Butt et al. [4] investigated face antimagic labelings on toroidal and Klein bottle grid graphs. Here, a face antimagic labeling on a toroidal or Klein bottle grid graph is a labeling of the vertices, edges and faces of an $m \times n$ toroidal grid graph $C_m \times C_n$ or an $m \times n$ Klein bottle grid graph $\mathcal{K}_{m,n}$ by the consecutive integers from 1 up to $|V(C_m \times C_n)| + |E(C_m \times C_n)| + |\mathcal{F}(C_m \times C_n)|$ or $|V(\mathcal{K}_{m,n})| + |E(\mathcal{K}_{m,n})| + |\mathcal{F}(\mathcal{K}_{m,n})|$, respectively, in such a way that the label of a 4-sided face and the labels of the vertices and edges surrounding that face all together add up to a weight of that face. These face-weights then form an arithmetic progression with common difference d.

Curran, Low and Locke [6, 7] investigated C_4-face-magic labelings on an $m \times n$ toroidal grid graph $C_m \times C_n$. They showed that $C_m \times C_n$ admits a C_4-face-magic labeling if and only if either $m = 2$, or $n = 2$, or both m and n are even. Curran, Low and Locke [8] also examined C_4-face-magic labelings on an $m \times n$ Klein bottle grid graph. They showed that an $m \times n$ Klein bottle grid graph admits a C_4-face-magic labeling if and only if n is even. In this paper, we consider C_4-face-magic labelings on an $m \times n$ projective grid graph. We show, in Theorem 2.7, that an $m \times n$ projective grid graph admits a C_4-face-magic labeling if and only if both m and n have the same parity. Also, when m and n are even, then the C_4-face-magic value must be $2mn+2$. Furthermore, when m and n are odd, then the C_4-face-magic value is either $2mn + 1$, $2mn + 2$, or $2mn + 3$.

In this paper, we investigate the C_4-face-magic labelings on $\mathcal{P}_{m,n}$ with C_4-face-magic value $2mn+2$ when m and n are odd. We show that a C_4-face-magic labeling $X = \{x_{i,j} : (i, j) \in V(\mathcal{P}_{m,n})\}$ with C_4-face-magic value $2mn + 2$ is *centrally balanced* in the sense that

$$x_{i,j} + x_{m+1-i,n+1-j} = mn + 1 \quad \text{for all } (i, j) \in V(\mathcal{P}_{m,n}).$$

Because of this additional structure on X, we are able to characterize and count these C_4-face-magic labelings on $\mathcal{P}_{m,n}$. Further, we pose an open problem related to C_4-face-magic labelings on $\mathcal{P}_{m,n}$ when m and n are even.

2 Preliminaries

Definition 2.1. For a graph $G = (V, E)$ embedded on the projective plane or plane or torus or Klein bottle, let $\mathcal{F}(G)$ denote the set of faces of G. Then G is called a C_n*-face-magic projective* or *planar* or *toroidal* or *Klein bottle* graph if there exists a bijection $f : V(G) \to \{1, 2, \ldots, |V(G)|\}$ such that for any $F \in \mathcal{F}(G)$ with $F \cong C_n$, the sum of all the vertex labels around C_n is a constant S. We call S the C_n*-face-magic value*.

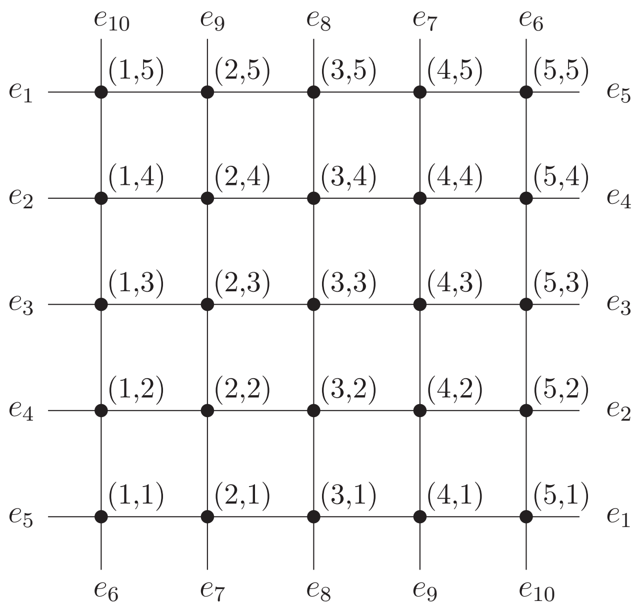

Figure 1: 5×5 projective grid graph $\mathcal{P}_{5,5}$.

Definition 2.2. Let m and n be integers such that $m, n \geqslant 2$. The $m \times n$ *projective grid graph*, denoted by $\mathcal{P}_{m,n}$, is the graph whose vertex set is

$$V(\mathcal{P}_{m,n}) = \{(i,j) : 1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n\},$$

and whose edge set consists of the following edges:

- there is an edge from (i,j) to $(i,j+1)$, for $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n-1$,
- there is an edge from (i,n) to $(m+1-i,1)$, for $1 \leqslant i \leqslant m$,
- there is an edge from (i,j) to $(i+1,j)$, for $1 \leqslant i \leqslant m-1$ and $1 \leqslant j \leqslant n$ and
- there is an edge from (m,j) to $(1,n+1-j)$, for $1 \leqslant j \leqslant n$.

The graph $\mathcal{P}_{m,n}$ has a natural embedding on the projective plane. This graph is a multigraph since there are double edges on the vertex sets $\{(1,1),(m,n)\}$ and $\{(m,1),(1,n)\}$.

Example 2.3. The 5×5 projective grid graph $\mathcal{P}_{5,5}$ is illustrated in Figure 1. Due to the orientation of the vertices in $\mathcal{P}_{m,n}$, we refer to the vertices $\{(i,j) : 1 \leqslant j \leqslant n\}$ as column i of $V(\mathcal{P}_{m,n})$ and $\{(i,j) : 1 \leqslant i \leqslant m\}$ as row j of $V(\mathcal{P}_{m,n})$.

Lemma 2.4. *Let m and n be integers such that $m, n \geqslant 2$. Suppose that $\mathcal{P}_{m,n}$ is a C_4-face-magic projective graph. Then m and n have the same parity.*

Proof. For the purposes of contradiction, we assume that $m \geqslant 2$ is even and $n \geqslant 3$ is odd. Let n_0 be the positive integer such that $n = 2n_0+1$. Let $\{x_{i,j} : (i,j) \in V(\mathcal{P}_{m,n})\}$

be a C_4-face-magic labeling on $\mathcal{P}_{m,n}$ Let $a = x_{1,n_0} + x_{1,n_0+1}$. When we set the two C_4-face sums given below equal to each other

$$x_{i,j} + x_{i,j+1} + x_{i+1,j} + x_{i+1,j+1} = S = x_{i+1,j} + x_{i+1,j+1} + x_{i+2,j} + x_{i+2,j+1},$$

we obtain

$$x_{i,j} + x_{i,j+1} = x_{i+2,j} + x_{i+2,j+1}.$$

Thus

$$x_{1,n_0} + x_{1,n_0+1} = x_{m-1,n_0} + x_{m-1,n_0+1}.$$

When we set the two C_4-face sums given below equal to each other

$$x_{m-1,n_0} + x_{m-1,n_0+1} + x_{m,n_0} + x_{m,n_0+1} = S = x_{m,n_0} + x_{m,n_0+1} + x_{1,n_0+1} + x_{1,n_0+2},$$

we obtain

$$x_{m-1,n_0} + x_{m-1,n_0+1} = x_{1,n_0+1} + x_{1,n_0+2}.$$

Thus

$$x_{1,n_0} + x_{1,n_0+1} = x_{1,n_0+1} + x_{1,n_0+2}$$

which, in turn, yields $x_{1,n_0} = x_{1,n_0+2}$. This is a contradiction. □

Lemma 2.5. *Suppose $m \geqslant 2$ and $n \geqslant 2$ are even integers. Let $\{x_{i,j} : (i,j) \in V(\mathcal{P}_{m,n})\}$ be a C_4-face-magic labeling on $\mathcal{P}_{m,n}$ with C_4-face-magic value S. Then $S = 2mn + 2$.*

Proof. Let m_0 and n_0 be positive integers such that $m = 2m_0$ and $n = 2n_0$. Consider the sum

$$\begin{aligned} \tfrac{1}{4}mnS = m_0n_0S &= \sum_{i=1}^{m_0}\sum_{j=1}^{n_0}(x_{2i-1,2j-1} + x_{2i-1,2j} + x_{2i,2j-1} + x_{2i,2j}) \\ &= \left(\sum_{k=1}^{mn} k\right) = \tfrac{1}{2}(mn)(mn+1). \end{aligned}$$

Thus $S = 2mn + 2$. □

Lemma 2.6. *Let $m \geqslant 3$ and $n \geqslant 3$ be odd integers. Let $\{x_{i,j} : (i,j) \in V(\mathcal{P}_{m,n})\}$ be a C_4-face-magic labeling on $\mathcal{P}_{m,n}$ with C_4-face-magic value S. Let $D_1 = x_{1,1} + x_{m,n}$ and $D_2 = x_{m,1} + x_{1,n}$ be the face sums of the two digons constructed from the pair of vertices at opposite corners of $\mathcal{P}_{m,n}$. Recall that a digon is a two-sided polygon. Then either*

1. *$S = 2mn + 1$ and $D_1 = D_2 = \frac{3}{2}mn + \frac{1}{2}$,*
2. *$S = 2mn + 2$ and $D_1 = D_2 = mn + 1$, or*
3. *$S = 2mn + 3$ and $D_1 = D_2 = \frac{1}{2}mn + \frac{3}{2}$.*

Proof. We first observe that, for $1 \leqslant j < n-1$, we have

$$x_{1,j} + x_{1,j+1} + x_{m,n+1-j} + x_{m,n-j} = S = x_{1,j+1} + x_{1,j+2} + x_{m,n-j} + x_{m,n-j-1}.$$

Thus, for $1 \leqslant j < n-1$,

$$x_{1,j} + x_{m,n+1-j} = x_{1,j+2} + x_{m,n-j-1}.$$

Hence, for $1 \leqslant j \leqslant (n-1)/2$, we have

$$x_{1,2j-1} + x_{m,n+2-2j} = x_{1,2j+1} + x_{m,n-2j}.$$

Thus

$$D_1 = x_{1,1} + x_{m,n} = x_{1,n} + x_{m,1} = D_2.$$

Hence,

$$2D_1 = x_{1,1} + x_{m,n} + x_{1,n} + x_{m,1}.$$

Therefore,

$$\begin{aligned} 2D_1 + (mn-1)S = & \sum_{i=1}^{m-1}\sum_{j=1}^{n-1}(x_{i,j} + x_{i+1,j} + x_{i,j+1} + x_{i+1,j+1}) \\ & + \sum_{i=1}^{m-1}(x_{i,n} + x_{i+1,n} + x_{m-i,1} + x_{m+1-i,1}) \\ & + \sum_{j=1}^{n-1}(x_{m,j} + x_{m,j+1} + x_{1,n-j} + x_{1,n+1-j}) \\ & + (x_{1,1} + x_{m,n} + x_{m,1} + x_{1,n}) \\ = & \, 4\left(\sum_{k=1}^{mn} k\right) = (2mn)(mn+1). \end{aligned}$$

Thus

$$(mn-1)S = 2m^2n^2 + 2mn - 2D_1.$$

Since

$$10 \leqslant 2D_1 \leqslant 4mn - 6,$$

we have

$$2m^2n^2 - 2mn + 6 \leqslant (mn-1)S \leqslant 2m^2n^2 + 2mn - 10.$$

Thus

$$2mn + \frac{6}{mn-1} \leqslant S \leqslant 2mn + 4 - \frac{6}{mn-1}.$$

Since $m \geqslant 3$ and $n \geqslant 3$, we have

$$2mn + 1 \leqslant S \leqslant 2mn + 3.$$

We observe that

$$D_1 = D_2 = m^2n^2 + mn - \tfrac{1}{2}(mn-1)S.$$

For $S = 2mn+1$, we have $D_1 = D_2 = \frac{3}{2}mn + \frac{1}{2}$. Similarly, for $S = 2mn+2$, we have $D_1 = D_2 = mn+1$. Also, for $S = 2mn+3$, we have $D_1 = D_2 = \frac{1}{2}mn + \frac{3}{2}$. $\square$

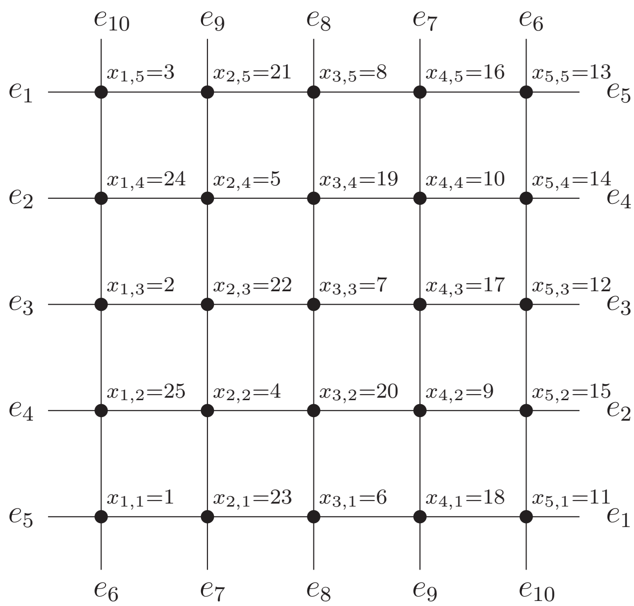

Figure 2: C_4-face-magic labeling on $\mathcal{P}_{5,5}$ having C_4-face-magic value 53.

Theorem 2.7. *Let m and n be integers such that $m, n \geqslant 2$. Then $\mathcal{P}_{m,n}$ admits a C_4-face-magic labeling if and only if m and n have the same parity.*

Proof. ($\Rightarrow$) Suppose $\mathcal{P}_{m,n}$ admits a C_4-face-magic labeling. Then, by Lemma 2.4, m and n have the same parity.

($\Leftarrow$) **Case 1.** Assume $m \geqslant 3$ and $n \geqslant 3$ are odd integers. Let m_0 and n_0 be integers such that $m = 2m_0 + 1$ and $n = 2n_0 + 1$. We define

- $x_{2i-1,2j-1} = n(i-1) + j$ for $1 \leqslant i \leqslant m_0 + 1$ and $1 \leqslant j \leqslant n_0 + 1$,
- $x_{2i,2j} = n(i-1) + n_0 + 1 + j$ for $1 \leqslant i \leqslant m_0$ and $1 \leqslant j \leqslant n_0$,
- $x_{2i-1,2j} = n(m-i+1) - j + 1$ for $1 \leqslant i \leqslant m_0 + 1$ and $1 \leqslant j \leqslant n_0$ and
- $x_{2i,2j-1} = n(m-i+1) - n_0 - j + 1$ for $1 \leqslant i \leqslant m_0$ and $1 \leqslant j \leqslant n_0 + 1$.

We observe that for the vertices (i,j) where $i+j$ even, we assign the labels $1, 2, \ldots, \frac{1}{2}mn + \frac{1}{2}$ in lexicographic order; however, for the vertices (i,j) where $i+j$ odd, we assign the labels $\frac{1}{2}mn + \frac{3}{2}, \frac{1}{2}mn + \frac{5}{2}, \ldots, mn$ in reverse lexicographic order. See Figure 2 for an example of this labeling on the 5×5 projective grid graph $\mathcal{P}_{5,5}$.

We have $x_{2i-1,2j-1} + x_{2i-1,2j} = mn + 1$ for $1 \leqslant i \leqslant m_0 + 1$ and $1 \leqslant j \leqslant n_0$. Also, we have $x_{2i,2j-1} + x_{2i,2j} = mn + 2$ for $1 \leqslant i \leqslant m_0$ and $1 \leqslant j \leqslant n_0$. Thus, for $1 \leqslant i \leqslant m-1$ and $1 \leqslant j \leqslant n_0$, we have

$$x_{i,2j-1} + x_{i,2j} + x_{i+1,2j-1} + x_{i+1,2j} = 2mn + 3.$$

Next, we have $x_{2i-1,2j} + x_{2i-1,2j+1} = mn + 2$ for $1 \leqslant i \leqslant m_0 + 1$ and $1 \leqslant j \leqslant n_0$. Also, we have $x_{2i,2j} + x_{2i,2j+1} = mn + 1$ for $1 \leqslant i \leqslant m_0$ and $1 \leqslant j \leqslant n_0$. Thus, for $1 \leqslant i \leqslant m - 1$ and $1 \leqslant j \leqslant n_0$, we have

$$x_{i,2j} + x_{i,2j+1} + x_{i+1,2j} + x_{i+1,2j+1} = 2mn + 3.$$

We observe that, for $1 \leqslant j \leqslant n_0 + 1$, $x_{1,2j-1} + x_{m,n+2-2j} = \frac{1}{2}mn + \frac{3}{2}$ and for $1 \leqslant j \leqslant n_0$, $x_{1,2j} + x_{m,n+1-2j} = \frac{3}{2}mn + \frac{3}{2}$. Thus, for $1 \leqslant j \leqslant n - 1$, we have

$$x_{1,j} + x_{m,n+1-j} + x_{1,j+1} + x_{m,n-j} = 2mn + 3.$$

Similarly, for $1 \leqslant i \leqslant m_0 + 1$, $x_{2i-1,1} + x_{m+2-2i,n} = \frac{1}{2}mn + \frac{3}{2}$ and for $1 \leqslant i \leqslant m_0$, $x_{2i,1} + x_{m+1-2i,n} = \frac{3}{2}mn + \frac{3}{2}$. Thus, for $1 \leqslant i \leqslant m - 1$, we have

$$x_{i,1} + x_{m+1-i,n} + x_{i+1,1} + x_{m-i,n} = 2mn + 3.$$

Case 2. Assume $m \geqslant 2$ and $n \geqslant 2$ are even integers. Let m_0 and n_0 be integers such that $m = 2m_0$ and $n = 2n_0$. We define

- $x_{2i-1,2j-1} = n(i-1) + j$ for $1 \leqslant i \leqslant m_0$ and $1 \leqslant j \leqslant n_0$,
- $x_{2i,2j} = n(i-1) + n_0 + j$ for $1 \leqslant i \leqslant m_0$ and $1 \leqslant j \leqslant n_0$,
- $x_{2i-1,2j} = n(m-i+1) - j + 1$ for $1 \leqslant i \leqslant m_0$ and $1 \leqslant j \leqslant n_0$ and
- $x_{2i,2j-1} = n(m-i+1) - n_0 - j + 1$ for $1 \leqslant i \leqslant m_0$ and $1 \leqslant j \leqslant n_0$.

We observe that for the vertices (i, j) where $i + j$ even, we assign the labels $1, 2, \ldots, \frac{1}{2}mn$ in lexicographic order; however, for the vertices (i, j) where $i + j$ odd, we assign the labels $\frac{1}{2}mn + 1, \frac{1}{2}mn + 2, \ldots, mn$ in reverse lexicographic order. See Figure 3 for an example of this labeling on the 6×6 projective grid graph $\mathcal{P}_{6,6}$.

We have $x_{2i-1,2j-1} + x_{2i-1,2j} = mn + 1$ for $1 \leqslant i \leqslant m_0$ and $1 \leqslant j \leqslant n_0$. Also, we have $x_{2i,2j-1} + x_{2i,2j} = mn + 1$ for $1 \leqslant i \leqslant m_0$ and $1 \leqslant j \leqslant n_0$. Thus, for $1 \leqslant i \leqslant m - 1$ and $1 \leqslant j \leqslant n_0$, we have

$$x_{i,2j-1} + x_{i,2j} + x_{i+1,2j-1} + x_{i+1,2j} = 2mn + 2.$$

Next, we have $x_{2i-1,2j} + x_{2i-1,2j+1} = mn + 2$ for $1 \leqslant i \leqslant m_0$ and $1 \leqslant j \leqslant n_0 - 1$. Also, we have $x_{2i,2j} + x_{2i,2j+1} = mn$ for $1 \leqslant i \leqslant m_0$ and $1 \leqslant j \leqslant n_0 - 1$. Thus, for $1 \leqslant i \leqslant m - 1$ and $1 \leqslant j \leqslant n_0 - 1$, we have

$$x_{i,2j} + x_{i,2j+1} + x_{i+1,2j} + x_{i+1,2j+1} = 2mn + 2.$$

We observe that, for $1 \leqslant j \leqslant n_0$, $x_{1,2j-1} + x_{m,n+2-2j} = \frac{1}{2}mn + 1$ and for $1 \leqslant j \leqslant n_0$, $x_{1,2j} + x_{m,n+1-2j} = \frac{3}{2}mn + 1$. Thus, for $1 \leqslant j \leqslant n - 1$, we have

$$x_{1,j} + x_{m,n+1-j} + x_{1,j+1} + x_{m,n-j} = 2mn + 2.$$

Similarly, for $1 \leqslant i \leqslant m_0$, $x_{2i-1,1} + x_{m+2-2i,n} = \frac{1}{2}mn + 1$ and for $1 \leqslant i \leqslant m_0$, $x_{2i,1} + x_{m+1-2i,n} = \frac{3}{2}mn + 1$. Thus, for $1 \leqslant i \leqslant m - 1$, we have

$$x_{i,1} + x_{m+1-i,n} + x_{i+1,1} + x_{m-i,n} = 2mn + 2.$$

$\square$

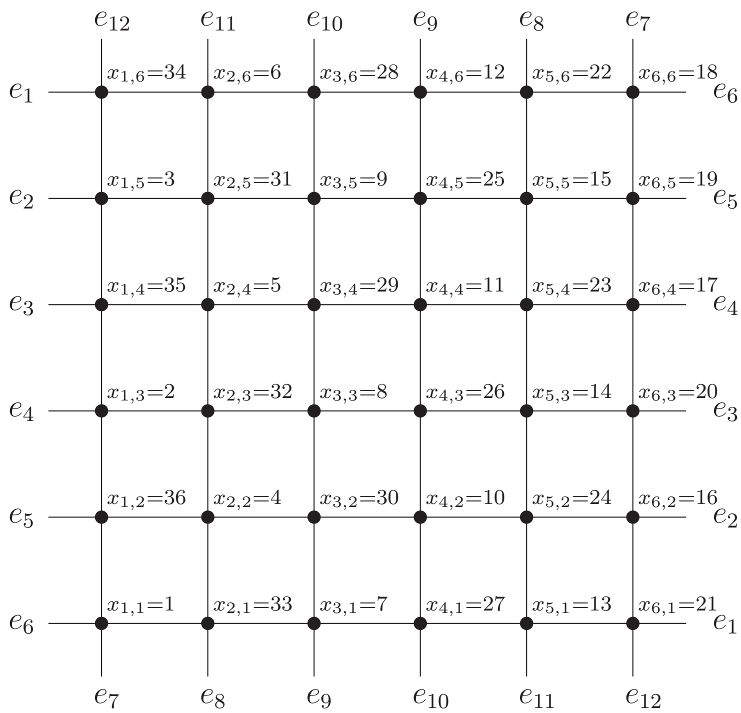

Figure 3: C_4-face-magic labeling on $\mathcal{P}_{6,6}$ having C_4-face-magic value 74.

3 C_4-face-magic projective grid graphs having an odd number of vertices and C_4-face-magic value $2mn + 2$

In this section we characterize the C_4-face-magic labelings on $\mathcal{P}_{m,n}$ having C_4-face-magic value $2mn + 2$ when m and n are odd. In Lemma 3.4, we show that a C_4-face-magic labeling $X = \{x_{i,j} : (i, j) \in V(\mathcal{P}_{m,n})\}$ on $\mathcal{P}_{m,n}$ having C_4-face-magic value $2mn + 2$ is *centrally balanced* in the sense that

$$x_{i,j} + x_{m+1-i,n+1-j} = mn + 1 \quad \text{for all } (i, j) \in V(\mathcal{P}_{m,n}).$$

In Definitions 3.6, 3.8, 3.10, and 3.12, we introduce permutations on the rows and columns of X, called *elementary projective labeling operations* (see Definition 3.14), that result in another C_4-face-magic labeling on $\mathcal{P}_{m,n}$. See Lemmas 3.7, 3.9, 3.11 and 3.13. Among all C_4-face-magic labelings that can be obtained by applying a sequence of elementary projective labeling operations to X, there is a unique labeling Z in which the labels on both the central row and the central column of Z are in ascending order. This labeling Z is called the *standard projective labeling associated with* X (see Definition 3.17). Thus, we only need to characterize the standard centrally balanced C_4-face-magic labelings on $\mathcal{P}_{m,n}$. See Theorem 3.16. In Definition 3.18, we introduce the concept of a *palindromic sequence labeling* on the $m \times n$ planar grid graph $P_m \times P_n$. In Propositions 3.20 and 3.21, we show that there is a one-to-one correspondence between the standard centrally balanced C_4-face-magic labelings on $\mathcal{P}_{m,n}$ and the palindromic sequence labelings on $P_m \times P_n$. We introduce the concept of an (m,n)-projective factorization sequence in Definition 3.28. In Theorem 3.32, we show that any palindromic sequence labeling on $P_m \times P_n$ can be constructed from an (m,n)-projective factorization sequence or an (n,m)-projective factorization sequence. Similarly, in Theorems 3.34 and 3.35, we show that any standard centrally balanced C_4-face magic labeling on $\mathcal{P}_{m,n}$ can be constructed from an (m,n)-projective factorization sequence or an (n,m)-projective factorization sequence. In fact, this is the only way to construct a standard centrally balanced C_4-face-magic labeling on $\mathcal{P}_{m,n}$. These results allow us to count the number of C_4-face-magic labelings on $\mathcal{P}_{m,n}$ having C_4-face-magic value $2mn+2$. See Theorems 3.40 and 3.41.

Notation 3.1. Throughout this section, we assume that both $m \geqslant 3$ and $n \geqslant 3$ are odd integers. We write $m = 2m_0 + 1$ and $n = 2n_0 + 1$ for integers m_0 and n_0. For any positive integer N, we let $N^+ = N + 1$. In particular, we have $m_0^+ = m_0 + 1$ and $n_0^+ = n_0 + 1$.

Notation 3.2. We refer to the vertex $(\frac{1}{2}(m+1), \frac{1}{2}(n+1)) = (m_0^+, n_0^+)$ as the *center* of the projective grid graph $\mathcal{P}_{m,n}$. The graph automorphisms of $\mathcal{P}_{m,n}$ that are induced by homeomorphisms of the projective plane are described in relation to the center of $\mathcal{P}_{m,n}$. We let R_θ denote the rotation by θ degrees in the counter-clockwise direction about the center. The symmetry H (V) is the reflection about the horizontal (vertical) axis passing through the center. Thus, for distinct integers m and n, the set of symmetries on $\mathcal{P}_{m,n}$ is $\{R_0, R_{180}, H, V\}$. We let D_+ (D_-) denote the reflection about the diagonal with positive (negative) slope passing through the center. When $m = n$, the set of symmetries on $\mathcal{P}_{m,m}$ is $D_4 = \{R_0, R_{90}, R_{180}, R_{270}, H, V, D_+, D_-\}$.

Definition 3.3. Let $X = \{x_{i,j} : (i,j) \in V(\mathcal{P}_{m,n})\}$ be a C_4-face-magic labeling on $\mathcal{P}_{m,n}$ with C_4-face value $S = 2(mn+1)$. We say that X is *centrally balanced* if, for all $(i,j) \in V(\mathcal{P}_{m,n})$,

$$x_{i,j} + x_{m+1-i,n+1-j} = \tfrac{1}{2}S = mn + 1.$$

Lemma 3.4. *Suppose $m \geqslant 3$ and $n \geqslant 3$ are odd integers. Let $X = \{x_{i,j} : (i,j) \in V(\mathcal{P}_{m,n})\}$ be a C_4-face-magic labeling on $\mathcal{P}_{m,n}$ with C_4-face-magic value $S = 2mn+2$. Then X is centrally balanced. Furthermore, $x_{m_0^+,n_0^+} = \frac{1}{2}mn + \frac{1}{2}$.*

Proof. By Lemma 2.6, the digons formed by the vertex sets $\{(1,1),(m,n)\}$ and $\{(m,1),(1,n)\}$ have face values

$$D_1 = x_{1,1} + x_{m,n} = \tfrac{1}{2}S = mn + 1$$

and

$$D_2 = x_{m,1} + x_{1,n} = \tfrac{1}{2}S = mn + 1.$$

Suppose that for some integer $1 \leqslant i < m$,

$$x_{i,1} + x_{m+1-i,n} = \tfrac{1}{2}S.$$

Since

$$x_{i,1} + x_{i+1,1} + x_{m+1-i,n} + x_{m-i,n} = S,$$

we have

$$x_{i+1,1} + x_{m-i,n} = \tfrac{1}{2}S.$$

Similarly, suppose that for some integer $1 \leqslant j < n$,

$$x_{1,j} + x_{m,n+1-j} = \tfrac{1}{2}S.$$

Since

$$x_{1,j} + x_{1,j+1} + x_{m,n+1-j} + x_{m,n-j} = S,$$

we have

$$x_{1,j+1} + x_{m,n-j} = \tfrac{1}{2}S.$$

Hence,

$$x_{i,1} + x_{m+1-i,n} = \tfrac{1}{2}S$$

for all $1 \leqslant i \leqslant m$ and

$$x_{1,j} + x_{m,n+1-j} = \tfrac{1}{2}S$$

for all $1 \leqslant j \leqslant n$.

Suppose there exist integers $1 < i < m$ and $1 < j < n$ such that

1. for all $1 \leqslant i' < i$ and $1 \leqslant j' \leqslant n$, $x_{i',j'} + x_{m+1-i',n+1-j'} = \frac{1}{2}S$ and
2. for all $1 \leqslant j' < j$, $x_{i,j'} + x_{m+1-i,n+1-j'} = \frac{1}{2}S$.

We need to show that $x_{i,j} + x_{m+1-i,n+1-j} = \frac{1}{2}S$. When we add the two C_4-face-values

$$x_{i-1,j-1} + x_{i-1,j} + x_{i,j-1} + x_{i,j} = S$$

and

$$x_{m+2-i,n+2-j} + x_{m+2-i,n+1-j} + x_{m+1-i,n+2-j} + x_{m+1-i,n+1-j} = S,$$

we obtain

$$\begin{aligned}&\big(x_{i-1,j-1} + x_{m+2-i,n+2-j}\big) + \big(x_{i-1,j} + x_{m+2-i,n+1-j}\big)\\ &+\big(x_{i,j-1} + x_{m+1-i,n+2-j}\big) + \big(x_{i,j} + x_{m+1-i,n+1-j}\big) = 2S.\end{aligned}$$

Since

$$\begin{aligned}x_{i-1,j-1} + x_{m+2-i,n+2-j} &= \tfrac{1}{2}S,\\ x_{i-1,j} + x_{m+2-i,n+1-j} &= \tfrac{1}{2}S, \text{ and}\\ x_{i,j-1} + x_{m+1-i,n+2-j} &= \tfrac{1}{2}S,\end{aligned}$$

we have

$$x_{i,j} + x_{m+1-i,n+1-j} = \tfrac{1}{2}S.$$

Since

$$2x_{m_0^+,n_0^+} = x_{m_0^+,n_0^+} + x_{m+1-m_0^+,n+1-n_0^+} = mn + 1,$$

we have

$$x_{m_0^+,n_0^+} = \tfrac{1}{2}mn + \tfrac{1}{2}.$$

This completes the proof. $\square$

Lemma 3.5. *Let $X = \{x_{i,j} : (i,j) \in V(\mathcal{P}_{m,n})\}$ be a centrally balanced C_4-face-magic labeling on $\mathcal{P}_{m,n}$ with C_4-face-magic value $S = 2mn + 2$. For $1 \leqslant j \leqslant n_0$, let*

$$a_j = x_{1,j} + x_{1,j+1}.$$

Then,

1. *for all $1 \leqslant i \leqslant m_0$ where i is odd and $1 \leqslant j \leqslant n_0$, we have*

$$\begin{aligned} x_{i,j} + x_{i,j+1} &= a_j, & x_{i,n+1-j} + x_{i,n-j} &= S - a_j, \\ x_{m+1-i,j} + x_{m+1-i,j+1} &= a_j, \textit{ and} & x_{m+1-i,n+1-j} + x_{m+1-i,n-j} &= S - a_j, \quad \textit{and} \end{aligned}$$

2. *for all $1 \leqslant i \leqslant m_0$ where i is even and $1 \leqslant j \leqslant n_0$, we have*

$$\begin{aligned} x_{i,j} + x_{i,j+1} &= S - a_j, & x_{i,n+1-j} + x_{i,n-j} &= a_j, \\ x_{m+1-i,j} + x_{m+1-i,j+1} &= S - a_j, \textit{ and} & x_{m+1-i,n+1-j} + x_{m+1-i,n-j} &= a_j. \end{aligned}$$

Proof. When we equate the two C_4-face sums

$$\begin{gathered} x_{i,j} + x_{i,j+1} + x_{i+1,j} + x_{i+1,j+1} = S \quad \text{and} \\ x_{i+1,j} + x_{i+1,j+1} + x_{i+2,j} + x_{i+2,j+1} = S, \end{gathered}$$

we obtain

$$x_{i,j} + x_{i,j+1} = x_{i+2,j} + x_{i+2,j+1}. \tag{1}$$

By (1), for all $1 \leqslant i \leqslant m_0$ where i is odd and $1 \leqslant j \leqslant n_0$, we have

$$x_{i,j} + x_{i,j+1} = a_j \quad \text{and} \quad x_{m+1-i,j} + x_{m+1-i,j+1} = a_j.$$

Since

$$a_j + x_{2,j} + x_{2,j+1} = x_{1,j} + x_{1,j+1} + x_{2,j} + x_{2,j+1} = S,$$

we have

$$x_{2,j} + x_{2,j+1} = S - a_j.$$

By (1), for all $1 \leqslant i \leqslant m_0$ where i is even and $1 \leqslant j \leqslant n_0$, we have

$$x_{i,j} + x_{i,j+1} = S - a_j \quad \text{and} \quad x_{m+1-i,j} + x_{m+1-i,j+1} = S - a_j.$$

Since

$$a_j + x_{1,n+1-j} + x_{1,n-j} = x_{m,j} + x_{m,j+1} + x_{1,n+1-j} + x_{1,n-j} = S,$$

we have

$$x_{1,n+1-j} + x_{1,n-j} = S - a_j.$$

By (1), for all $1 \leqslant i \leqslant m_0$ where i is odd and $1 \leqslant j \leqslant n_0$, we have

$$x_{i,n+1-j} + x_{i,n-j} = S - a_j \quad \text{and} \quad x_{m+1-i,n+1-j} + x_{m+1-i,n-j} = S - a_j.$$

Since

$$(S - a_j) + x_{2,n+1-j} + x_{2,n-j} = x_{1,n+1-j} + x_{1,n-j} + x_{2,n+1-j} + x_{2,n-j} = S,$$

we have

$$x_{2,n+1-j} + x_{2,n-j} = a_j.$$

By (1), for all $1 \leqslant i \leqslant m_0$ where i is even and $1 \leqslant j \leqslant n_0$, we have

$$x_{i,n+1-j} + x_{i,n-j} = a_j \quad \text{and} \quad x_{m+1-i,n+1-j} + x_{m+1-i,n-j} = a_j.$$

□

Definition 3.6. Let $X = \{x_{i,j} : (i,j) \in V(\mathcal{P}_{m,n})\}$ be a centrally balanced C_4-face-magic labeling on $\mathcal{P}_{m,n}$. Let η be a permutation on the set $\{1, 2, \ldots, m_0\}$. We define a labeling on $\mathcal{P}_{m,n}$, $Z = \{z_{i,j} : (i,j) \in V(\mathcal{P}_{m,n})\}$, such that for all $1 \leqslant i \leqslant m_0$ and $1 \leqslant j \leqslant n$, we have

$$\begin{aligned}
z_{i,j} &= x_{\eta(i),j}, && \text{if } \eta(i) - i \text{ is even,}\\
z_{i,j} &= x_{\eta(i),n+1-j}, && \text{if } \eta(i) - i \text{ is odd,}\\
z_{m_0^+,j} &= x_{m_0^+,j}, && \\
z_{m+1-i,j} &= x_{m+1-\eta(i),j}, && \text{if } \eta(i) - i \text{ is even and}\\
z_{m+1-i,j} &= x_{m+1-\eta(i),n+1-j}, && \text{if } \eta(i) - i \text{ is odd.}
\end{aligned}$$

We let $\mathcal{E}_\eta$ denote the labeling operation given by $\mathcal{E}_\eta(X) = Z$.

Lemma 3.7. *Let $X = \{x_{i,j} : (i,j) \in V(\mathcal{P}_{m,n})\}$ be a centrally balanced C_4-face-magic labeling on $\mathcal{P}_{m,n}$ and let η be a permutation on the set $\{1, 2, \ldots, m_0\}$. Let $\mathcal{E}_\eta$ be the labeling operation defined in Definition 3.6. Then the labeling $Z = \mathcal{E}_\eta(X)$ is a centrally balanced C_4-face-magic labeling on $\mathcal{P}_{m,n}$.*

Proof. We first verify that Z is centrally balanced. Suppose that $1 \leqslant i \leqslant m_0$ and $1 \leqslant j \leqslant n$. If $\eta(i) - i$ is even, then

$$z_{i,j} + z_{m+1-i,n+1-j} = x_{\eta(i),j} + x_{m+1-\eta(i),n+1-j} = \tfrac{1}{2}S.$$

If $\eta(i) - i$ is odd, then

$$z_{i,j} + z_{m+1-i,n+1-j} = x_{\eta(i),n+1-j} + x_{m+1-\eta(i),j} = \tfrac{1}{2}S.$$

Furthermore, we have

$$z_{m_0^+,j} + z_{m+1-m_0^+,n+1-j} = x_{m_0^+,j} + x_{m+1-m_0^+,n+1-j} = \tfrac{1}{2}S.$$

Next, we show that Z is a C_4-face-magic labeling on $\mathcal{P}_{m,n}$. For all $1 \leqslant i < m$ and $1 \leqslant j < n$, one may use Lemma 3.5 to verify that

$$z_{i,j} + z_{i,j+1} = x_{i,j} + x_{i,j+1} \quad \text{and} \quad z_{i+1,j} + z_{i+1,j+1} = x_{i+1,j} + x_{i+1,j+1}.$$

Thus

$$\begin{aligned}&z_{i,j} + z_{i,j+1} + z_{i+1,j} + z_{i+1,j+1}\\&\qquad = x_{i,j} + x_{i,j+1} + x_{i+1,j} + x_{i+1,j+1} = S.\end{aligned}$$

Since Z is centrally balanced, for $1 \leqslant i < m$, we have

$$x_{i,n} + x_{m+1-i,1} + x_{i+1,n} + x_{m-i,1} = \tfrac{1}{2}S + \tfrac{1}{2}S = S.$$

Also, since Z is centrally balanced, for $1 \leqslant j < n$, we have

$$x_{m,j} + x_{1,n+1-j} + x_{m,j+1} + x_{1,n-j} = \tfrac{1}{2}S + \tfrac{1}{2}S = S.$$

□

Definition 3.8. Let $X = \{x_{i,j} : (i,j) \in V(\mathcal{P}_{m,n})\}$ be a centrally balanced C_4-face-magic labeling on $\mathcal{P}_{m,n}$. Let κ be a permutation on the set $\{1, 2, \ldots, n_0\}$. We define a labeling on $\mathcal{P}_{m,n}$, $Z = \{z_{i,j} : (i,j) \in V(\mathcal{P}_{m,n})\}$, such that for all $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n_0$, we have

$$\begin{aligned}
z_{i,j} &= x_{i,\kappa(j)}, && \text{if } \kappa(j) - j \text{ is even,}\\
z_{i,j} &= x_{m+1-i,\kappa(j)}, && \text{if } \kappa(j) - j \text{ is odd,}\\
z_{i,n_0^+} &= x_{i,n_0^+}, &&\\
z_{i,n+1-j} &= x_{i,n+1-\kappa(j)}, && \text{if } \kappa(j) - j \text{ is even and}\\
z_{i,n+1-j} &= x_{m+1-i,n+1-\kappa(j)}, && \text{if } \kappa(j) - j \text{ is odd.}
\end{aligned}$$

We let $\mathcal{E}_\kappa$ denote the labeling operation given by $\mathcal{E}_\eta(X) = Z$.

Lemma 3.9. *Let $X = \{x_{i,j} : (i,j) \in V(\mathcal{P}_{m,n})\}$ be a centrally balanced C_4-face-magic labeling on $\mathcal{P}_{m,n}$ and let κ be a permutation on the set $\{1, 2, \ldots, n_0\}$. Let $\mathcal{E}_\kappa$ be the labeling operation defined in Definition 3.8. Then the labeling $Z = \mathcal{E}_\kappa(X)$ is a centrally balanced C_4-face-magic labeling on $\mathcal{P}_{m,n}$.*

The proof of Lemma 3.9 is similar to the proof of Lemma 3.7; we leave the details of the proof to the reader.

Definition 3.10. Let $X = \{x_{i,j} : (i,j) \in V(\mathcal{P}_{m,n})\}$ be a centrally balanced C_4-face-magic labeling on $\mathcal{P}_{m,n}$. Let $\alpha : \{1, 2, \ldots, m_0\} \to \{0, 1\}$. We define a labeling on $\mathcal{P}_{m,n}$, $Z = \{z_{i,j} : (i,j) \in V(\mathcal{P}_{m,n})\}$, such that for all $1 \leqslant i \leqslant m_0$ and $1 \leqslant j \leqslant n$, we have

$$\begin{aligned}
z_{i,j} &= x_{(1-\alpha(i))i+\alpha(i)(m+1-i),j}, \quad \text{and}\\
z_{m+1-i,j} &= x_{\alpha(i)i+(1-\alpha(i))(m+1-i),j}.
\end{aligned}$$

We let $\mathcal{E}_\alpha$ denote the labeling operation given by $\mathcal{E}_\alpha(X) = Z$. The labeling operation $\mathcal{E}_\alpha$ has the effect of keeping the labelings on the vertices of columns i and $m+1-i$ the same if $\alpha(i) = 0$ and swapping the labelings on the vertices of column i with those of column $m+1-i$ if $\alpha(i) = 1$.

Lemma 3.11. *Let $X = \{x_{i,j} : (i,j) \in V(\mathcal{P}_{m,n})\}$ be a centrally balanced C_4-face-magic labeling on $\mathcal{P}_{m,n}$ and let $\alpha : \{1, 2, \ldots, m_0\} \to \{0, 1\}$. Let $\mathcal{E}_\alpha$ be the labeling operation defined in Definition 3.10. Then the labeling $Z = \mathcal{E}_\alpha(X)$ is a centrally balanced C_4-face-magic labeling on $\mathcal{P}_{m,n}$.*

Proof. First, we show that Z is centrally balanced. Suppose $\alpha(i) = 0$. Then

$$z_{i,j} = x_{i,j} \quad \text{and} \quad z_{m+1-i,j} = x_{m+1-i,j}.$$

Thus

$$z_{i,j} + z_{m+1-i,n+1-j} = x_{i,j} + x_{m+1-i,n+1-j} = \tfrac{1}{2}S.$$

Suppose $\alpha(i) = 1$. Then

$$z_{i,j} = x_{m+1-i,j} \quad \text{and} \quad z_{m+1-i,j} = x_{i,j}.$$

Thus

$$z_{i,j} + z_{m+1-i,n+1-j} = x_{m+1-i,j} + x_{i,n+1-j} = \tfrac{1}{2}S.$$

The proof that Z is a C_4-face-magic labeling on $\mathcal{P}_{m,n}$ is similar to that in the proof of Lemma 3.7. ☐

Definition 3.12. Let $X = \{x_{i,j} : (i,j) \in V(\mathcal{P}_{m,n})\}$ be a centrally balanced C_4-face-magic labeling on $\mathcal{P}_{m,n}$. Let $\beta : \{1, 2, \ldots, n_0\} \to \{0, 1\}$. We define a labeling on $\mathcal{P}_{m,n}$, $Z = \{z_{i,j} : (i,j) \in V(\mathcal{P}_{m,n})\}$, such that for all $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n_0$, we have

$$\begin{aligned} z_{i,j} &= x_{i,(1-\beta(j))j+\beta(j)(n+1-j)}, \quad \text{and} \\ z_{i,n+1-j} &= x_{i,\beta(j)j+(1-\beta(j))(n+1-j)}. \end{aligned}$$

We let $\mathcal{E}_\beta$ denote the labeling operation given by $\mathcal{E}_\beta(X) = Z$. The labeling operation $\mathcal{E}_\beta$ has the effect of keeping the labelings on the vertices of rows j and $n+1-j$ the same if $\beta(j) = 0$ and swapping the labelings on the vertices of row j with those of row $n+1-j$ if $\beta(j) = 1$.

Lemma 3.13. *Let $X = \{x_{i,j} : (i,j) \in V(\mathcal{P}_{m,n})\}$ be a centrally balanced C_4-face-magic labeling on $\mathcal{P}_{m,n}$ and let $\beta : \{1, 2, \ldots, n_0\} \to \{0, 1\}$. Let $\mathcal{E}_\beta$ be the labeling operation defined in Definition 3.12. Then the labeling $Z = \mathcal{E}_\beta(X)$ is a centrally balanced C_4-face-magic labeling on $\mathcal{P}_{m,n}$.*

The proof of Lemma 3.13 is similar to the proof of Lemma 3.11; we leave the details of the proof to the reader.

Definition 3.14. We call each of the labeling operations $\mathcal{E}_\eta$ in Definition 3.6, $\mathcal{E}_\kappa$ in Definition 3.8, $\mathcal{E}_\alpha$ in Definition 3.10 and $\mathcal{E}_\beta$ in Definition 3.12 an *elementary projective labeling operation.*

Definition 3.15. We say that two centrally balanced C_4-face-magic labelings on $\mathcal{P}_{m,n}$ are *projective labeling equivalent* if one labeling can be obtained from the other by applying a sequence of elementary projective labeling operations.

Given a centrally balanced C_4-face-magic labeling X on $\mathcal{P}_{m,n}$, the next theorem identifies a canonical centrally balanced C_4-face-magic labeling on $\mathcal{P}_{m,n}$ that is projective labeling equivalent to X.

Theorem 3.16. *Let $X = \{x_{i,j} : (i,j) \in V(\mathcal{P}_{m,n})\}$ be a centrally balanced C_4-face-magic labeling on $\mathcal{P}_{m,n}$. Then there is a unique centrally balanced C_4-face-magic labeling $Z = \{z_{i,j} : (i,j) \in V(\mathcal{P}_{m,n})\}$ on $\mathcal{P}_{m,n}$ that is projective labeling equivalent to X such that*

1. $z_{1,n_0^+} < z_{2,n_0^+} < \cdots < z_{m,n_0^+}$ *and*
2. $z_{m_0^+,1} < z_{m_0^+,2} < \cdots < z_{m_0^+,n}$.

Proof. By Lemma 3.4, we have $x_{m_0^+,n_0^+} = \frac{1}{2}(mn+1) = \frac{1}{4}S$. It is easy to check that this value remains the same for any elementary projective labeling operation that we apply to X. Since X is centrally balanced, for all $1 \leqslant i \leqslant m_0$, we have

$$x_{i,n_0^+} + x_{m+1-i,n_0^+} = \tfrac{1}{2}S.$$

Thus, either $x_{i,n_0^+} < \frac{1}{4}S$ or $x_{m+1-i,n_0^+} < \frac{1}{4}S$. We define a function $\alpha : \{1,2,\ldots,m_0\} \to \{0,1\}$ as follows. For each $1 \leqslant i \leqslant m_0$, we define

$$\alpha(i) = \begin{cases} 0, & \text{if } x_{i,n_0^+} < \frac{1}{4}S, \\ 1, & \text{if } x_{m+1-i,n_0^+} < \frac{1}{4}S. \end{cases}$$

We replace X with $\mathcal{E}_\alpha(X)$. This new centrally balanced C_4-face-magic labeling on $\mathcal{P}_{m,n}$ satisfies, for all $1 \leqslant i \leqslant m_0$,

$$\begin{aligned} x_{i,n_0^+} &< \tfrac{1}{4}S, \quad \text{and} \\ x_{m+1-i,n_0^+} &> \tfrac{1}{4}S. \end{aligned}$$

Choose a permutation η of $\{1,2,\ldots,m_0\}$ such that

$$x_{\eta(1),n_0^+} < x_{\eta(2),n_0^+} < \cdots < x_{\eta(m_0),n_0^+}.$$

We replace X with $\mathcal{E}_\eta(X)$. This new centrally balanced C_4-face-magic labeling on $\mathcal{P}_{m,n}$ satisfies,

$$x_{1,n_0^+} < x_{2,n_0^+} < \cdots < x_{m,n_0^+}$$

A similar argument allows us to choose a function $\beta : \{1,2,\ldots,n_0\} \to \{0,1\}$ and a permutation κ on $\{1,2,\ldots,n_0\}$ such that $Z = \mathcal{E}_\kappa(\mathcal{E}_\beta(X))$ satisfies

$$\begin{aligned} z_{1,n_0^+} &< z_{2,n_0^+} < \cdots < z_{m,n_0^+}, \quad \text{and} \\ z_{m_0^+,1} &< z_{m_0^+,2} < \cdots < z_{m_0^+,n}. \end{aligned}$$

□

Definition 3.17. We refer to the centrally balanced C_4-face-magic labeling Z in Theorem 3.16 as the *standard projective labeling associated with* X. We say that Z is a standard centrally balanced C_4-face-magic labeling on $\mathcal{P}_{m,n}$.

As a result of Theorem 3.16, we need only find all standard centrally balanced C_4-face-magic projective labelings on $\mathcal{P}_{m,n}$. See Table 2 for an example of a standard centrally balanced C_4-face-magic projective labeling on $\mathcal{P}_{9,9}$.

Definition 3.18. Let $Y = \{y_{i,j} : (i,j) \in V(P_m \times P_n)\}$ be a labeling on the planar grid graph $P_m \times P_n$. Suppose there exist palindromic sequences of positive integers $(a_1, a_2, \ldots, a_{m-1})$ and $(b_1, b_2, \ldots, b_{n-1})$. For convenience, let $a_0 = 0$ and $b_0 = 0$. We say that Y is a *palindromic sequence labeling* on $P_m \times P_n$ provided that,

1. $Y = \{1, 2, \ldots, mn\}$ and
2. for all $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$, we have

$$y_{i,j} = y_{1,1} + \left(\sum_{k=0}^{i-1} a_k\right) + \left(\sum_{\ell=0}^{j-1} b_\ell\right).$$

Definition 3.19. Let $X = \{x_{i,j} : (i,j) \in V(\mathcal{P}_{m,n})\}$ be a standard centrally balanced C_4-face-magic labeling on $\mathcal{P}_{m,n}$. The *palindromic sequence labeling associated with* X is the labeling on the planar grid graph $P_m \times P_n$ given by $Y = \{y_{i,j} : (i,j) \in V(P_m \times P_n)\}$ where

$$y_{m_0^+ + \sigma_1 i, n_0^+ + \sigma_2 j} = x_{m_0^+ + (-1)^j \sigma_1 i, n_0^+ + (-1)^i \sigma_2 j}$$

for all $0 \leqslant i \leqslant m_0$, $0 \leqslant j \leqslant n_0$ and $\sigma_1, \sigma_2 \in \{-1, 1\}$. We refer to the transformation $\mathcal{T}$ defined by $\mathcal{T}(X) = Y$ as the *projective to palindromic sequence transformation.*

Proposition 3.20. *Suppose $X = \{x_{i,j} : (i,j) \in V(\mathcal{P}_{m,n})\}$ is a standard centrally balanced C_4-face-magic labeling on $\mathcal{P}_{m,n}$. Let $Y = \mathcal{T}(X) = \{y_{i,j} : (i,j) \in V(P_m \times P_n)\}$ where*

$$y_{m_0^+ + \sigma_1 i, n_0^+ + \sigma_2 j} = x_{m_0^+ + (-1)^j \sigma_1 i, n_0^+ + (-1)^i \sigma_2 j}$$

for all $0 \leqslant i \leqslant m_0$, $0 \leqslant j \leqslant n_0$ and $\sigma_1, \sigma_2 \in \{-1, 1\}$. Then Y is a palindromic sequence labeling on $P_m \times P_n$.

Proof. We first observe that $Y = X = \{1, 2, \ldots, mn\}$.

For each $1 \leqslant i \leqslant m_0$, let $c_i = x_{m_0^+ + 1 - i, n_0^+} - x_{m_0^+ - i, n_0^+}$. Similarly, for each $1 \leqslant j \leqslant m_0$, let $d_j = x_{m_0^+, n_0^+ + 1 - j} - x_{m_0^+, n_0^+ - j}$. Since X is a standard centrally balanced C_4-face-magic labeling on $\mathcal{P}_{m,n}$, we have

$$\begin{aligned} x_{m_0^+ + 1 - i, n_0^+} &> x_{m_0^+ - i, n_0^+}, && \text{for all } 1 \leqslant i \leqslant m_0, \text{ and} \\ x_{m_0^+, n_0^+ + 1 - j} &> x_{m_0^+, n_0^+ - j}, && \text{for all } 1 \leqslant i \leqslant n_0. \end{aligned}$$

Thus, c_i is positive for all $1 \leqslant i \leqslant m_0$ and d_j is positive for all $1 \leqslant j \leqslant n_0$.

Let $c_0 = 0$ and $d_0 = 0$. By Lemma 3.4, we have $x_{m_0^+,n_0^+} = \frac{1}{2}(mn+1) = \frac{1}{4}S$. By the definitions of c_i and d_j, we have

$$x_{m_0^+-i,n_0^+} = x_{m_0^+,n_0^+} - \sum_{k=0}^{i} c_k, \qquad \text{for } 0 \leqslant i \leqslant m_0, \text{ and}$$

$$x_{m_0^+,n_0^+-j} = x_{m_0^+,n_0^+} - \sum_{\ell=0}^{j} d_\ell, \qquad \text{for } 0 \leqslant j \leqslant n_0.$$

Since X is centrally balanced, we have

$$x_{m_0^++i,n_0^+} = x_{m_0^+,n_0^+} + \sum_{k=0}^{i} c_k, \qquad \text{for } 0 \leqslant i \leqslant m_0, \text{ and}$$

$$x_{m_0^+,n_0^++j} = x_{m_0^+,n_0^+} + \sum_{\ell=0}^{j} d_\ell, \qquad \text{for } 0 \leqslant j \leqslant n_0.$$

Hence,

$$x_{m_0^++\sigma_1 i,n_0^+} = x_{m_0^+,n_0^+} + \sigma_1\left(\sum_{k=0}^{i} c_k\right), \qquad \text{for } 0 \leqslant i \leqslant m_0 \text{ and } \sigma_1 \in \{-1,1\}, \text{ and} \tag{2}$$

$$x_{m_0^+,n_0^++\sigma_2 j} = x_{m_0^+,n_0^+} + \sigma_2\left(\sum_{\ell=0}^{j} d_\ell\right), \qquad \text{for } 0 \leqslant j \leqslant n_0 \text{ and } \sigma_2 \in \{-1,1\}. \tag{3}$$

Since X is a C_4-face-magic labeling on $\mathcal{P}_{m,n}$, (2) and (3) uniquely determine the values of X which are given by

$$x_{m_0^++\sigma_1 i,n_0^++\sigma_2 j} = \tfrac{1}{4}S + (-1)^j\sigma_1\left(\sum_{k=0}^{i} c_k\right) + (-1)^i\sigma_2\left(\sum_{\ell=0}^{j} d_\ell\right), \tag{4}$$

for all $0 \leqslant i \leqslant m_0$, $0 \leqslant j \leqslant n_0$ and $\sigma_1, \sigma_2 \in \{-1,1\}$. In order to verify (4), we need to show that the face sums of each C_4-face on $\mathcal{P}_{m,n}$ is $S = 2mn+2$. We replace i with $i \pm 1$ and j with $j \pm 1$ in (4) to obtain

$$x_{m_0^++\sigma_1(i\pm1),n_0^++\sigma_2(j\pm1)} = \tfrac{1}{4}S + (-1)^{j\pm1}\sigma_1\left(\sum_{k=0}^{i\pm1} c_k\right) + (-1)^{i\pm1}\sigma_2\left(\sum_{\ell=0}^{j\pm1} d_\ell\right), \tag{5}$$

$$x_{m_0^++\sigma_1(i\pm1),n_0^++\sigma_2 j} = \tfrac{1}{4}S + (-1)^{j}\sigma_1\left(\sum_{k=0}^{i\pm1} c_k\right) + (-1)^{i\pm1}\sigma_2\left(\sum_{\ell=0}^{j} d_\ell\right), \text{ and} \tag{6}$$

$$x_{m_0^++\sigma_1 i,n_0^++\sigma_2(j\pm1)} = \tfrac{1}{4}S + (-1)^{j\pm1}\sigma_1\left(\sum_{k=0}^{i} c_k\right) + (-1)^{i}\sigma_2\left(\sum_{\ell=0}^{j\pm1} d_\ell\right), \tag{7}$$

where $0 < i \leqslant m_0$ if $i \pm 1$ represents $i-1$, $0 \leqslant i < m_0$ if $i \pm 1$ represents $i+1$, $0 < j \leqslant n_0$ if $j \pm 1$ represents $j-1$ and $0 \leqslant j < n_0$ if $j \pm 1$ represents $j+1$. Adding

(4), (5), (6) and (7), yields

$$x_{m_0^+ +\sigma_1(i\pm 1),n_0^+ +\sigma_2(j\pm 1)} + x_{m_0^+ +\sigma_1(i\pm 1),n_0^+ +\sigma_2 j} \\ + x_{m_0^+ +\sigma_1 i,n_0^+ +\sigma_2(j\pm 1)} + x_{m_0^+ +\sigma_1 i,n_0^+ +\sigma_2 j} = S.$$

For convenience, let $c_{m_0^+} = 0$ and $d_{n_0^+} = 0$. By (4) and

$$y_{m_0^+ +\sigma_1 i,n_0^+ +\sigma_2 j} = x_{m_0^+ +(-1)^j\sigma_1 i,n_0^+ +(-1)^i\sigma_2 j},$$

we have

$$y_{m_0^+ +\sigma_1 i,n_0^+ +\sigma_2 j} = y_{m_0^+,n_0^+} + \sigma_1\left(\sum_{s=0}^{i} c_s\right) + \sigma_2\left(\sum_{t=0}^{j} d_t\right), \tag{8}$$

for all $0 \leqslant i \leqslant m_0^+$, $0 \leqslant j \leqslant n_0^+$ and $\sigma_1, \sigma_2 \in \{-1, 1\}$.

We need to show that $Y = \{y_{i,j} : (i, j) \in V(P_m \times P_n)\}$ is a palindromic sequence labeling on $P_m \times P_n$. Let

$$a_k = c_{m_0^+ -k}, \qquad \text{for } 0 \leqslant k \leqslant m_0, \tag{9}$$

$$a_k = c_{k-m_0}, \qquad \text{for } m_0^+ \leqslant k \leqslant m-1, \tag{10}$$

$$b_\ell = d_{n_0^+ -\ell}, \qquad \text{for } 0 \leqslant \ell \leqslant n_0, \text{ and} \tag{11}$$

$$b_\ell = d_{\ell-n_0}, \qquad \text{for } n_0^+ \leqslant \ell \leqslant n-1. \tag{12}$$

Then $(a_1, a_2, \ldots, a_{m-1})$ and $(b_1, b_2, \ldots, b_{n-1})$ are palindromic sequences. Observe that $a_0 = c_{m_0^+}, = 0$ and $b_0 = d_{n_0^+} = 0$. We need to show that for all $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$, we have

$$y_{i,j} = y_{1,1} + \sum_{k=0}^{i-1} a_k + \sum_{\ell=0}^{j-1} b_\ell. \tag{13}$$

Case 1. Assume $1 \leqslant i \leqslant m_0^+$ and $1 \leqslant j \leqslant n_0^+$. Let $\sigma_1 = -1$ and $\sigma_2 = -1$. From (8), we have

$$y_{m_0^+ -i',n_0^+ -j'} = y_{m_0^+,n_0^+} - \left(\sum_{s=0}^{i'} c_s\right) - \left(\sum_{t=0}^{j'} d_t\right), \text{ and} \tag{14}$$

$$y_{1,1} = y_{m_0^+,n_0^+} - \left(\sum_{s=0}^{m_0^+} c_s\right) - \left(\sum_{t=0}^{n_0^+} d_t\right). \tag{15}$$

Recall that $c_{m_0^+} = 0$ and $d_{n_0^+} = 0$. Subtracting (15) from (14) yields

$$y_{m_0^+ -i',n_0^+ -j'} - y_{1,1} = \left(\sum_{s=i'+1}^{m_0^+} c_s\right) + \left(\sum_{t=j'+1}^{n_0^+} d_t\right).$$

Replacing i' with $m_0^+ - i$ and j' with $n_0^+ - j$ yields

$$\begin{aligned} y_{i,j} - y_{1,1} &= \sum_{s=m_0^+-i+1}^{m_0^+} c_s + \sum_{t=n_0^+-j+1}^{n_0^+} d_t \\ &= \sum_{k=0}^{i-1} c_{m_0^+-k} + \sum_{\ell=0}^{j-1} d_{n_0^+-\ell}. \end{aligned}$$

Hence, by (9) and (11), (13) holds for $1 \leqslant i \leqslant m_0^+$ and $1 \leqslant j \leqslant n_0^+$.

Case 2. Assume $m_0^+ < i \leqslant m$ and $n_0^+ < j \leqslant n$. From (8), we have

$$y_{m_0^++i',n_0^++j'} = y_{m_0^+,n_0^+} + \left(\sum_{s=1}^{i'} c_s\right) + \left(\sum_{t=1}^{j'} d_t\right), \text{ and} \tag{16}$$

$$y_{1,1} = y_{m_0^+,n_0^+} - \left(\sum_{s=1}^{m_0^+} c_s\right) - \left(\sum_{t=1}^{n_0^+} d_t\right). \tag{17}$$

Again, recall that $c_{m_0^+} = 0$ and $d_{n_0^+} = 0$. Subtracting (17) from (16) yields

$$y_{m_0^++i',n_0^++j'} - y_{1,1} = \left(\sum_{s=1}^{m_0^+} c_s\right) + \left(\sum_{s=1}^{i'} c_s\right) + \left(\sum_{t=1}^{n_0^+} d_t\right) + \left(\sum_{t=1}^{j'} d_t\right).$$

Replacing i' with $i - m_0^+$ and j' with $j - n_0^+$ yields

$$\begin{aligned} y_{i,j} - y_{1,1} &= \sum_{s=1}^{m_0^+} c_s + \sum_{s=1}^{i-m_0^+} c_s + \sum_{t=1}^{n_0^+} d_t + \sum_{t=1}^{j-n_0^+} d_t \\ &= \sum_{k=0}^{m_0} c_{m_0^+-k} + \sum_{k=m_0^+}^{i-1} c_{k-m_0} + \sum_{\ell=0}^{n_0} d_{n_0^+-\ell} + \sum_{\ell=n_0^+}^{j-1} d_{\ell-n_0} \end{aligned}$$

Hence, by (9), (10), (11) and (12), (13) holds for $m_0^+ < i \leqslant m$ and $n_0^+ < j \leqslant n$.

A similar argument to those in Cases 1 and 2 shows that (13) holds when either $1 \leqslant i \leqslant m_0^+$ and $n_0^+ < j \leqslant n$, or $m_0^+ < i \leqslant m$ and $1 \leqslant j \leqslant n_0^+$. □

Proposition 3.21. *Suppose $Y = \{y_{i,j} : (i,j) \in V(P_m \times P_n)\}$ is a palindromic sequence labeling on $P_m \times P_n$. Let $X = \mathcal{T}(Y) = \{x_{i,j} : (i,j) \in V(\mathcal{P}_{m,n})\}$ where*

$$x_{m_0^++\sigma_1 i,n_0^++\sigma_2 j} = y_{m_0^++(-1)^j\sigma_1 i,n_0^++(-1)^i\sigma_2 j}$$

for all $0 \leqslant i \leqslant m_0$, $0 \leqslant j \leqslant n_0$ and $\sigma_1, \sigma_2 \in \{-1, 1\}$. Then X is a standard centrally balanced C_4-face-magic labeling on $\mathcal{P}_{m,n}$.

Proof. Let $(a_1, a_2, \ldots, a_{m-1})$ and $(b_1, b_2, \ldots, b_{n-1})$ be the palindromic sequences used in Y. Let $a_0 = 0$ and $b_0 = 0$. Then, for all $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$,

$$y_{i,j} = y_{1,1} + \sum_{k=0}^{i-1} a_k + \sum_{\ell=0}^{j-1} b_\ell. \tag{18}$$

Since $y_{1,1}$ is the smallest label in Y, we have $y_{1,1} = 1$. Also, $y_{m,n}$ is the largest label in Y. Thus $y_{m,n} = mn$. By equation (18),

$$mn = 1 + \sum_{k=0}^{m-1} a_k + \sum_{\ell=0}^{n-1} b_\ell = 1 + 2\left(\sum_{k=1}^{m_0} a_k\right) + 2\left(\sum_{\ell=1}^{n_0} b_\ell\right).$$

Thus

$$y_{m_0^+,n_0^+} = 1 + \sum_{k=1}^{m_0} a_k + \sum_{\ell=1}^{n_0} b_\ell = \tfrac{1}{2}(mn+1).$$

Let $S = 2(mn+1)$. Then $y_{m_0^+,n_0^+} = \frac{1}{4}S$.

Let $c_0 = 0$ and $d_0 = 0$. For $1 \leqslant k \leqslant m_0$ and $1 \leqslant \ell \leqslant n_0$, Let

$$\begin{aligned} c_k &= a_{m_0^+ - k} = a_{k+m_0} \text{ and} \\ d_\ell &= b_{n_0^+ - \ell} = b_{\ell+n_0}. \end{aligned}$$

We can show that

$$y_{m_0^+ + \sigma_1 i, n_0^+ + \sigma_2 j} = y_{m_0^+,n_0^+} + \sigma_1\left(\sum_{s=0}^{i} c_s\right) + \sigma_2\left(\sum_{t=0}^{j} d_t\right), \tag{19}$$

for all $0 \leqslant i \leqslant m_0^+$, $0 \leqslant j \leqslant n_0^+$ and $\sigma_1, \sigma_2 \in \{-1, 1\}$. The proof that equation (19) follows from equation (18) is similar to the proof that equation (13) follows from equation (8) in Proposition 3.20.

Since $X = \mathcal{T}(Y)$ and $y_{m_0^+,n_0^+} = \frac{1}{4}S$, we have

$$x_{m_0^+ + \sigma_1 i, n_0^+ + \sigma_2 j} = \tfrac{1}{4}S + (-1)^j \sigma_1\left(\sum_{k=0}^{i} c_k\right) + (-1)^i \sigma_2\left(\sum_{\ell=0}^{j} d_\ell\right),$$

for all $0 \leqslant i \leqslant m_0$, $0 \leqslant j \leqslant n_0$ and $\sigma_1, \sigma_2 \in \{-1, 1\}$. It is straight forward to show that X is a standard centrally balanced C_4-face-magic labeling on $\mathcal{P}_{m,n}$. We leave the details to the reader. □

Remark 3.22. The graphs $\mathcal{P}_{m,n}$ and $P_m \times P_n$ have the same vertex set. We observe that the projective to palindromic sequence transformation $\mathcal{T}$ has the effect of applying the symmetry $H^i V^j$ to the set of vertices $\{(m_0^+ \pm i, n_0^+ \pm j)\}$ in $V(\mathcal{P}_{m,n}) = V(P_m \times P_n)$.

By Proposition 3.20, when we apply this transformation to a standard centrally balanced C_4-face-magic labeling X on $\mathcal{P}_{m,n}$, the result is a palindromic sequence labeling $Y = \mathcal{T}(X)$ on $P_m \times P_n$.

By Proposition 3.21, when we apply this transformation to a palindromic sequence labeling Y on $P_m \times P_n$, the result is a standard centrally balanced C_4-face-magic labeling $X = \mathcal{T}(Y)$ on $\mathcal{P}_{m,n}$.

Since $\mathcal{T}$ is an involution, $\mathcal{T}$ is a one-to-one correspondence between standard centrally balanced C_4-face-magic projective labelings on $\mathcal{P}_{m,n}$ and palindromic sequence labelings on $P_m \times P_n$.

Definition 3.23. The *horizontal lexicographic labeling* $\mathrm{HLL}(m, n) = \{x_{i,j} : (i, j) \in V(P_m \times P_n)\}$ on $P_m \times P_n$ is defined by

$$x_{i,j} = i + m(j - 1)$$

for all $(i, j) \in V(P_m \times P_n)$.

Similarly, the *vertical lexicographic labeling* $\mathrm{VLL}(m, n) = \{x_{i,j} : (i, j) \in V(P_m \times P_n)\}$ on $P_m \times P_n$ is defined by

$$x_{i,j} = j + n(i - 1)$$

for all $(i, j) \in V(P_m \times P_n)$.

Notation 3.24. Let $(a_1, a_2, \dots, a_n)$ be a sequence of positive integers and let r be a positive integer. The *concatenation* of r copies of $(a_1, a_2, \dots, a_n)$ is denoted by

$$(a_1, a_2, \dots, a_n)^r = (a_1, a_2, \dots, a_n, a_1, a_2, \dots, a_n, \dots, a_1, a_2, \dots, a_n)$$

where there are r copies of $(a_1, a_2, \dots, a_n)$ in this sequence. For example,

$$(1, 5, 8)^3 = (1, 5, 8, 1, 5, 8, 1, 5, 8).$$

Remark 3.25. The palindromic sequences related to the horizontal lexicographic labeling $\mathrm{HLL}(m, n)$ on $P_m \times P_n$ are

$$(a_1, a_2, \dots, a_{m-1}) = (1)^{m-1}, \quad \text{and}$$
$$(b_1, b_2, \dots, b_{n-1}) = (m)^{n-1}.$$

The palindromic sequences related to the vertical lexicographic labeling $\mathrm{VLL}(m,n)$ on $P_m \times P_n$ are

$$(a_1, a_2, \dots, a_{m-1}) = (n)^{m-1}, \quad \text{and}$$
$$(b_1, b_2, \dots, b_{n-1}) = (1)^{n-1}.$$

Definition 3.26. Let $X = \{x_{i,j} : (i, j) \in V(P_m \times P_n)\}$ be a palindromic sequence labeling on $P_m \times P_n$, and let r be a positive integer. The *r-horizontal connected sum of X*, denoted by $Y = \mathrm{HCS}^r(X)$, is the palindromic sequence labeling on $P_{mr} \times P_n$ given by

$$y_{mk+i,j} = (mn)k + x_{i,j}, \quad \text{for all } 0 \leqslant k < r, \ 1 \leqslant i \leqslant m, \text{ and } 1 \leqslant j \leqslant n.$$

Similarly, the *r-vertical connected sum of X*, denoted by $Y = \mathrm{VCS}^r(X)$, is the palindromic sequence labeling on $P_m \times P_{nr}$ given by

$$y_{i,nk+j} = (mn)k + x_{i,j}, \quad \text{for all } 0 \leqslant k < r, \ 1 \leqslant i \leqslant m, \text{ and } 1 \leqslant j \leqslant n.$$

Remark 3.27. Let $X = \{x_{i,j} : (i,j) \in V(P_m \times P_n)\}$ be a palindromic sequence labeling on $P_m \times P_n$ that uses the palindromic sequences $(a_1, a_2, \ldots, a_{m-1})$ and $(b_1, b_2, \ldots, b_{n-1})$. Then the r-horizontal connected sum of X is a palindromic sequence labeling on $P_{mr} \times P_n$ that uses the palindromic sequences

$$(a'_1, a'_2, \ldots, a'_{mr-1}) = (a_1, a_2, \ldots, a_{m-1}, (A, a_1, a_2, \ldots, a_{m-1})^{r-1}) \text{ and}$$
$$(b'_1, b'_2, \ldots, b'_{n-1}) = (b_1, b_2, \ldots, b_{n-1})$$

where $A = 1 + b_1 + b_2 \cdots + b_{n-1}$.

Similarly, the r-vertical connected sum of X is a palindromic sequence labeling on $P_{mr} \times P_n$ that uses the palindromic sequences

$$(a'_1, a'_2, \ldots, a'_{m-1}) = (a_1, a_2, \ldots, a_{m-1}) \text{ and}$$
$$(b'_1, b'_2, \ldots, b'_{nr-1}) = (b_1, b_2, \ldots, b_{n-1}, (B, b_1, b_2, \ldots, b_{n-1})^{r-1})$$

where $B = 1 + a_1 + a_2 \cdots + a_{m-1}$.

We introduce the following definition in order to discuss the main results of this paper.

Definition 3.28. Suppose there exists a positive integer k such that one of the two following conditions holds.

1. There are factorizations of $m = m_1 m_2 \ldots m_k$ and $n = n_1 n_2 \ldots n_k$, where $m_i > 1$ and $n_i > 1$ for all $1 \leqslant i \leqslant k$.

2. There are factorizations of $m = m'_1 m'_2 \ldots m'_k m'_{k+1}$ and $n = n'_1 n'_2 \ldots n'_k$, where $m'_i > 1$ for all $1 \leqslant i \leqslant k+1$ and $n'_i > 1$ for all $1 \leqslant i \leqslant k$.

We say that $(m_1, n_1, m_2, n_2, \ldots, m_k, n_k)$ is an (m,n)-*projective factorization sequence of length* $2k$. Also, we say $(m'_1, n'_1, m'_2, n'_2, \ldots, m'_k, n'_k, m'_{k+1})$ is an (m,n)-*projective factorization sequence of length* $2k+1$. For convenience, we let $n'_{k+1} = 1$ and refer to $(m'_1, n'_1, m'_2, n'_2, \ldots, m'_{k+1}, n'_{k+1})$ as an (m,n)-*projective factorization sequence of length* $2k+1$. In addition, we say that $(m_1, n_1, m_2, n_2, \ldots, m_k, n_k)$ and $(m'_1, n'_1, m'_2, n'_2, \ldots, m'_{k+1}, n'_{k+1})$ are (m,n)-*projective factorization sequences.*

Furthermore, we let $\tau(m,n)$ denote the number of distinct (m,n)-*projective factorization sequences.*

Notation 3.29. Let k be a positive integer. Let $m_1, m_2, \ldots, m_k$ and $n_1, n_2, \ldots, n_k$ be integers greater than 1 except possibly n_k (for which $n_k \geqslant 1$). Let $X_1 = \mathrm{HLL}(m_1, n_1)$. For $2 \leqslant i \leqslant k$, let $X_i = \mathrm{VCS}^{n_i}(\mathrm{HCS}^{m_i}(X_{i-1}))$. Let $M = m_1 m_2 \cdots m_k$ and $N = n_1 n_2 \cdots n_k$. By Remarks 3.25 and 3.27, X_k is a palindromic sequence labeling on $P_M \times P_N$.

Let $X = \{x_{i,j} : (i,j) \in V(P_m \times P_n)\}$ be a palindromic sequence labeling on $P_m \times P_n$. Let $\mathrm{Grid}(m', n') = \{(i,j) : 1 \leqslant i \leqslant m' \text{ and } 1 \leqslant j \leqslant n'\}$. Let $\mathrm{Label}_X(\mathrm{Grid}(m', n')) = \{x_{i,j} : (i,j) \in \mathrm{Grid}(m', n')\}$.

Definition 3.30. Let $m \geqslant 3$ and $n \geqslant 3$ be odd integers.

1. Let $F = (m_i, n_i : 1 \leqslant i \leqslant k)$ be an (m, n)-projective factorization sequence. Let $X_1 = \mathrm{HLL}(m_1, n_1)$. For $2 \leqslant i \leqslant k$, let $Y_i = \mathrm{HCS}^{m_i}(X_{i-1})$ and $X_i = \mathrm{VCS}^{n_i}(Y_i)$. The *horizontal palindromic sequence labeling associated with* F is given by $\mathrm{HPSL}(F) = X_k$.

2. Let $F' = (n'_i, m'_i : 1 \leqslant i \leqslant k)$ be an (n, m)-projective factorization sequence. Let $X'_1 = \mathrm{VLL}(m'_1, n'_1)$. For $2 \leqslant i \leqslant k$, let $Y'_i = \mathrm{VCS}^{n'_i}(X'_{i-1})$ and $X'_i = \mathrm{HCS}^{m'_i}(Y'_i)$. The *vertical palindromic sequence labeling associated with* F' is given by $\mathrm{VPSL}(F') = X'_k$.

Lemma 3.31. *Let $X = \{x_{i,j} : (i, j) \in V(P_m \times P_n)\}$ be a palindromic sequence labeling on $P_m \times P_n$. Let $F = (m_i, n_i : 1 \leqslant i \leqslant k)$ be an (m, n)-projective factorization sequence and let $W = \mathrm{HPSL}(F)$. Suppose $x_{i,j} = w_{i,j}$ for all $(i, j) \in \mathrm{Grid}(m', n')$. Let z be smallest positive integer such that $z \in \mathrm{Label}_X(\mathrm{Grid}(m', n'))$ and $z + 1 \notin \mathrm{Label}_X(\mathrm{Grid}(m', n'))$. Then either $x_{m'+1,1} = z + 1$ or $x_{1,n'+1} = z + 1$.*

Proof. The labels $1, 2, \ldots, z$ appear in $\mathrm{Label}_X(\mathrm{Grid}(m', n'))$. We observe that $x_{m'+1,1} < x_{i,j}$ for all $i > m'+1$, or $i = m'+1$ and $j > 1$. Also, $x_{1,n'+1} < x_{i,j}$ for all $j > n'+1$, or $j = n' + 1$ and $i > 1$. Hence, among all vertices in $V(P_m \times P_n) \setminus \mathrm{Grid}(m', n')$, the vertex with the smallest label from X is either $(m' + 1, 1)$ or $(1, n' + 1)$. Since $z + 1$ does not appear in $\mathrm{Label}_X(\mathrm{Grid}(m', n'))$, either $x_{m'+1,1} = z+1$ or $x_{1,n'+1} = z+1$. □

Theorem 3.32. *Let $m \geqslant 3$ and $n \geqslant 3$ be odd integers. Let $X = \{x_{i,j} : (i, j) \in V(P_m \times P_n)\}$ be a palindromic sequence labeling on $P_m \times P_n$. Then X is constructed in one of the following two ways.*

1. *There exists an (m, n)-projective factorization sequence $F = (m_1, n_1, m_2, n_2, \ldots, m_{k'}, n_{k'})$ such that $X = \mathrm{HPSL}(F)$.*

2. *There exists an (n, m)-projective factorization sequence $F' = (n'_1, m'_1, n'_2, m'_2, \ldots, n'_{k'}, m'_{k'})$ such that $X = \mathrm{VPSL}(F')$. (See Definition 3.30.)*

Furthermore, distinct (m, n)-projective factorization sequences F_1 and F_2 give rise to distinct palindromic sequence labelings $\mathrm{HPSL}(F_1)$ and $\mathrm{HPSL}(F_2)$ on $P_m \times P_n$. Similarly, distinct (n, m)-projective factorization sequences F'_1 and F'_2 give rise to distinct palindromic sequence labelings $\mathrm{VPSL}(F'_1)$ and $\mathrm{VPSL}(F'_2)$ on $P_m \times P_n$.

Proof. By Remarks 3.25 and 3.27, the constructions of $\mathrm{HPSL}(F)$ and $\mathrm{VPSL}(F')$ are palindromic sequence labelings on $P_m \times P_n$.

We need to show that X must necessarily be either $\mathrm{HPSL}(F)$ or $\mathrm{VPSL}(F')$. Let $(a_1, a_2, \ldots, a_{m-1})$ and $(b_1, b_2, \ldots, b_{n-1})$ be the palindromic sequences used in X. Then

$$x_{i,j} = x_{1,1} + \sum_{k=0}^{i-1} a_i + \sum_{\ell=0}^{j-1} b_\ell, \quad \text{for all } 1 \leqslant i \leqslant m \text{ and } 1 \leqslant j \leqslant n,$$

where $a_0 = 0$ and $b_0 = 0$.

In the proof we will choose the assumption that leads to the conclusion that $X = \text{HPSL}(F)$ in part (1) of Theorem 3.32. The proof of part (2) of Theorem 3.32 is similar to the proof of part (1) of Theorem 3.32. By Definition 3.30, for the (m,n)-projective factorization sequence $F = (m_k, n_k : 1 \leqslant k \leqslant k')$, we have $X_1 = \text{HLL}(m_1, n_1)$ and for $2 \leqslant k \leqslant k'$, $Y_k = \text{HCS}^{m_k}(X_{k-1})$ and $X_k = \text{VCS}^{n_k}(Y_k)$. Then $\text{HPSL}(F) = X_{k'}$.

Given a palindromic sequence S of positive integers, we let $\nu(S)$ denote the number of terms in S and $\sigma(S)$ denote the sum of the terms in S. Define

$$\begin{aligned}
A_1 &= 1,\\
B_1 &= m_1,\\
\text{HPS}(1) &= (A_1)^{m_1-1} = (1)^{m_1-1}, \text{ and}\\
\text{VPS}(1) &= (B_1)^{n_1-1} = (m_1)^{n_1-1}.
\end{aligned}$$

Then

$$\begin{aligned}
\nu(\text{HPS}(1)) &= m_1 - 1, &(20)\\
\nu(\text{VPS}(1)) &= n_1 - 1, &(21)\\
\sigma(\text{HPS}(1)) &= m_1 - 1, \text{ and}\\
\sigma(\text{VPS}(1)) &= (n_1 - 1)m_1.
\end{aligned}$$

For all $k \geqslant 2$, define

$$\begin{aligned}
A_k &= \sigma(\text{VPS}(k-1)) + 1,\\
\text{HPS}(k) &= \big(\text{HPS}(k-1), (A_k, \text{HPS}(k-1))^{m_k-1}\big), &(22)\\
B_k &= \sigma(\text{HPS}(k)) + 1, \text{ and}\\
\text{VPS}(k) &= \big(\text{VPS}(k-1), (B_k, \text{VPS}(k-1))^{n_k-1}\big). &(23)
\end{aligned}$$

Then

$$\begin{aligned}
\nu(\text{HPS}(k)) &= m_k\big(\nu(\text{HPS}(k-1)) + 1\big) - 1, &(24)\\
\nu(\text{VPS}(k)) &= n_k\big(\nu(\text{VPS}(k-1)) + 1\big) - 1, &(25)\\
\sigma(\text{HPS}(k)) &= m_k\sigma(\text{HPS}(k-1)) + (m_k - 1)A_k, \text{ and}\\
\sigma(\text{VPS}(k)) &= n_k\sigma(\text{VPS}(k-1)) + (n_k - 1)B_k.
\end{aligned}$$

We observe that $\text{HPS}(k)$ and $\text{VPS}(k-1)$ are the palindromic sequences used in Y_k for all $1 < k \leqslant k'$ and $\text{HPS}(k)$ and $\text{VPS}(k)$ are the palindromic sequences used in X_k for all $1 \leqslant k \leqslant k'$. Further, it should be pointed out that the integers m_k and n_k, for all $1 \leqslant k \leqslant k'$, are arbitrary positive integers with no assumption that m_k is a factor of m or n_k is a factor of n. We will demonstrate that m_k is a factor of m and n_k is a factor of n, for all $1 \leqslant k \leqslant k'$, at the end of the proof.

Let $M_k = \nu(\mathrm{HPS}(X_k))$ and $N_k = \nu(\mathrm{VPS}(X_k))$. By (20), (21), (24) and (25), we have

$$M_k^+ = m_1 m_2 \cdots m_k \text{ and} \tag{26}$$
$$N_k^+ = n_1 n_2 \cdots n_k. \tag{27}$$

Then Y_k is a palindromic sequence labeling on $P_{M_k^+} \times P_{N_{k-1}^+}$ and X_k is a palindromic sequence labeling on $P_{M_k^+} \times P_{N_k^+}$.

Let $W = \mathrm{HPSL}(F)$. We let $(c_1, c_2, \ldots, c_{M_{k'}})$ and $(d_1, d_2, \ldots, d_{N_{k'}})$ be the palindromic sequences used in W. Then

$$w_{i,j} = w_{1,1} + \sum_{k=0}^{i-1} c_i + \sum_{\ell=0}^{j-1} d_\ell, \quad \text{for all } 1 \leqslant i \leqslant M_{k'}^+ \text{ and } 1 \leqslant j \leqslant N_{k'}^+,$$

where $c_0 = 0$ and $d_0 = 0$.

We assume that $x_{i,j} = w_{i,j}$ for all $(i,j) \in \mathrm{Grid}(m', n')$. We want to show under various assumptions that either

$$x_{i,j} = w_{i,j} \;\; \text{for all } (i,j) \in \mathrm{Grid}(m'+1, n'), \text{ or}$$
$$x_{i,j} = w_{i,j} \;\; \text{for all } (i,j) \in \mathrm{Grid}(m', n'+1).$$

Let z be smallest positive integer such that $z \in \mathrm{Label}_X(\mathrm{Grid}(m', n'))$ and $z+1 \notin \mathrm{Label}_X(\mathrm{Grid}(m', n'))$. By Lemma 3.31, either $x_{m'+1,1} = z+1$ or $x_{1,n'+1} = z+1$.

Since $x_{1,1}$ is the smallest label in X, we have $x_{1,1} = 1$. Thus $1 \in \mathrm{Label}_X(\mathrm{Grid}(1,1))$, but $2 \notin \mathrm{Label}_X(\mathrm{Grid}(1,1))$ By Lemma 3.31, either $x_{1,2} = 2$ or $x_{2,1} = 2$. We will assume that $x_{2,1} = 2$. We will see that the choice $x_{2,1} = 2$ leads to the conclusion in part (1) of Theorem 3.32.

The choice $x_{1,2} = 2$ leads to the conclusion in part (2) of Theorem 3.32. Since the proof of part (2) of Theorem 3.32 is similar to the proof of part (1) of Theorem 3.32, we leave the details of the proof of part (2) of Theorem 3.32 to the reader.

Since $x_{2,1} = 2$, we have $\mathrm{Label}_X(\mathrm{Grid}(2,1)) = \{1,2\}$. By Lemma 3.31, either $x_{1,2} = 3$ or $x_{3,1} = 3$. We may continue to argue in this fashion. Let m_1 be the largest positive integer such that $x_{m_1,1} = m_1$, but $x_{m_1+1,1} \neq m_1+1$. Thus $\mathrm{Label}_X(\mathrm{Grid}(m_1,1)) = \{1,2,\ldots,m_1\}$. By Lemma 3.31, either $x_{1,2} = m_1+1$ or $x_{m_1+1,1} = m_1+1$. Since $x_{m_1+1,1} \neq m_1+1$, $x_{1,2} = m_1+1$.

We observe that $a_i = 1$ for all $1 \leqslant i < m_1$ and $a_{m_1} > 1$. Thus $x_{i,2} = m_1 + i$ for all $1 \leqslant i \leqslant m_1$ and $x_{m_1+1,2} \neq 2m_1+1$. Hence, $\mathrm{Label}_X(\mathrm{Grid}(m_1,2)) = \{1,2,\ldots,2m_1\}$ and $x_{m_1+1,2} \neq 2m_1+1$. By Lemma 3.31, we have $x_{3,1} = 2m_1+1$. We continue to argue in this fashion. We let n_1 be the largest integer such that $x_{m_1,n_1} = m_1 n_1$, but $x_{1,n_1+1} \neq m_1 n_1 + 1$. Thus the labels from X coincide with $X_1 = \mathrm{HLL}(m_1, n_1)$ on $\mathrm{Grid}(m_1, n_1)$, but the labels from X do not coincide with $\mathrm{HLL}(m_1, n_1+1)$ on $\mathrm{Grid}(m_1, n_1+1)$. Thus $z = m_1 n_1 = M_1^+ N_1^+$ is the largest label such that $z \in \mathrm{Label}_X(\mathrm{Grid}(m_1, n_1))$, but $z+1 \notin \mathrm{Label}_X(\mathrm{Grid}(m_1, n_1))$. Also, $x_{1,n_1+1} \neq M_1^+ N_1^+ + 1$. By Lemma 3.31, $x_{m_1+1,1} = M_1^+ N_1^+ + 1$. Hence, $a_{m_1} = A_2 = \sigma(\mathrm{VPS}(1)) + 1 =$

$(n_1 - 1)m_1 + 1$. Therefore, the labels from X coincide with the labels from $X_1 = \mathrm{HLL}(m_1, n_1)$ on the vertices in $\mathrm{Grid}(M_1^+, N_1^+)$, $z = M_1^+ N_1^+$ is the smallest positive integer such that $z \in \mathrm{Label}_X(\mathrm{Grid}(M_1^+, N_1^+))$ and $z + 1 \notin \mathrm{Label}_X(\mathrm{Grid}(M_1^+, N_1^+))$ and $x_{M_1^+ +1,1} = z + 1$.

In order to complete the proof by Mathematical Induction, we assume that for some positive integer k that the labels from X coincide with the labels from X_{k-1} on the vertices in $\mathrm{Grid}(M_{k-1}^+, N_{k-1}^+)$, $z = M_{k-1}^+ N_{k-1}^+$ is the smallest positive integer such that $z \in \mathrm{Label}_X(\mathrm{Grid}(M_{k-1}^+, N_{k-1}^+))$ and $z+1 \notin \mathrm{Label}_X(\mathrm{Grid}(M_{k-1}^+, N_{k-1}^+))$ and $x_{M_{k-1}^+ +1,1} = z + 1$.

For convenience, let $n' = N_{k-1}^+$. We first establish the following claim. Suppose there exists a positive integer s such that, for some integer m' with $sM_{k-1}^+ < m' < (s+1)M_{k-1}^+$, the labels from X coincide with the labels from $\mathrm{HCS}^{s+1}(X_{k-1})$ on the vertices in $\mathrm{Grid}(m', n')$. I.e., we have $x_{i,j} = w_{i,j}$ for all $(i, j) \in \mathrm{Grid}(m', n')$. We want to show that the labels from X must coincide with the labels from $\mathrm{HCS}^{s+1}(X_{k-1})$ on the vertices in $\mathrm{Grid}(m' + 1, n')$. I.e., we want to show $x_{i,j} = w_{i,j}$ for all $(i, j) \in \mathrm{Grid}(m' + 1, n')$. Then $a_{m'} = A_t$ for some $1 \leqslant t < k$. Let z be the smallest positive integer such that $z \in \mathrm{Label}_X(\mathrm{Grid}(m', n'))$ and $z + 1 \notin \mathrm{Label}_X(\mathrm{Grid}(m', n'))$. Then $x_{i',j'} = z$ for some $(i', j') \in \mathrm{Grid}(m', n')$. By Lemma 3.31, either $x_{m'+1,1} = z + 1$ or $x_{1,n'+1} = z + 1$. We want to show that $x_{m'+1,1} = z + 1$. For the purposes of contradiction, assume $x_{1,n'+1} = z + 1$. Since the labels from X coincide with the labels from $\mathrm{HCS}^s(X_{k-1})$ on $\mathrm{Grid}(m', n')$, and $\mathrm{HCS}^s(X_{k-1}) = \{1, 2, \ldots, sM_{k-1}^+ N_{k-1}^+\}$, we have $i' > sM_{k-1}^+$.

Case 1. Assume $c_{i'} = A_1 = 1$. We want to show $i' = m'$ and $a_{m'} = 1 = A_1 = c_{m'}$. For the purposes of contradiction, we assume $i' < m'$. Since X and W coincide on $\mathrm{Grid}(m', n')$, we have $a_{i'} = c_{i'} = 1$. Then $z + 1 = x_{i',j'} + a_{i'} = x_{i'+1,j'} \in \mathrm{Label}_X(\mathrm{Grid}(m', n'))$ which contradicts $z+1 \notin \mathrm{Label}_X(\mathrm{Grid}(m', n'))$. Hence, $i' = m'$, $c_{m'} = A_1 = 1$ and $x_{m',j'} = z$.

We observe that $x_{m',1} \in \mathrm{Label}_X(\mathrm{Grid}(m', n'))$, but $x_{m',1} + 1 = w_{m',1} + c_{m'} = w_{m'+1,1} \notin \mathrm{Label}_X(\mathrm{Grid}(m', n'))$. Since z is the smallest positive integer with the property that $z \in \mathrm{Label}_X(\mathrm{Grid}(m', n'))$ and $z + 1 \notin \mathrm{Label}_X(\mathrm{Grid}(m', n'))$, we have $x_{m',1} = z$.

We next observe that $x_{m',2} = x_{m',1} + b_1 = z + m_1$. For the purposes of contradiction, we assume $x_{1,n'+1} = z + 1$. Thus $x_{m_1,n'+1} = x_{1,n'+1} + \sigma(\mathrm{HPS}(1)) = z + m_1$. This contradicts the condition that each of the labels from $\{1, 2, \ldots, mn\}$ is used exactly once in X. Hence, $z + 1 = x_{m'+1,1} = x_{m',1} + a_{m'} = z + a_{m'}$. Therefore, $a_{m'} = 1 = A_1 = c_{m'}$. Thus $x_{m'+1,j} = x_{m',j} + a_{m'} = w_{m',j} + c_{m'} = w_{m'+1,j}$ for all $1 \leqslant j \leqslant n'$. Hence, $x_{i,j} = w_{i,j}$ for all $(i, j) \in \mathrm{Grid}(m' + 1, n')$.

Case 2. Assume $c_{i'} = A_t$ for some integer $2 \leqslant t \leqslant k - 1$. We want to show $i' = m'$ and $a_{m'} = \sigma(\mathrm{VPS}(t - 1)) + 1 = A_t = c_{m'}$. Let $p_1 = i'/M_{t-1}^+$ and $p_2 = N_{k-1}^+/N_{t-1}^+$. We show that the labels from X on $\mathrm{Grid}(m', n')$ is a $p_1 \times p_2$ array of copies of X_{t-1} such that the labels in any two copies of X_{t-1} differ by some constant. From (22) and (23), we have

$$\begin{aligned}\mathrm{HPS}(k-1) &= \big((\mathrm{HPS}(t-1), a_{iM^+_{t-1}} : 1 \leqslant i < p_1), \mathrm{HPS}(t-1)\big) \quad \text{and}\\ \mathrm{VPS}(k-1) &= \big((\mathrm{VPS}(t-1), b_{jN^+_{t-1}} : 1 \leqslant j < p_2), \mathrm{VPS}(t-1)\big),\end{aligned}$$

where $a_{iM^+_{t-1}} = A_{s_i}$ such that $s_i \geqslant t$ for all $1 \leqslant i < p_1$ and $b_{jN^+_{t-1}} = B_{t_j}$ such that $t_j \geqslant t$ for all $1 \leqslant j < p_2$. Since $\mathrm{HPS}(t-1) = (a_1, a_2, \ldots, a_{M_{t-1}})$ and $\mathrm{VPS}(t-1) = (b_1, b_2, \ldots, b_{N_{t-1}})$, we have

$$x_{i_5 M^+_{t-1}+i_4, j_5 N^+_{t-1}+j_4} = x_{i_5 M^+_{t-1}+1, j_5 N^+_{t-1}+1} + \sum_{i=0}^{i_4-1} a_i + \sum_{j=0}^{j_4-1} b_j$$

for all $0 \leqslant i_5 < p_1$, $0 \leqslant j_5 < p_2$, $1 \leqslant i_4 \leqslant M^+_{t-1}$ and $1 \leqslant j_4 \leqslant N^+_{t-1}$.

For the purposes of contradiction, we assume $i' < m'$. Since X and W coincide on $\mathrm{Grid}(m', n')$, we have $a_{i'} = c_{i'} = A_t$. Also, $x_{i',j'} = z$ is the label on the rightmost column of a copy of X_{t-1} lying within $\mathrm{Grid}(m', n')$. Let (i_0, j_0) be the lower leftmost vertex and (i_1, j_1) be the upper rightmost vertex in this copy of X_{t-1}. Then $i_1 = i_0 + M_{t-1} = i_0 + \nu(\mathrm{HPS}(t-1))$ and $j_1 = j_0 + N_{t-1} = i_0 + \nu(\mathrm{VPS}(t-1))$. Also, $x_{i_1,j_1} = x_{i_0,j_0} + \sigma(\mathrm{HPS}(t-1)) + \sigma(\mathrm{VPS}(t-1))$ and $x_{i_1,j_1} = x_{i_1,j_0} + \sigma(\mathrm{VPS}(t-1))$. Since $\{x_{i,j} : i_0 \leqslant i \leqslant i_1 \text{ and } j_0 \leqslant i \leqslant j_1\}$ contains the labels $\{x_{i_0,j_0}, x_{i_0,j_0}+1, x_{i_0,j_0}+2, \ldots, x_{i_1,j_1}\}$, we have $(i', j') = (i_1, j_1)$. Thus $x_{i_1,j_1} = z$ and $x_{i_1,j_0} = x_{i_1,j_1} - \sigma(\mathrm{VPS}(t-1))$. We have $x_{i_1+1,j_0} = x_{i_1,j_0} + A_t = (x_{i_1,j_1} - \sigma(\mathrm{VPS}(t-1))) + (\sigma(\mathrm{VPS}(t-1))) + 1) = z + 1$. Thus $z+1 = x_{i_1+1,j_0} \in \mathrm{Label}_X(\mathrm{Grid}(m'n'))$ which contradicts $z+1 \notin \mathrm{Label}_X(\mathrm{Grid}(m'n'))$. Hence, $i' = m'$, $c_{m'} = A_t = \sigma(\mathrm{VPS}(t-1)) + 1$ and $x_{m',j'} = z$.

We have $p_1 = m'/M^+_{t-1}$ and $p_2 = N^+_{k-1}/N^+_{t-1}$. We observed that the labels from X on $\mathrm{Grid}(m', n')$ is a $p_1 \times p_2$ array of copies of X_{t-1} such that the labels in any two copies of X_{t-1} differ by some constant. Thus the labels from X on the vertices of column m' from $\mathrm{Grid}(m', n')$ are a stack of copies of the rightmost column of X_{t-1} that lie one atop another such that the labels in any two copies differ by some constant. Let $j_2 = N_{t-1} + 1$. We observe that $x_{m',j_2} = x_{m',1} + \sigma(\mathrm{VPS}(t-1))$ and $w_{m'+1,1} = w_{m',1} + c_{m'} = x_{m',1} + A_t = x_{m',j_2} + 1$. Thus $x_{m',j_2} \in \mathrm{Label}_X(\mathrm{Grid}(m', n'))$ and $x_{m',j_2} + 1 = w_{m'+1,1} \notin \mathrm{Label}_X(\mathrm{Grid}(m', n'))$. Let $i_2 = m' - M_{t-1} = m' - \nu(\mathrm{HPS}(t-1))$. Then $\{x_{i,j} : i_2 \leqslant i \leqslant m' \text{ and } 1 \leqslant i \leqslant j_2\}$ contains the labels $\{x_{i_2,1}, x_{i_2,1}+1, x_{i_2,1}+2, \ldots, x_{m',j_2}\}$. Since z is the smallest positive integer such that $z \in \mathrm{Label}_X(\mathrm{Grid}(m', n'))$ and $z+1 \notin \mathrm{Label}_X(\mathrm{Grid}(m', n'))$, we have $j' = j_2$ and $x_{m',j_2} = z$. Thus $b_{j_2} = B_t$. Hence, $x_{m',j_2+1} = x_{m',j_2} + B_t = z + \sigma(\mathrm{HPS}(t)) + 1$. For the purposes of contradiction, we assume $x_{1,n'+1} = z + 1$. Let $i_3 = M_t + 1$. Since $i_3 = M^+_t \leqslant M^+_{k-1} < m'$, we have $x_{i_3,n'+1} = x_{1,n'+1} + \sigma(\mathrm{HPS}(t)) = z + 1 + \sigma(\mathrm{HPS}(t))$. Thus $x_{i_3,n'+1} = z + \sigma(\mathrm{HPS}(t)) + 1$ duplicates the label $x_{m',j_2+1} = z + \sigma(\mathrm{HPS}(t)) + 1$. This contradicts the condition that each of the labels from $\{1, 2, \ldots, mn\}$ is used exactly once in X. Thus $z + 1 = x_{m'+1,1} = x_{m',1} + a_{m'} = z - \sigma(\mathrm{VPS}(t-1)) + a_{m'}$. Therefore, $a_{m'} = \sigma(\mathrm{VPS}(t-1)) + 1 = A_t = c_{m'}$. Thus $x_{m'+1,j} = x_{m',j} + a_{m'} = w_{m',j} + c_{m'} = w_{m'+1,j}$ for all $1 \leqslant j \leqslant n'$. Hence, $x_{i,j} = w_{i,j}$ for all $(i, j) \in \mathrm{Grid}(m'+1, n')$.

Therefore, the only time that we can choose $x_{1,n'+1} = z + 1$ is when the labels from X on column m' of $\mathrm{Grid}(m', n')$ is the rightmost column of the labels from

$\mathrm{HCS}^s(X_{k-1})$ for some positive integer s. We let m_k be the largest positive integer such that the labels of X match the labels of $\mathrm{HCS}^{m_k}(X_{k-1})$ on $\mathrm{Grid}(m_kM^+_{k-1}, N^+_{k-1})$, but the labels of X do not match the labels of $\mathrm{HCS}^{m_k+1}(X_{k-1})$ on $\mathrm{Grid}((m_k + 1)M^+_{k-1}, N^+_{k-1})$. By assumption, there are at least two copies of X_{k-1} in this horizontal connected sum of X_{k-1}. Thus, we have $m_k > 1$.

If we have $X = \mathrm{HCS}^{m_k}(X_{k-1})$, then $X = Y_k$. We set $k' = k$ and $n_{k'} = 1$. Then $X_{k'} = \mathrm{VCS}^1(Y_{k'}) = Y_{k'}$. Hence, $X = X_{k'}$.

Otherwise, X is a labeling on a larger set of vertices than the labeling $Y_k = \mathrm{HCS}^{m_k}(X_{k-1})$ on $\mathrm{Grid}(M^+_k, N^+_{k-1})$. Thus the labels from X coincide with the labels from Y_k on the vertices in $\mathrm{Grid}(M^+_k, N^+_{k-1})$. Since $Y_k = \mathrm{HCS}^{m_k}(X_{k-1})$ is a palindromic sequence labeling on $P_{M^+_k} \times P_{N^+_{k-1}}$, $z = M^+_kN^+_{k-1}$ is the smallest positive integer such that $z \in \mathrm{Label}_X(\mathrm{Grid}(M^+_k, N^+_{k-1}))$ but $z+1 \notin \mathrm{Label}_X\mathrm{Grid}(M^+_k, N^+_{k-1}))$. Furthermore, $x_{1,N^+_{k-1}+1} = z + 1$.

In order to further complete the proof by Mathematical Induction, we assume that for some positive integer k that the labels from X coincide with the labels from Y_k on $\mathrm{Grid}(M^+_k, N^+_{k-1})$, $z = M^+_kN^+_{k-1}$ is the smallest positive integer such that $z \in \mathrm{Label}_X(\mathrm{Grid}(M^+_k, N^+_{k-1}))$ and $z+1 \notin \mathrm{Label}_X(\mathrm{Grid}(M^+_k, N^+_{k-1}))$ and $x_{1,N^+_{k-1}+1} = z + 1$.

For convenience, let $m' = M^+_k$. We establish the following claim. Suppose there exists a positive integer s such that, for some integer n' with $sN^+_{k-1} < n' < (s + 1)N^+_{k-1}$, the labels from X coincide with the labels from $\mathrm{VCS}^{s+1}(Y_k)$ on $\mathrm{Grid}(m', n')$. I.e., we have $x_{i,j} = w_{i,j}$ for all $(i, j) \in \mathrm{Grid}(m', n')$. We want to show that the labels from X must coincide with the labels from $\mathrm{VCS}^{s+1}(Y_k)$ on $\mathrm{Grid}(m', n' + 1)$. I.e., we want to show $x_{i,j} = w_{i,j}$ for all $(i, j) \in \mathrm{Grid}(m', n' + 1)$. Then $b_{n'} = B_t$ for some $1 \leqslant t < k$. Let z be the smallest positive integer such that $z \in \mathrm{Label}_X(\mathrm{Grid}(m', n'))$ and $z + 1 \notin \mathrm{Label}_X(\mathrm{Grid}(m', n'))$. Then $x_{i',j'} = z$ for some $(i', j') \in \mathrm{Grid}(m', n')$. By Lemma 3.31, either $x_{m'+1,1} = z + 1$ or $x_{1,n'+1} = z + 1$. We want to show that $x_{1,n'+1} = z + 1$. For the purposes of contradiction, assume $x_{m'+1,1} = z + 1$. Since the labels from X coincide with the labels from $\mathrm{VCS}^s(Y_k)$ on $\mathrm{Grid}(m', n')$ and $\mathrm{VCS}^s(Y_k) = \{1, 2, \ldots, sM^+_kN^+_{k-1}\}$, we have $j' > sN^+_{k-1}$.

Case 3. Assume $d_{j'} = B_1 = m_1$. We want to show $j' = n'$ and $b_{n'} = m_1 = B_1 = d_{n'}$. For the purposes of contradiction, we assume $j' < n'$. Since X and W coincide on $\mathrm{Grid}(m', n')$, we have $b_{j'} = d_{j'} = m_1$. Thus the labels of X on the vertices from row j' of $\mathrm{Grid}(m', n')$ correspond to copies of a non-topmost row of $X_1 = \mathrm{HLL}(m_1, n_1)$ laid side by side such that the labels in any two copies differ by some constant. Furthermore, $a_{i'} \neq A_1 = 1$; otherwise, $z + 1 = x_{i',j'} + a_{i'} = x_{i'+1,j'} \in \mathrm{Label}_X(\mathrm{Grid}(m', n'))$ which contradicts $z+1 \notin \mathrm{Label}_X(\mathrm{Grid}(m', n'))$. Thus $a_{i'} = A_r$ for some $r > 1$ and $a_{i'-i} = A_1 = 1$ for $1 \leqslant i \leqslant m_1 - 1$. So $x_{i'-m_1+1,j'} = z - m_1 + 1$ and $x_{i'-m_1+1,j'+1} = x_{i'-m_1+1,j'} + b_{j'} = z + 1$. Thus $z + 1 = x_{i'-m_1+1,j'+1} \in \mathrm{Label}_X(\mathrm{Grid}(m', n'))$ which contradicts $z+1 \notin \mathrm{Label}_X(\mathrm{Grid}(m', n'))$. Hence, $j' = n'$, $d_{n'} = B_1 = m_1$ and $x_{i',n'} = z$.

We observe that $x_{i+1,n'} = x_{1,n'} + i$ for all $1 \leqslant i \leqslant m_1 - 1$. Thus $x_{m_1,n'} \in$

$\mathrm{Label}_X(\mathrm{Grid}(m', n'))$, but

$$x_{m_1,n'} + 1 = x_{1,n'} + m_1 = w_{1,n'} + d_{n'} = w_{1,n'+1} \notin \mathrm{Label}_X(\mathrm{Grid}(m', n')).$$

Also, $\{x_{i,n'} : 1 \leqslant i \leqslant m_1\}$ is the set $\{x_{1,n'}, x_{1,n'} + 1, x_{1,n'} + 2, \ldots, x_{m_1,n'}, \}$. Since z is the smallest positive integer with the property that $z \in \mathrm{Label}_X(\mathrm{Grid}(m', n'))$ and $z + 1 \notin \mathrm{Label}_X(\mathrm{Grid}(m', n'))$, we have $x_{m_1,n'} = z$.

We observe that $x_{m_1+1,n'} = x_{m_1,n'} + a_{m_1} = z + A_2 = z + \sigma(\mathrm{VPS}(1)) + 1$. For the purposes of contradiction, we assume $x_{m'+1,1} = z + 1$. Since $n_1 = \nu(\mathrm{VPS}(1)) + 1$, $x_{m'+1,n_1} = x_{m'+1,1} + \sigma(\mathrm{VPS}(1)) = z + 1 + \sigma(\mathrm{VPS}(1))$. This contradicts the condition that each of the labels from $\{1, 2, \ldots, mn\}$ is used exactly once in X. Hence, $x_{1,n'+,1} = z + 1$. Since $z = x_{m_1,n'} = x_{1,n'} + (m_1 - 1)$, we have $z + 1 = x_{1,n'+,1} = x_{1,n'} + b_{n'} = (z - m_1 + 1) + b_{n'}$. Therefore, $b_{n'} = m_1 = B_1 = d_{n'}$. Thus $x_{i,n'+1} = x_{i,n'} + b_{n'} = w_{i,n'} + d_{n'} = w_{i,n'+1}$ for all $1 \leqslant i \leqslant m'$. Hence, $x_{i,j} = w_{i,j}$ for all $(i, j) \in \mathrm{Grid}(m', n' + 1)$.

Case 4. Assume $d_{j'} = B_t$ for some integer $2 \leqslant t \leqslant k - 1$. We want to show $j' = n'$ and $b_{n'} = \sigma(\mathrm{HPS}(t)) + 1 = B_t = d_{n'}$. Let $p_1 = M_k^+/M_t^+$ and $p_2 = j'/N_{t-1}^+$. An argument similar to that in Case 2 shows that the labels from X on $\mathrm{Grid}(m', n')$ is a $p_1 \times p_2$ array of copies of Y_t such that the labels in any two copies of Y_t differ by some constant.

For the purposes of contradiction, we assume $j' < n'$. Since the labels of X and W are the same on $\mathrm{Grid}(m', n')$, $b_{j'} = d_{j'} = B_t$. Then $x_{i',j'} = z$ is the label on the topmost row of a copy of Y_t that lies in $\mathrm{Grid}(m', n')$. Let (i_0, j_0) be the lower leftmost vertex and (i_1, j_1) be the upper rightmost vertex in this copy of Y_t. Then $i_1 = i_0 + \nu(\mathrm{HPS}(t))$ and $j_1 = j_0 + \nu(\mathrm{VPS}(t - 1))$. Also, $x_{i_1,j_1} = x_{i_0,j_0} + \sigma(\mathrm{HPS}(t)) + \sigma(\mathrm{VPS}(t-1))$ and $x_{i_1,j_1} = x_{i_0,j_1} + \sigma(\mathrm{HPS}(t))$. Since $\{x_{i,j} : i_0 \leqslant i \leqslant i_1 \text{ and } j_0 \leqslant i \leqslant j_1\}$ is the set of labels $\{x_{i_0,j_0}, x_{i_0,j_0}+1, x_{i_0,j_0}+2, \ldots, x_{i_1,j_1}\}$, we have $(i', j') = (i_1, j_1)$. Thus $x_{i_1,j_1} = z$ and $x_{i_0,j_1} = x_{i_1,j_1} - \sigma(\mathrm{HPS}(t))$. We have $x_{i_0,j_1+1} = x_{i_0,j_1} + B_t = z + 1$. Thus $z+1 = x_{i_0,j_1+1} \in \mathrm{Label}_X(\mathrm{Grid}(m'n'))$ which contradicts $z+1 \notin \mathrm{Label}_X(\mathrm{Grid}(m', n'))$. Hence, $j' = n'$, $d_{n'} = B_t$ and $x_{i',n'} = z$.

We have $p_1 = M_k^+/M_t^+$ and $p_2 = n'/N_{t-1}^+$. We observed that the labels from X on $\mathrm{Grid}(m', n')$ is a $p_1 \times p_2$ array of copies of Y_t such that the labels in any two copies of Y_t differ by some constant. Thus the labels from X on the vertices of row n' of $\mathrm{Grid}(m', n')$ are a list of copies of the topmost row of Y_t laid side by side such that the labels in any two copies differ by some constant.

Let $i_2 = \nu(\mathrm{HPS}(t)) + 1$. Then $x_{i_2,n'} = x_{1,n'} + \sigma(\mathrm{HPS}(t))$. We observe that $x_{i_2,n'} \in \mathrm{Label}_X(\mathrm{Grid}(m', n'))$. Also, $w_{1,n'+1} = w_{1,n'} + d_{n'} = x_{1,n'} + B_t = x_{i_2,n'} + 1$. Hence, $x_{i_2,n'} + 1 = w_{1,n'+1} \notin \mathrm{Label}_X(\mathrm{Grid}(m', n'))$. Let $j_2 = n' - \nu(\mathrm{VPS}(t - 1))$. We observe that $\{x_{i,j} : 1 \leqslant i \leqslant i_2 \text{ and } j_2 \leqslant j \leqslant n'\}$ is the set of labels $\{x_{1,j_2}, x_{1,j_2} + 1, x_{1,j_2} + 2, \ldots, x_{i_2,n'}\}$. Since z is the smallest positive integer with the property that $z \in \mathrm{Label}_X(\mathrm{Grid}(m', n'))$ and $z + 1 \notin \mathrm{Label}_X(\mathrm{Grid}(m', n'))$, we have $x_{i_2,n'} = z$.

Since $i_2 = M_t^+$, we have $a_{i_2} = A_{t+1} = \sigma(\mathrm{VPS}(t)) + 1$. Thus $x_{i_2+1,n'} = x_{i_2,n'} + a_{i_2} = z + \sigma(\mathrm{VPS}(t)) + 1$. Let $j_3 = N_t + 1 = \nu(\mathrm{VPS}(t)) + 1$. We observe that $j_3 = N_t^+ \leqslant N_{k-1}^+$. For the purposes of contradiction, we assume $x_{m'+1,1} = z + 1$, Hence, $x_{m'+1,j_3} = x_{m'+1,1} + \sigma(\mathrm{VPS}(t)) = z + 1 + \sigma(\mathrm{VPS}(t))$. This contradicts the condition that each of the labels from $\{1, 2, \ldots, mn\}$ is used exactly once in X. Thus $z+1 = x_{1,n'+1} = x_{1,n'} + b_{n'} = z - \sigma(\mathrm{HPS}(t)) + b_{n'}$. Therefore, $b_{n'} = \sigma(\mathrm{HPS}(t)) + 1 = B_t = d_{n'}$. Thus $x_{i,n'+1} = x_{i,n'} + b_{n'} = w_{i,n'} + d_{n'} = w_{i,n'+1}$ for all $1 \leqslant i \leqslant m'$. Hence, $x_{i,j} = w_{i,j}$ for all $(i,j) \in \mathrm{Grid}(m', n'+1)$.

Therefore, the only time that we can choose $x_{m'+1,1} = z+1$ is when the labels of X on row n' of $\mathrm{Grid}(m', n')$ is the topmost row of the labels on $\mathrm{VCS}^s(Y_k)$ for some positive integer s. We let n_k be the largest positive integer such that the labels of X match the labels of $\mathrm{VCS}^{n_k}(Y_k)$ on $\mathrm{Grid}(M_k^+, n_k N_{k-1}^+)$, but the labels of X do not match the labels of $\mathrm{VCS}^{n_k+1}(Y_k)$ on $\mathrm{Grid}(M_k^+, (n_k+1)N_{k-1}^+)$. By assumption, there are at least two copies of Y_k in this vertical connected sum of Y_k. Thus, we have $n_k > 1$.

If X is a labeling on a larger set of vertices than the labeling $X_k = \mathrm{VCS}^{n_k}(Y_k)$, then we need to continue the inductive step. Thus the labels from X coincide with the labels from X_k on $\mathrm{Grid}(M_k^+, N_k^+)$. Since $X_k = \mathrm{VCS}^{n_k}(Y_k)$ is a palindromic sequence labeling on $P_{M_k^+} \times P_{N_k^+}$, $z = M_k^+ N_k^+$ is the smallest positive integer such that $z \in \mathrm{Label}_X(\mathrm{Grid}(M_k^+, N_k^+))$ but $z+1 \notin \mathrm{Label}_X(\mathrm{Grid}(M_k^+, N_k^+))$. Furthermore, $x_{M_k^+ +1,1} = z+1$.

Otherwise, we have $X = \mathrm{VCS}^{n_k}(Y_k)$. We set $k' = k$. Then $n_{k'} = n_k$ and $X = X_{k'}$. This completes the inductive step of the proof.

From equations (26) and (27), we have

$$\begin{aligned} m &= M_{k'}^+ = m_1 m_2 \cdots m_{k'} \text{ and} \\ n &= N_{k'}^+ = n_1 n_2 \cdots n_{k'}. \end{aligned}$$

Thus m_k is a factor of m and n_k is a factor of n for all $1 \leqslant k \leqslant k'$. Also, each of the factors m_k and n_k are greater than 1, for all $1 \leqslant k \leqslant k'$, except possibly $n_{k'}$ (for which $n_{k'} \geqslant 1$). We let F denote the (m,n)-projective factorization sequence $F = (m_k, n_k : 1 \leqslant k \leqslant k')$. Then $X = \mathrm{HPSL}(F)$.

Let $F_i = (m_{i,1}, n_{i,1}, m_{i,2}, n_{i,2}, \ldots, m_{i,k_i}, n_{i,k_i})$, for $i = 1$ and 2, be distinct (m,n)-projective factorization sequences. We need to show that $\mathrm{HPSL}(F_1)$ and $\mathrm{HPSL}(F_2)$ are distinct palindromic sequence labelings on $P_m \times P_n$. Let $(a_{i,j} : 1 \leqslant j \leqslant m)$ and $(b_{i,j} : 1 \leqslant j \leqslant n)$ be the palindromic sequences used in $\mathrm{HPSL}(F_i)$ for $i = 1$ and 2. If either the sequences $(a_{i,j} : 1 \leqslant j \leqslant m)$, for $i = 1$ and 2, are different, or the sequences $(b_{i,j} : 1 \leqslant j \leqslant n)$, for $i = 1$ and 2, are different, then the palindromic sequence labelings $\mathrm{HPSL}(F_1)$ and $\mathrm{HPSL}(F_2)$ are distinct. If F_1 and F_2 have different lengths, then either the number of distinct values in the sequences $(a_{i,j} : 1 \leqslant j \leqslant m)$, for $i = 1$ and 2, are different, or the number of distinct values in the sequences $(b_{i,j} : 1 \leqslant j \leqslant n)$, for $i = 1$ and 2, are different.

Suppose F_1 and F_2 have the same length $k' = k_1 = k_2$. Let k be the smallest positive integer such that either $m_{1,k} \neq m_{2,k}$ or $n_{1,k} \neq n_{2,k}$. Since two factorizations of m (or n) with exactly k factors each cannot have exactly $k-1$ factors that are the same, we have $k \leqslant k'-1$. In case $n_{1,k'} = n_{2,k'} = 1$, we have $k \leqslant k'-2$. Let $\mathrm{HPS}(i,j)$ and $\mathrm{VPS}(i,j)$ be the palindromic sequences used in $X_{i,j}$ for $i = 1$ and 2.

If $m_{1,k} \neq m_{2,k}$, then the vertical palindromic sequence for $X_{i,k} = \mathrm{VCS}^{n_{i,k}}(\mathrm{HCS}^{m_{i,k}}$

$(X_{i,k-1}))$ is

$$\mathrm{VPS}(i,k) = \big(\mathrm{VPS}(i,k-1), (B_{i,k}, \mathrm{VPS}(i,k-1))^{n_{i,k}-1}\big),$$

where

$$B_{i,k} = m_{i,k}[\sigma(\mathrm{HPS}(i,k-1)) + \sigma(\mathrm{VPS}(i,k-1)) + 1] - \sigma(\mathrm{VPS}(i,k-1)).$$

Since $\mathrm{HPS}(1,k-1) = \mathrm{HPS}(2,k-1)$, $\mathrm{VPS}(1,k-1) = \mathrm{VPS}(2,k-1)$ and $m_{1,k} \neq m_{2,k}$, we have $B_{1,k} \neq B_{2,k}$.

If $n_{1,k} \neq n_{2,k}$, then the horizontal palindromic sequence for $Y_{i,k+1} = \mathrm{HCS}^{m_{i,k+1}}(\mathrm{VCS}^{n_{i,k}}(Y_{i,k}))$ is

$$\mathrm{HPS}(i,k+1) = \big(\mathrm{HPS}(i,k), (A_{i,k+1}, \mathrm{HPS}(i,k))^{m_{i,k+1}-1}\big),$$

where

$$A_{i,k+1} = n_{i,k}[\sigma(\mathrm{VPS}(i,k-1)) + \sigma(\mathrm{HPS}(i,k)) + 1] - \sigma(\mathrm{HPS}(i,k)).$$

Since $\mathrm{VPS}(1,k-1) = \mathrm{VPS}(2,k-1)$, $\mathrm{HPS}(1,k) = \mathrm{HPS}(2,k)$ and $n_{1,k} \neq n_{2,k}$, we have $A_{1,k+1} \neq A_{2,k+1}$. In all three cases, $\mathrm{HPSL}(F_1)$ and $\mathrm{HPSL}(F_2)$ are distinct palindromic sequence labelings on $P_m \times P_n$.

A similar argument shows that if F_1' and F_2' are distinct (n,m)-projective factorization sequences, then $\mathrm{VPSL}(F_1')$ and $\mathrm{VPSL}(F_2')$ are distinct palindromic sequence labelings on $P_m \times P_n$. □

Example 3.33. Table 1 illustrates the palindromic sequence labeling $\mathrm{HPSL}(3,3,3,3)$ on $P_9 \times P_9$ that uses the palindromic sequences $\mathrm{HPS}(2) = (1,1,7,1,1,7,1,1)$ and $\mathrm{VPS}(2) = (3,3,21,3,3,21,3,3)$. This labeling corresponds to the $(9,9)$-projective factorization sequence $(3,3,3,3)$.

Theorem 3.34. *Let $m \geqslant 3$ and $n \geqslant 3$ be odd integers. Let $X = \{x_{i,j} : (i,j) \in V(\mathcal{P}_{m,n})\}$ be a standard centrally balance C_4-face-magic labeling on $\mathcal{P}_{m,n}$. Then X is constructed in one of the following two ways.*

1. *There exists an (m,n)-projective factorization sequence $F = (m_1, n_1, m_2, n_2, \ldots, m_{k'}, n_{k'})$ such that $X = \mathcal{T}(\mathrm{HPSL}(F))$.*

2. *There exists an (n,m)-projective factorization sequence $F' = (n_1', m_1', n_2', m_2', \ldots, n_{k'}', m_{k'}')$ such that $X = \mathcal{T}(\mathrm{VPSL}(F'))$.*

Furthermore, distinct (m,n)-projective factorization sequences F_1 and F_2 give rise to distinct palindromic sequence labelings $\mathcal{T}(\mathrm{HPSL}(F_1))$ and $\mathcal{T}(\mathrm{HPSL}(F_2))$ on $\mathcal{P}_{m,n}$. Similarly, distinct (n,m)-projective factorization sequences F_1' and F_2' give rise to distinct palindromic sequence labelings $\mathcal{T}(\mathrm{VPSL}(F_1'))$ and $\mathcal{T}(\mathrm{VPSL}(F_2'))$ on $\mathcal{P}_{m,n}$.

Proof. By Proposition 3.20, $\mathcal{T}(X)$ is a palindromic sequence labeling on $P_m \times P_n$. By Theorem 3.32, $\mathcal{T}(X)$ is constructed in one of the following two ways.

61	62	63	70	71	72	79	80	81
58	59	60	67	68	69	76	77	78
55	56	57	64	65	66	73	74	75
34	35	36	43	44	45	52	53	54
31	32	33	40	41	42	49	50	51
28	29	30	37	38	39	46	47	48
7	8	9	16	17	18	25	26	27
4	5	6	13	14	15	22	23	24
1	2	3	10	11	12	19	20	21

Table 1: The palindromic sequence labeling HPSL$(3, 3, 3, 3)$ on $P_9 \times P_9$. For convenience, we display $P_9 \times P_9$ as a 9×9 checkerboard.

1. There exists an (m, n)-projective factorization sequence $F = (m_1, n_1, m_2, n_2, \ldots, m_{k'}, n_{k'})$ such that $\mathcal{T}(X) = \mathrm{HPSL}(F)$.
2. There exists an (n, m)-projective factorization sequence $F' = (n'_1, m'_1, n'_2, m'_2, \ldots, n'_{k'}, m'_{k'})$ such that $\mathcal{T}(X) = \mathrm{VPSL}(F')$.

Since $\mathcal{T}$ is an involution, we have either $X = \mathcal{T}(\mathrm{HPSL}(F))$ or $X = \mathcal{T}(\mathrm{VPSL}(F'))$.

By Theorem 3.32, given distinct (m, n)-projective factorization sequences F_1 and F_2, the palindromic sequence labelings $\mathrm{HPSL}(F_1)$ and $\mathrm{HPSL}(F_2)$ on $P_m \times P_n$ are distinct. Thus $\mathcal{T}(\mathrm{HPSL}(F_1))$ and $\mathcal{T}(\mathrm{HPSL}(F_2))$ are distinct standard centrally balanced C_4-face-magic projective labelings on $\mathcal{P}_{m,n}$. Similarly, if F'_1 and F'_2 are distinct (n, m)-projective factorization sequences, then $\mathcal{T}(\mathrm{VPSL}(F'_1))$ and $\mathcal{T}(\mathrm{VPSL}(F'_2))$ are distinct standard centrally balanced C_4-face-magic projective labelings on $\mathcal{P}_{m,n}$. $\square$

Theorem 3.35. *Let $m \geqslant 3$ be an odd integer. Let $X = \{x_{i,j} : (i, j) \in V(\mathcal{P}_{m,m})\}$ be a standard centrally balance C_4-face-magic labeling on $\mathcal{P}_{m,m}$. There exists an (m, m)-projective factorization sequence $F = (m_1, n_1, m_2, n_2, \ldots, m_{k'}, n_{k'})$ such that $X = \mathcal{T}(\mathrm{HPSL}(F))$.*

Proof. By Theorem 3.34, X is constructed in one of the following two ways.

1. There exists an (m, m)-projective factorization sequence $F = (m_1, n_1, m_2, n_2, \ldots, m_{k'}, n_{k'})$ such that $X = \mathcal{T}(\mathrm{HPSL}(F))$.
2. There exists an (m, m)-projective factorization sequence $F' = (n'_1, m'_1, n'_2, m'_2, \ldots, n'_{k'}, m'_{k'})$ such that $X = \mathcal{T}(\mathrm{VPSL}(F'))$.

If X is constructed in the first of these two ways, we are done. Otherwise, there exists an (m, m)-projective factorization sequence $F' = (n'_1, m'_1, n'_2, m'_2, \ldots, n'_{k'}, m'_{k'})$ such that $X = \mathcal{T}(\mathrm{VPSL}(F'))$. We apply the reflection D_+ about the diagonal line with positive slope passing through the center of $\mathcal{P}_{m,m}$ to X to obtain the labeling Y. Then $Y = \mathcal{T}(\mathrm{HPSL}(F'))$. □

Example 3.36. Table 2 illustrates the standard centrally balanced C_4-face-magic projective labeling $\mathcal{T}(\mathrm{HPSL}(3,3,3,3))$ on $\mathcal{P}_{9,9}$ with C_4-face-magic value $S = 164$. This labeling corresponds to the $(9,9)$-projective factorization sequence $(3,3,3,3)$.

61	2	63	10	71	12	79	20	81
78	23	76	15	68	13	60	5	58
55	8	57	16	65	18	73	26	75
54	47	52	39	44	37	36	29	34
31	32	33	40	41	42	49	50	51
48	53	46	45	38	43	30	35	28
7	56	9	64	17	66	25	74	27
24	77	22	69	14	67	6	59	4
1	62	3	70	11	72	19	80	21

Table 2: The standard centrally balanced C_4-face magic labeling $\mathcal{T}(\mathrm{HPSL}(3,3,3,3))$ on $\mathcal{P}_{9,9}$. For convenience, we display $\mathcal{P}_{9,9}$ as a 9×9 projective checkerboard.

Notation 3.37. Let $m \geqslant 3$ be an odd integer. We define the function β given by

$$\beta(m) = \begin{cases} \left((\frac{m-1}{4})!\right)^2, & \text{if } m \equiv 1 \pmod 4, \\ (\frac{m-3}{4})!(\frac{m+1}{4})!, & \text{if } m \equiv 3 \pmod 4. \end{cases}$$

The following theorem gives us the minimum number of distinct C_4-face-magic projective labelings on $\mathcal{P}_{m,n}$ having C_4-face-magic value $2mn+1$ or $2mn+3$ for distinct odd integers m and n.

Theorem 3.38. *[9] Let $m \geqslant 3$ and $n \geqslant 3$ be distinct odd integers. Then the number of distinct C_4-face-magic projective labelings on $\mathcal{P}_{m,n}$ having C_4-face-magic value $2mn+1$ or $2mn+3$ (up to symmetries on the projective plane) is at least*

$$\big(\tau(m,n) + \tau(n,m)\big)2^{m/2+n/2-3}\beta(m)\beta(n).$$

The next theorem gives us the minimum number of distinct C_4-face-magic projective labelings on $\mathcal{P}_{m,m}$ having C_4-face-magic value $2m^2+1$ or $2m^2+3$.

Theorem 3.39. *[9] Let $m \geqslant 3$ be an odd integer. Then the number of distinct C_4-face-magic projective labelings on $\mathcal{P}_{m,m}$ having C_4-face-magic value $2m^2+1$ or $2m^2+3$ (up to symmetries on the projective plane) is at least*

$$\tau(m,m)2^{m-3}\big(\beta(m)\big)^2.$$

We determine the number of centrally balanced C_4-face-magic projective labelings on $\mathcal{P}_{m,n}$ when m and n are distinct odd integers in the theorem below.

Theorem 3.40. *Let $m \geqslant 3$ and $n \geqslant 3$ be distinct odd integers. Then the number of distinct centrally balanced C_4-face-magic projective labelings on $\mathcal{P}_{m,n}$ (up to symmetries on the projective plane) is*

$$\big(\tau(m,n)+\tau(n,m)\big)2^{m/2+n/2-3}(\tfrac{m-1}{2})!(\tfrac{n-1}{2})!.$$

Proof. We first count the number of centrally balanced C_4-face-magic projective labelings on $\mathcal{P}_{m,n}$. For each standard centrally balanced C_4-face-magic labeling X on $\mathcal{P}_{m,n}$, there are $2^{m_0}m_0!2^{n_0}n_0!$ elementary projective labeling operations that give rise to $2^{m_0}m_0!2^{n_0}n_0!$ centrally balanced C_4-face-magic projective labelings on $\mathcal{P}_{m,n}$ associated with X. Of these elementary projective labeling operations, 4 of them result from the symmetries R_0, R_{180}, H and V. Thus there are $\frac{1}{4}(2^{m_0}m_0!2^{n_0}n_0!) = 2^{m/2+n/2-3}(\frac{m-1}{2})!(\frac{n-1}{2})!$ distinct centrally balanced C_4-face-magic projective labelings on $\mathcal{P}_{m,n}$ associated with X (up to symmetries on the projective plane). By Theorem 3.34, each (m,n)-projective factorization sequence F and each (n,m)-projective factorization sequence F' are associated with unique standard centrally balanced C_4-face-magic projective labelings X on $\mathcal{P}_{m,n}$ given by $\mathcal{T}(\mathrm{HPSL}(F))$ and $\mathcal{T}(\mathrm{VPSL}(F'))$. Thus there are $\big(\tau(m,n)+\tau(n,m)\big)2^{m/2+n/2-3}(\frac{m-1}{2})!(\frac{n-1}{2})!$ distinct standard centrally balanced C_4-face-magic labeling X on $\mathcal{P}_{m,n}$ (up to symmetries on the projective plane). □

We determine the number of centrally balanced C_4-face-magic projective labelings on $\mathcal{P}_{m,m}$ in the next theorem below.

Theorem 3.41. *Let $m \geqslant 3$ be an odd integer. Then the number of distinct centrally balanced C_4-face-magic projective labelings on $\mathcal{P}_{m,m}$ (up to symmetries on the projective plane) is*

$$\tau(m,m)2^{m-3}\big((\tfrac{m-1}{2})!\big)^2.$$

The proof is similar to that of Theorem 3.40. We now state the minimum number of C_4-face-magic labelings on $\mathcal{P}_{m,n}$ when m and n are distinct odd integers.

Theorem 3.42. *Let $m \geqslant 3$ and $n \geqslant 3$ be distinct odd integers. Then the number of distinct C_4-face-magic projective labelings on $\mathcal{P}_{m,n}$ (up to symmetries on the projective plane) is at least*

$$\big(\tau(m,n)+\tau(n,m)\big)2^{m/2+n/2-3}\big((\tfrac{m-1}{2})!(\tfrac{n-1}{2})!+2\beta(m)\beta(n)\big).$$

Proof. By Theorem 3.40, there are $\big(\tau(m,n)+\tau(n,m)\big)2^{m/2+n/2-3}(\frac{m-1}{2})!(\frac{n-1}{2})!$ distinct standard centrally balanced C_4-face-magic labeling X on $\mathcal{P}_{m,n}$ (up to symmetries on the projective plane). By Theorem 3.38, for each value $S = 2mn+1$ and $S = 2mn+3$, there are at least $\big(\tau(m,n)+\tau(n,m)\big)2^{m/2+n/2-3}\beta(m)\beta(n)$ C_4-face-magic projective labelings on $\mathcal{P}_{m,n}$ with C_4-face-magic value S. Therefore, by Lemma 2.6, there are at least

$$\big(\tau(m,n)+\tau(n,m)\big)2^{m/2+n/2-3}\big((\tfrac{m-1}{2})!(\tfrac{n-1}{2})!+2\beta(m)\beta(n)\big)$$

distinct C_4-face-magic projective labelings on $\mathcal{P}_{m,n}$ (up to symmetries on the projective plane). $\square$

Theorem 3.43. *Let $m \geqslant 3$ be an odd integer. Then the number of distinct C_4-face-magic projective labelings on $\mathcal{P}_{m,m}$ (up to symmetries on the projective plane) is at least*

$$\tau(m,m)2^{m-3}\Big(\big((\tfrac{m-1}{2})!\big)^2 + 2\big(\beta(m)\big)^2\Big).$$

Proof. We make use of Theorems 3.39 and 3.41 to verify this theorem. The proof is similar to that of Theorem 3.42. $\square$

These results lead us to ask the following question.

Problem 3.44. Can one characterize the C_4-face-magic labelings on the $m \times n$ projective grid graph $\mathcal{P}_{m,n}$ when m and n are even?

Due to Lemma 2.5, the C_4-face-magic value of a labeling in Problem 3.44 must be $2mn + 2$. Curran and Locke [10] have characterized the C_4-face-magic projective labelings on the 4×4 projective grid graph $\mathcal{P}_{4,4}$. They show that there are 144 C_4-face-magic projective labelings on $\mathcal{P}_{4,4}$ up to symmetries on the projective plane.

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