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On degree conditions of semi-balanced 3-partite Hamiltonian graphs

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Abstract

A k-partite graph is said to be a semi-balanced k-partite graph if each partite set has either n or m vertices. We deal with semi-balanced 3-partite graphs. If $G = (V_1 \cup V_2 \cup V_3, E)$ is a semi-balanced 3-partite graph with $|V_1| = |V_2| \geq |V_3| \geq 2$ which satisfies the following conditions: (1) for all $x \in V_i$ $(i = 1, 2)$, $|N(x) \cap V_j| \geq \frac{|V_j|+2}{2}$ $(j = 1, 2, 3,\ j \neq i)$, and (2) for all $x \in V_3$, $|N(x) \cap V_j| \geq \frac{2|V_j|-|V_3|+2}{2}$ $(j = 1, 2)$, then G is Hamiltonian. And we also show that a semi-balanced 3-partite graph $G = (V_1 \cup V_2 \cup V_3, E)$, where $|V_1| = |V_2| \geq |V_3|$, is pancyclic if for all $x \in V_i$, $|N(x) \cap V_j| \geq \frac{2|V_j|}{3}$ (for all $j \neq i$).

1 Introduction

In this paper, we deal with simple graphs. For a vertex v of a graph G, the *neighborhood* of v in G is $N_G(v) = \{u \in V(G) \mid uv \in E(G)\}$. Let $\delta(G)$ denote the *minimum degree* of G. For a subset $S \subset V(G)$, $\langle S \rangle$ denotes the subgraph induced by S. A *Hamiltonian cycle* (respectively, *Hamiltonian path*) in G is a cycle (respectively, path) which contains every vertex of G. Furthermore, for a subgraph H of G, a cycle (respectively, path) which contains every vertex of H is said to be an *H-Hamiltonian cycle* (respectively, *H-Hamiltonian path*).

A graph G is said to be *pancyclic* if G contains a cycle of length l, for all $3 \leq l \leq |V(G)|$. A bipartite graph G with $2n$ vertices is said to be *bipancyclic* if G contains a cycle of length $2l$ for all $2 \leq l \leq n$. A k-partite graph is said to be a *balanced k-partite graph* if each partite set has the same number of vertices. A k-partite graph is said to be a *semi-balanced k-partite graph* if each partite set has either n or m vertices. A k-regular spanning subgraph of G is said to be a k-$factor$. And for a subgraph H of G, a k-regular spanning subgraph of H is said to be an H-k-factor. For two graphs G_1 and G_2, the union of G_1 and G_2, denoted by $G_1 \cup G_2$, is the graph with $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$.

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The earliest works on degree conditions of Hamiltonian graphs are given by Dirac [3] and Ore [7].

Theorem 1.1 (Dirac [3]) *Let G be a graph with $n \geq 3$ vertices. If $\delta(G) \geq \frac{n}{2}$, then G is Hamiltonian.*

Theorem 1.2 (Ore [7]) *Let G be a graph with $n \geq 3$ vertices. If $d(u) + d(v) \geq n$ for any two non-adjacent vertices u and v of G, then G is Hamiltonian.*

For balanced bipartite graphs, Moon and Moser [6] gave the following result.

Theorem 1.3 (Moon-Moser [6]) *Let G be a balanced bipartite graph with $2n \geq 4$ vertices. If $\delta(G) \geq \frac{n+1}{2}$, then G is Hamiltonian.*

A graph G with a 1-factor is said to be a *Hamiltonian alternating cycle graph* (*HAC-graph*) if every 1-factor is contained in a Hamiltonian cycle of G. A graph with a 1-factor is said to be a *Hamiltonian alternating path graph* (*HAP-graph*) if every 1-factor is contained in a Hamiltonian path of G.

For balanced bipartite graphs, Las Vergnas [5] gave the following result.

Theorem 1.4 (Las Vergnas [5]) *Let G be a balanced bipartite graph with partite sets V_1, V_2 ($|V_i| = n$). If for each pair x, y of nonadjacent vertices with $x \in V_1, y \in V_2$, we have*

$$\begin{aligned} &\textit{Case 1:} \quad d(x) + d(y) \geq n + 2, \\ &\textit{Case 2:} \quad d(x) + d(y) \geq n + 1, \end{aligned}$$

then, in Case 1, G is an HAC-graph, and in Case 2, G is an HAP-graph.

Moreover, Yokomura [9] gave the following Ore-type condition for a balanced 3-partite graph to be Hamiltonian.

Theorem 1.5 (Yokomura [9]) *Let G be a balanced 3-partite graph with partite sets V_1, V_2 and V_3, where $|V_i| = n$ for $i = 1, 2, 3$. If $|N(u) \cap V_j| + |N(v) \cap V_i| \geq n + 1$ for any two nonadjacent vertices $u \in V_i$ and $v \in V_j$ $(1 \leq i < j \leq 3)$, then G is Hamiltonian.*

Also Chen et al. [2] gave the following Dirac-type condition for a balanced k-partite graph to be Hamiltonian.

Theorem 1.6 (Chen et al. [2]) *Let G be a balanced k-partite graph with kn vertices. If the minimum degree satisfies*

$$\delta(G) > \begin{cases} \left(\dfrac{k}{2} - \dfrac{1}{k+1}\right) n & \textit{if } k \textit{ is odd}, \\ \left(\dfrac{k}{2} - \dfrac{2}{k+2}\right) n & \textit{if } k \textit{ is even}, \end{cases}$$

then G is Hamiltonian.

Almost all known sufficient conditions for a graph to have a Hamiltonian cycle imply that their graphs have many edges. Some sufficient conditions for a graph to be Hamiltonian also imply that it is pancyclic. For example, Ore's result [7] was generalized by Bondy [1].

Theorem 1.7 (Bondy [1]) *Let G be a graph with $n \geq 3$ vertices. If $d(u)+d(v) \geq n$ for any two non-adjacent vertices u and v of G, then G is either pancyclic or the graph $K_{\frac{n}{2},\frac{n}{2}}$.*

Moreover, for balanced bipartite graphs, Schmeichel and Mitchem [8] showed the following result:

Theorem 1.8 (Schmeichel and Mitchem [8]) *Let G be a balanced bipartite graph with $2n$ vertices, $(n > 3)$. If $d(v) \geq \frac{n+1}{2}$ for all $v \in V(G)$, then G is bi-pancyclic.*

In each of the above results on k-partite graphs, the graphs are balanced and thus have partite sets of the same size. In this paper, we consider some sufficient conditions that 3-partite graphs such that one partite set consists of a different number of vertices from the other partite sets to be Hamiltonian or pancyclic.

2 Hamiltonian semi-balanced 3-partite graphs

In this section, we give a degree condition for a semi-balanced 3-partite graph to be Hamiltonian.

Theorem 2.1 *Let G be a semi-balanced 3-partite graph with partite sets V_1, V_2, V_3 and $|V_1| = |V_2| \geq |V_3| \geq 2$. If G satisfies the conditions:*
(1) *for all $x \in V_i$ $(i = 1, 2)$,*

$$|N(x) \cap V_j| \geq \frac{|V_j| + 2}{2} \qquad (j = 1, 2, 3, \quad j \neq i)\text{, and}$$

(2) *for all $x \in V_3$,*

$$|N(x) \cap V_j| \geq \frac{2\,|V_j| - |V_3| + 2}{2} \qquad (j = 1, 2)\,,$$

then G is Hamiltonian.

Proof of Theorem 2.1. Let $|V_1| = |V_2| = n$ and $|V_3| = m$, where $n \geq m \geq 2$. Now G is a semi-balanced 3-partite graph with $2n + m$ vertices.

For $n \leq 3$, the graphs that satisfy the condition in the theorem are $K_{2,2,2}$, $K_{3,3,2}$ and $K_{3,3,3}$. It is obvious that these graphs contain Hamiltonian cycles.

So we assume that $n \geq 4$.

In $\langle V_1 \cup V_2 \rangle$, each vertex $v \in V_1$ is adjacent to $\frac{n+2}{2}$ vertices of V_2 and each vertex $u \in V_2$ is adjacent to $\frac{n+2}{2}$ vertices of V_1. So $\langle V_1 \cup V_2 \rangle$ is a balanced bipartite graph satisfying the conditions of Theorem 1.3. Thus $\langle V_1 \cup V_2 \rangle$ is Hamiltonian and has a 1-factor $F^{1,2}$.

Let A be a set of m edges in $F^{1,2}$, and let $S_1 = V_1 \cap V(A)$, $S_2 = V_2 \cap V(A)$. For $\langle S_1 \cup V_3 \rangle$, each vertex $w \in V_3$ is adjacent to at least

$$\frac{2n - m + 2}{2} - (n - m) = \frac{m + 2}{2}$$

vertices in S_1 and each vertex $v \in S_1$ is adjacent to at least $\frac{m+2}{2}$ vertices in V_3 by condition (1) in the theorem. So $\langle S_1 \cup V_3 \rangle$ is a balanced bipartite graph satisfying the condition of Theorem 1.3. Thus $\langle S_1 \cup V_3 \rangle$ is Hamiltonian and has a 1-factor $F^{1,3} = \{v_i w_i \mid v_i \in S_1, w_i \in V_3,\ i = 1, 2, \ldots, m\}$. Similarly, $\langle S_2 \cup V_3 \rangle$ has a 1-factor $F^{2,3} = \{w_i u_i \mid w_i \in V_3, u_i \in S_2,\ i = 1, 2, \ldots, m\}$. From two 1-factors $F^{1,3}$ and $F^{2,3}$, G has m paths:

$$P_{(i)} = v_i w_i u_i \ \ (v_i \in S_1,\ u_i \in S_2,\ w_i \in V_3,\ i = 1, 2, \ldots, m).$$

We create a new semi-balanced 3-partite graph H by adding each edge $v_i u_i$ $(v_i \in S_1,\ u_i \in S_2)$ if there is no edge $v_i u_i$ in G. Let B be the set of these added edges and $M = \{v_i u_i \mid v_i \in S_1, u_i \in S_2, i = 1, 2, \ldots, m\}$. Then H is a simple semi-balanced 3-partite graph with a 1-factor $F_*^{1,2} = M \cup (F^{1,2} - A)$ in $\langle V_1 \cup V_2 \rangle$. And by Theorem 1.4, $\langle V_1 \cup V_2 \rangle$ has a Hamiltonian cycle $C^{1,2}$ which contains all edges of $F_*^{1,2}$. Then by replacing m edges $u_i v_i$ of M with m paths $P_{(i)}$, respectively, and deleting all edges of B, we can obtain a Hamiltonian cycle of G. □

3 Pancyclic semi-balanced 3-partite graphs

In this section, we give a degree condition for a semi-balanced 3-partite graph to be pancyclic. In the proof of Theorem 3.2, we use following result, that is, Hall's Theorem [4].

Given any sets $S_1, S_2, \ldots, S_k$, we say that an element $s_i \in S_i$ is a *representative* for the set S_i which contains it. If $s_i \neq s_j$ for each i, j with $1 \leq i < j \leq k$, then $\{s_1, s_2, \ldots, s_k\}$ are said to be a *system of distinct representatives* for the sets $S_1, S_2, \ldots, S_k$.

Theorem 3.1 (Hall [4]) *A collection $S_1, S_2, \ldots, S_k$ $(k \geq 1)$ of finite nonempty sets has a system of distinct representatives if and only if the union of every t $(1 \leq t \leq k)$ sets of these sets contains at least t elements.*

Theorem 3.2 *Let G be a semi-balanced 3-partite graph with partite sets V_1, V_2, V_3 and $|V_1| = |V_2| \geq |V_3|$. If G satisfies the conditions:*

for all $x \in V_i$,

$$|N(x) \cap V_j| \geq \frac{2|V_j|}{3} \quad (\textit{for all } j \neq i),$$

then G is pancyclic.

Proof of Theorem 3.2. Let $|V_1| = |V_2| = n$ and $|V_3| = m$, where $n \geq m$. Let G be a semi-balanced 3-partite graph on $2n + m$ vertices with partite sets V_1, V_2 and V_3 that satisfies the condition in the theorem.

Let $n = m = 1$. For all $x \in V_i$,

$$|N(x) \cap V_j| \geq \frac{2\,|V_j|}{3} = \frac{2 \cdot 1}{3} = \frac{2}{3} \quad (\text{for all } j \neq i)\,.$$

Thus $G = K_{1,1,1} = C_3$, and G is pancyclic.

So we assume that $n \geq 2$.

First, we prove that a semi-balanced 3-partite graph has a cycle of length l for all numbers l such that $2n \leq l \leq 2n + m$.

Let $n = 2$. For all $x \in V_i$ $(i = 1, 2)$,

$$|N(x) \cap V_{3-i}| = \frac{2\,|V_{3-i}|}{3} \geq \frac{2 \cdot 2}{3} = \frac{4}{3}.$$

Thus $\langle V_1 \cup V_2 \rangle = K_{2,2}$, and there exists a $\langle V_1 \cup V_2 \rangle$-Hamiltonian cycle of length $2n = 4$. If $n \geq 3$, for all vertices $v \in V_i$ $(i = 1, 2)$, v is adjacent to at least $\frac{2n}{3}$ vertices of V_{3-i}. Thus, from Theorem 1.3, there exists a $\langle V_1 \cup V_2 \rangle$-Hamiltonian cycle of length $2n$ in G.

Let

$$C^{1,2} = v_1 u_1 v_2 u_2 \cdots v_n u_n v_1 \quad (v_i \in V_1, u_i \in V_2)$$

be a $\langle V_1 \cup V_2 \rangle$-Hamiltonian cycle of length $2n$ and for $w \in V_3$, let $S_w = \{e = xy \in E(C^{1,2}) \mid \{x, y\} \subset N(w)\}$. We assume that $\{S_w \mid w \in V_3\}$ has a system of distinct representatives $S = \{e_w = x_w y_w \mid e_w \in S_w, w \in V_3\}$. And let $P = \{p_w = x_w y_w \mid x_w y_w \in S,\ w \in V_3\}$. Then for all $s \in \{1, \ldots, m\}$, by replacing s edges on $C^{1,2}$ with s disjoint paths of length 2 in P, we can expand $C^{1,2}$ into a cycle of length $2n + s$ in G.

For each $w_i \in V_3$, since $|V_j - N(w_i)| \leq \frac{n}{3}$ for $j = 1, 2$, it follows that $|V_1 \cup V_2 - N(w_i)| \leq \frac{2n}{3}$. If all vertices of $V_1 \cup V_2 - N(w_i)$ are removed from $C^{1,2}$, the remaining edges on $C^{1,2}$ are edges whose two end vertices are adjacent to w_i. By deleting one vertex in $C^{1,2}$, at most two edges are removed from $C^{1,2}$. Since $|E\,(C^{1,2})| = 2n$, $C^{1,2} - (V_1 \cup V_2 - N(w_i))$ has at least $\frac{2n}{3}$ $\left(= 2n - \left(\frac{2n}{3} \times 2\right)\right)$ edges for each $w_i \in V_3$. Therefore, for each vertex $w_i \in V_3$, there exist at least $\frac{2n}{3}$ edges on $C^{1,2}$ whose two end vertices are adjacent to w_i.

Let T be a subset of V_3 which has more than $\frac{2m}{3}$ vertices and $|T| = \frac{2m}{3} + \alpha$. We assume that there exists an edge f of $C^{1,2}$ whose two end vertices have no common adjacent vertices in T. Thus, at least one of the end vertices of f, say v, is adjacent to at most $\frac{m}{3} + \frac{\alpha}{2}$ vertices of T. And v is adjacent to at most

$$\left(\frac{m}{3} + \frac{\alpha}{2}\right) + \left(\frac{m}{3} - \alpha\right) = \frac{2m}{3} - \frac{\alpha}{2} \ \left(< \frac{2m}{3}\right)$$

vertices of V_3. This contradicts the condition of our theorem. Therefore, both end vertices of each edge of $C^{1,2}$ have a common adjacent vertex $w_i \in T$.

Now, for each $w_i \in V_3$, let $E(w_i)$ be the set of edges on $C^{1,2}$ whose two end vertices are adjacent to w_i. Then for $W \subseteq V_3$,

$$\text{if} \quad |W| \leq \frac{2m}{3}, \text{ then } \left|\bigcup_{w_i \in W} E(w_i)\right| \geq \frac{2n}{3} \ \left(\geq \frac{2m}{3}\right),$$

$$\text{if} \quad \frac{2m}{3} < |W|, \text{ then } \left|\bigcup_{w_i \in W} E(w_i)\right| = 2n.$$

Thus, by Theorem 3.1, the collection $\{S_w \mid w \in V_3\}$ has a system of distinct representatives and G has a cycle of length of $2n + s$ for all $s \in \{1, \ldots, m\}$.

To complete the proof of Theorem 3.2, we prove that G contains an l-cycle for every $3 \leq l \leq 2n + 1$.

Let $vu \in E(\langle V_1 \cup V_2 \rangle)$ $(v \in V_1,\ u \in V_2)$. Since $|N(v) \cap N(u) \cap V_3| \geq \frac{m}{3}$, v and u have a common adjacent vertex, say w, in V_3. Then $C = vwuv$ is a 3-cycle of G.

If $n = 3$, $\langle V_1 \cup V_2 \rangle$ has a Hamiltonian cycle $C^{1,2} = v_1u_1v_2u_2v_3u_3v_1$ and thus has a path $v_1u_1v_2u_2$. Since $|N(v_1) \cap N(v_2) \cap V_3| \geq \frac{m}{3}$, v_1 and v_2 have a common adjacent vertex, say w', in V_3. Then $C = v_1u_1v_2w'v_1$ is a 4-cycle of G. Similarly, since $|N(v_1) \cap N(u_2) \cap V_3| \geq \frac{m}{3}$, v_1 and u_2 have a common adjacent vertex, say w'', in V_3. Then $C = v_1u_1v_2u_2w''v_1$ is a 5-cycle of G.

If $n \geq 4$, each vertex $v \in V_1$ is adjacent to at least $\frac{2n}{3}$ vertices in V_2 and each vertex $u \in V_2$ is adjacent to at least $\frac{2n}{3}$ vertices in V_1. By Theorem 1.8, $\langle V_1 \cup V_2 \rangle$ has a cycle

$$C^{1,2}_{2t} = v_1u_1v_2u_2 \cdots v_tu_tv_1 \ \ (v_i \in V_1,\ u_i \in V_2)$$

of length $2t$ for all $t \in \{2, \ldots, n\}$. Since $|N(v_1) \cap N(u_1) \cap V_3| \geq \frac{m}{3}$, v_1 and u_1 have a common adjacent vertex, say w^*, in V_3. Then

$$v_1w^*u_1v_2u_2 \cdots v_tu_tv_1 \ \ (v_i \in V_1,\ u_i \in V_2)$$

is a $(2t + 1)$-cycle. Thus G contains an l-cycle for every $3 \leq l \leq 2n + 1$.

We have already proved that a semi-balanced 3-partite graph G contains a cycle of length of $2n+s$ for all $s \in \{1, \ldots, m\}$; thus for all numbers l such that $3 \leq l \leq 2n+m$, G has a cycle of length l. Therefore G is pancyclic. □

A split graph is a graph in which the vertex set can be partitioned into a clique and an independent set. We have the following result by Theorem 3.2.

Proposition 3.3 *Let G be a split graph with a clique K of $2n$ vertices and an independent set V_3 of m vertices, where $m < n$. If the clique K can be divided into two subsets V_1 and V_2 such that $|V_1| = |V_2| = n$, and*

$$|N(x) \cap V_3| \geq \frac{2m}{3} \qquad \text{for all } x \in V_1 \cup V_2, \text{ and}$$

$$|N(x) \cap V_i| \geq \frac{2n}{3} \quad (i = 1,\ 2) \quad \text{for all } x \in V_3,$$

then G is pancyclic.

Remarks

Hamiltonicity and pancyclicity in semi-balanced 3-partite graphs with $|V_1| \geq |V_2| = |V_3|$ are open problems, and should be considered by other approaches.

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