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On the fifth chromatic coefficient

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Abstract

In 1980, E. J. Farrell derived an explicit expression for the third and fourth coefficients, $\mathbf{c}_{n-3}(\mathcal{G})$ and $\mathbf{c}_{n-4}(\mathcal{G})$, in the chromatic polynomial of a general connected graph $\mathcal{G}$ (the zeroth, first and second terms were known). In this paper we explicitly calculate the fifth chromatic coefficient $\mathbf{c}_{n-5}(\mathcal{G})$. The method involves a synthesis of Farrell's original approach, together with some combinatorics associated to block-counting and purification processes.

1 Introduction

Birkhoff's original motivation behind introducing the chromatic polynomial in 1912 was to find an algebraic proof of the Four Colour Theorem for planar graphs [4]. Throughout $\mathcal{G}$ will denote a general connected graph with n vertices and m edges. The chromatic polynomial $\chi(\mathcal{G};\lambda)$ interpolates the number of λ-colourings of $\mathcal{G}$, and is a polynomial of degree n with integer coefficients. If one expands this into

$$\chi(\mathcal{G};\lambda) \;=\; \sum_{j=0}^{n} \mathbf{c}_j(\mathcal{G}) \cdot \lambda^j \;=\; \mathbf{c}_n(\mathcal{G})\lambda^n + \mathbf{c}_{n-1}(\mathcal{G})\lambda^{n-1} + \mathbf{c}_{n-2}(\mathcal{G})\lambda^{n-2} \;+\; \dots$$

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then it is known that $\mathbf{c}_n(\mathcal{G}) = 1$, $\mathbf{c}_{n-1}(\mathcal{G}) = -m$, $\mathbf{c}_{n-2}(\mathcal{G}) = \binom{m}{2} - t$, with t the number of subgraphs of $\mathcal{G}$ which are triangles. Whitney [19] alternatively expresses

$$\chi(\mathcal{G};\lambda) \;=\; \sum_{k,s\in\mathbb{N}} (-1)^s (k,s)_{\mathcal{G}} \cdot \lambda^k \tag{1.1}$$

where $(k,s)_{\mathcal{G}}$ counts the number of subgraphs of $\mathcal{G}$ with k components and s edges.

1.1 Chromatic Roots

The roots of chromatic polynomials (i.e. *chromatic roots*) have been widely studied as they are not only of mathematical interest, but are of interest in statistical mechanics where the chromatic polynomial is related to the zero-temperature antiferromagnetic Potts model partition function [3]. The integer chromatic roots of a graph G are $\{0, 1, \ldots, \chi(G) - 1\}$ where $\chi(G)$ is the chromatic number of G. There are no real negative chromatic roots, but the real roots are dense in $[32/27, \infty)$ from [17, 11], and the complex roots are dense in $\mathbb{C}$ by [16]. Birkhoff and Lewis [5] proved that all chromatic roots of planar graphs are less than 5. The chromatic roots of planar graphs are dense in the interval $[32/27, 4]$, aside from a small interval around $\tau + 2$ where τ is the golden ratio [17, 15, 13].

Every chromatic root is an algebraic integer, but not every algebraic integer is a chromatic root. The question is: which algebraic integers are chromatic roots? The following two conjectures arose at an Isaac Institute program on 'Combinatorics and Statistical Mechanics':

Conjecture 1.1 (The $\alpha+n$ conjecture) *Let α be an algebraic integer. Then there exists a natural number n such that $\alpha + n$ is a chromatic root.*

Conjecture 1.2 (The $n\alpha$ conjecture) *Let α be an algebraic integer. Then $n\alpha$ is a chromatic root for any natural number n.*

The $\alpha + n$ conjecture has been shown to be true for the quadratic and cubic cases [7], and evidence towards the quartic case is given in [8]. Expressions for specific chromatic coefficients may provide information on which algebraic integers can be chromatic roots. For example, they are used in [8] to identify bicliques of small order that have $\alpha+n$ as a chromatic root for a given algebraic integer α. Expressions for the coefficients $\mathbf{c}_{n-i}(\mathcal{G})$, $i \in [0, 4]$ in terms of linear combinations of numbers of induced subgraphs are known [9], and also in terms of linear combinations of homomorphisms [2]. In this paper, we give an expression for the fifth coefficient $\mathbf{c}_{n-5}(\mathcal{G})$ in terms of a linear combination of numbers of induced subgraphs, expanding the work in Farrell's paper.

1.2 Overview of Farrell's work

Henceforth we shall use the graph label for a particular graph to also indicate the number of copies of that graph inside $\mathcal{G}$. For example, using this notation $m = K_2$

because the number of edges in $\mathcal{G}$ is the number of copies of K_2 inside $\mathcal{G}$; likewise $t = K_3$ is the number of triangles, etc. To introduce some further terminology, a subgraph S of $\mathcal{G}$ is called an *induced subgraph* (or often by Farrell a *pure subgraph*) if $\widetilde{S} = S$, where $\widetilde{S}$ denotes the subgraph of $\mathcal{G}$ induced by the set of vertices of S. The following two beautiful results were established by Farrell in early 1980.

Theorem 1.3 *[9, Thm 1] The coefficient $\mathbf{c}_{n-3}(\mathcal{G})$ of λ^{n-3} in $\chi(\mathcal{G};\lambda)$ equals*

$$-\binom{m}{3} + (m-2)\cdot t + B - 2\cdot K_4$$

*where B indicates the number of **induced** four-cycles C_4 contained inside $\mathcal{G}$.*

Theorem 1.4 *[9, Thm 2] The coefficient $\mathbf{c}_{n-4}(\mathcal{G})$ of λ^{n-4} in $\chi(\mathcal{G};\lambda)$ equals*[1]

$$\binom{m}{4} - \binom{m-2}{2}t + \binom{t}{2} - (m-3)B + (2m-9)\cdot K_4 - 6\cdot K_5 - D + M + 3R + 2P$$

*where D denotes the number of **induced** five-cycles in $\mathcal{G}$, M is equal to the number of **induced** theta graphs $\theta_{2,2,2}$, R counts the number of **induced** wheels on five vertices, and lastly P is the number of **induced** wheels on five vertices missing a single spoke.*

For the λ^{n-i}-coefficient above, the result is proved by systematically counting all subgraphs with $k = n - i$ components and s edges that appear in Equation (1.1). By [9, Lemma 1] the only subgraphs containing n vertices and $n - i$ components are those with $n - c$ isolated vertices and $c - i$ non-empty components of edges, where the integer c lies inside the range $i < c \leqslant 2i$.

If $i = 3$ then $c \in \{4, 5, 6\}$: there are six graphs with 4 vertices and 1 non-trivial component, two graphs with 5 vertices and 2 non-trivial components, and a single graph with 6 vertices and 3 non-trivial components; this yields $6 + 2 + 1 = 9$ graphs.

If $i = 4$ then $c \in \{5, 6, 7, 8\}$: there are twenty-one graphs with 5 vertices and 1 non-trivial component, nine graphs with 6 vertices and 2 non-trivial components, two graphs with 7 vertices and 3 non-trivial components, and a single graph with 8 vertices and 4 non-trivial components; this gives $21 + 9 + 2 + 1 = 33$ possibilities.

Lastly for $i = 3$ (respectively $i = 4$), Farrell determines the multiplicity of all the induced subgraphs inside the 9 (respectively 33) graphs above with $c - i$ non-trivial components.

1.3 Bicliques and the $\alpha + n$-conjecture

One approach to solving the $\alpha + n$ conjecture is to construct a graph that has a chromatic polynomial which is a product of some linear factors, and an additional *interesting* factor $f(\lambda)$ where $f(\alpha) = 0$. An (r, k)*-biclique* is a graph consisting of two

[1] Note there is a typographical error in [9, Thm 2], namely "$-\binom{t}{2}$" should actually be "$+\binom{t}{2}$".

cliques of order r and k respectively, and some additional bridging edges connecting these cliques. The chromatic polynomial of the (r, k)-biclique is $\lambda(\lambda-1)(\lambda-2)\dots(\lambda-k)f(\lambda)$ where $f(\lambda)$ is a polynomial of degree r.

Biclique constructions have proved useful in showing that the $\alpha + n$ conjecture holds for quadratic [7] and cubic algebraic integers [6], as well as providing strong evidence that the conjecture holds for quartic integers [8]. Furthermore, for every pair of integers r, s and finite group $\mathcal{G}$ such that there is a quintic number field of signature (r, s) and Galois group $\mathcal{G}$, there exists a $(5, k)$-biclique whose chromatic polynomial has a splitting field with signature (r, s) and Galois group $\mathcal{G}$ by [10].

The expressions for chromatic coefficients $\mathbf{c}_{n-i}(\mathcal{G})$ provide both algebraic information about the chromatic polynomial and structural information about the graph. The expressions for the first four coefficients have played an important role in constructing bicliques that have a chromatic polynomial with the required interesting factor. Identifying an expression for the fifth coefficient in terms of induced subgraphs provides yet more information on this interesting factor, and enables the investigation of the chromatic roots of (r, k)-bicliques for $r > 4$.

1.4 Notation and conventions

Let H_1 and H_2 be two arbitrary graphs. We use the terminology $H_1 \bullet H_2$ to indicate the graph with components H_1 and H_2. Furthermore, if both H_1 and H_2 contain a copy of K_r then we shall write $H_1 \cup_r H_2$ for the graph obtained by identifying an r-clique inside H_1 with an r-clique in H_2. In certain cases there may be non-isomorphic graphs obtained via this process of gluing together two graphs on an r-clique, whenever there are multiple copies of K_r to choose from. However we note that all graphs $K_{n_1} \cup_r K_{n_2}$ must be isomorphic.

Name	Description
K_n	The complete graph of order n
$K_n - e$	The graph obtained by deleting an edge from K_n
K_{n_1,n_2}	The complete bipartite graph on n_1 and n_2 vertices
C_n	The cycle graph of order n
W_n	The wheel graph with a single centre vertex and n rim vertices
$W_n -$ rim	The wheel graph on $n+1$ vertices with a rim edge removed
$W_n -$ spoke	The wheel graph on $n+1$ vertices with a spoke edge removed
Z	The zig-zag graph on four vertices, namely $K_3 \cup_2 K_3$
$\theta_{n_1,n_2,\dots,n_i}$	The theta-graph with $i \geqslant 3$ paths of length $(n_1, n_2, \dots, n_i)$
$d \star K_3$	The graph obtained by gluing $d \geqslant 2$ triangles onto a single edge

Table 1: Common graph types that appear throughout the remainder of the text.

1.5 The main result

Our primary task is to compute the next chromatic coefficient in the alternating sequence of integers $1, -m, \binom{m}{2} - t,\ \mathbf{c}_{n-3}(\mathcal{G}),\ \mathbf{c}_{n-4}(\mathcal{G})$. Bearing in mind Farrell's approach in [9], we shall first introduce a labelling '$G^{\bullet}$' for graphs having c vertices with $5 < c \leqslant 2 \times 5$ and $c-5$ non-trivial components; these are exhaustively listed in Appendix A together with diagrams of each graph. We will also use the non-negative integer $\mathcal{P}(H)$ to indicate the number of times H occurs as an *induced* subgraph of $\mathcal{G}$.

Theorem 1.5 *The chromatic coefficient* $\mathbf{c}_{n-5}(\mathcal{G})$ *of* λ^{n-5} *in* $\chi(\mathcal{G};\lambda)$ *decomposes*[2] *into the following thirty-three (induced) terms:*

$$\begin{aligned}\mathbf{c}_{n-5}(\mathcal{G}) = &-\binom{m}{5} + \binom{m-2}{3}t - \binom{t}{2}(m-4) + 6(m-8)K_5 - 24K_6 \\ &+ \left(2t - 3m + 12 - 2\binom{m-6}{2}\right)K_4 + \left(\binom{m-3}{2} - t\right)\mathcal{P}(C_4) \\ &+ (m-4)\mathcal{P}(C_5) + \mathcal{P}(C_6) - 3(m-6)\mathcal{P}(W_4) - 4\mathcal{P}(W_5) \\ &- 2(m-5)\mathcal{P}\big(W_4 - \text{spoke}\big) - 3\mathcal{P}\big(W_5 - \text{spoke}\big) - \mathcal{P}(G^{105}) \\ &- (m-3)\mathcal{P}(\theta_{2,2,2}) - \mathcal{P}(\theta_{2,2,3}) + \mathcal{P}(\theta_{2,2,2,2}) - 4\mathcal{P}(K_{3,3}) + 2\mathcal{P}(G^{65}) \\ &+ 3\mathcal{P}(G^{80}) + 6\mathcal{P}(G^{95}) + 4\mathcal{P}(G^{97}) + 4\mathcal{P}(G^{102}) + 5\mathcal{P}(G^{107}) + 8\mathcal{P}(G^{108}) \\ &+ 4\mathcal{P}(G^{109}) + 12\mathcal{P}(G^{110}) + 16\mathcal{P}(G^{111}) - 2\mathcal{P}(G^{75}) - \mathcal{P}(G^{78}) \\ &- 2\mathcal{P}(G^{90}) - 4\mathcal{P}(G^{106}).\end{aligned}$$

The proof of this result is given in Section 2 and is an extension of Farrell's method. However it also relies on laborious computer calculation of subgraph structures, which would certainly have been unavailable in 1980 during the preparation of [9]. For instance, the initial number of graphs which appear in Equation (1.1) is 153 and our various methods allow us to further eliminate approximately 120 graphs, thereby leaving us with the expression above which has comparatively fewer terms. This formula for $\mathbf{c}_{n-5}(\mathcal{G})$ is best possible in the sense that any alternative expression would then, of necessity, involve exactly the same collection of induced subgraphs.

1.6 Applications

There exist several known inequalities for the standard basis expansion of $\chi(\mathcal{G};\lambda) = \sum_{j=0}^{n} \mathbf{c}_j(\mathcal{G}) \cdot \lambda^j$. The cleanest (and perhaps easiest) bounds on $\mathbf{c}_{n-i}(\mathcal{G})$ to state are those given by

$$\begin{aligned}&\binom{n-1}{n-i} + (m-n+1)\binom{n-2}{n-i-1} \\ &\qquad \leqslant \left|\mathbf{c}_{n-i}(\mathcal{G})\right| \leqslant \binom{m}{i} - \binom{m-g+2}{i-g+2} + \binom{m-\eta_g-g+2}{i-g+2}\end{aligned} \tag{1.2}$$

[2] If $\mathcal{G}$ is a regular map then this expression simplifies to yield Whitney's formula from [18].

assuming the graph $\mathcal{G}$ is both connected and simple.

Here $g \geqslant 1$ denotes the girth of $\mathcal{G}$ i.e. the length of the smallest cycle in $\mathcal{G}$, and η_g is the number of circuits in $\mathcal{G}$ of length g. The lower bound was obtained by Aigner [1] building on an earlier result of Read [14], while the upper bound is from Li and Tian's seminal paper [12]. In certain special cases, these bounds have been further sharpened by the efforts of Weickert, Gernert, Chang and others.

The result below is a consequence of Theorem 1.5 and Farrell's formulae [9].

Corollary 1.6 *If $\mathcal{G}$ denotes a graph with girth $g > 3$, then:*

(i) $\left|\mathbf{c}_{n-3}(\mathcal{G})\right| = \binom{m}{3} - \mathcal{P}(C_4)$;

(ii) $\left|\mathbf{c}_{n-4}(\mathcal{G})\right| = \binom{m}{4} - (m-3)\cdot\mathcal{P}(C_4) - \mathcal{P}(C_5) + \mathcal{P}(\theta_{2,2,2})$;

(iii)
$$\left|\mathbf{c}_{n-5}(\mathcal{G})\right| = \binom{m}{5} - \binom{m-3}{2}\mathcal{P}(C_4) - (m-4)\mathcal{P}(C_5) - \mathcal{P}(C_6) + \mathcal{P}(G^{78}) + (m-3)\mathcal{P}(\theta_{2,2,2}) \ + \ \mathcal{P}(\theta_{2,2,3}) - \mathcal{P}(\theta_{2,2,2,2}) + 4\cdot\mathcal{P}(K_{3,3}).$$

Proof: Since $\mathcal{G}$ is triangle-free, none of K_n, W_n, (W_n − spoke) or (W_n − rim) with $n \geqslant 3$ can occur as subgraphs of $\mathcal{G}$ because each of these contains triangles; assertions (i) and (ii) now follow directly from Theorems 1.3 and 1.4 respectively. Furthermore, by the same reasoning $\mathcal{G}$ contains none of the possible subgraphs

$$\left\{G^{65}, G^{75}, G^{79}, G^{80}, G^{90}, G^{95}, G^{93}, G^{97}, G^{102}, G^{106}, G^{107}, G^{108}, G^{109}, G^{110}, G^{111}\right\}$$

which all have triangles (see Appendix A), hence assertion (iii) immediately follows from the explicit formula stated in Theorem 1.5. □

The advantage of using Theorems 1.3, 1.4 and 1.5 over the bounds in (1.2) is that they give an exact formula for $\mathbf{c}_{n-i}(\mathcal{G})$ at every $i \leqslant 5$; the disadvantage is that we have not yet found explicit formulae for $\mathbf{c}_{n-6}(\mathcal{G})$, $\mathbf{c}_{n-7}(\mathcal{G})$, and beyond. The techniques presented in this paper should make their calculation possible, although we estimate that an exact formula for $\mathbf{c}_{n-6}(\mathcal{G})$ will contain at least 130 induced terms!

2 Calculating the fifth chromatic coefficient

We now give the proof of Theorem 1.5. Applying Equation (1.1), the coefficient $\mathbf{c}_{n-5}(\mathcal{G})$ of λ^{n-5} in the chromatic polynomial can be expressed via subgraphs as

$$\begin{aligned}\mathbf{c}_{n-5}(\mathcal{G}) = &-\big(F_1+F_2+F_6+F_9+F_{11}+F_{13}+F_{14}+F_{15}+F_{16}+F_{22}\big)\\&+\big(F_3+F_5+F_{10}+F_{12}+F_{17}+F_{18}+F_{19}+F_{20}+F_{21}\big)\\&+\big(G^1+G^3+G^{56}+G^{58}+G^{62}+G^{64}+G^{66}+G^{68}+G^{70}+G^{72}\\&+G^{75}+G^{78}+G^{86}+G^{96}+G^{100}+G^{108}+G^{111}+G^{76}+G^{80}\\&+G^{82}+G^{87}+G^{92}+G^{94}+G^{98}+G^{102}+G^{104}+G^{105}+G^{90}\big)\\&-\big(G^0+G^2+G^{110}+G^{101}+G^{55}+G^{81}+G^{95}+G^{97}+G^{103}+G^{106}\\&+G^{107}+G^{109}+G^{57}+G^{59}+G^{65}+G^{67}+G^{71}+G^{74}+G^{77}\\&+G^{79}+G^{83}+G^{89}+G^{91}+G^{93}+G^{99}+G^{63}+G^{69}+G^{73}\big).\end{aligned} \tag{2.1}$$

Here the graphs $G^\bullet$ are those listed in Appendix A. Also, to simplify the length of the equations, the families $F_1,\dots,F_{22}$ are disjoint subsets of $\{G^0,\dots,G^{156}\}$ consisting of graphs with a common block structure; in other words each F_i is a sum of G_i's, and no G_i can occur in two different F_i's – see Appendix B for details.

2.1 Deforestation

We first eliminate the forest term F_1 from our formula.

Lemma 2.1 *(i)* $F_1 = \binom{m}{5} - (F_8 + Z + C_5 + F_4)$;
(ii) $F_4 = t \times \binom{m-3}{2} - 2Z$;
(iii) $F_8 = 3\mathcal{P}(K_4)\cdot(m-6) + \mathcal{P}(Z)\cdot(m-5) + \mathcal{P}(C_4)\cdot(m-4)$.

PROOF: To begin with, if we choose 5 edges from m edges then the possible subgraphs that arise are (a) a forest on five edges, (b) a four-cycle and an edge, (c) a five-cycle, (d) graphs with three blocks: two K_2's and a K_3. As a graph equation,

$$\begin{aligned}\binom{m}{5} \;=\; &\big(\text{all forests on 5 edges}\big) \;+\; \big(C_4 \cup_1 K_2 + C_4 \bullet K_2 + Z\big)\\&+ C_5 \;+\; \big(\text{all graphs with 3 blocks: two } K_2\text{'s and a } K_3\big)\end{aligned}$$

so that $\binom{m}{5} = F_1 + (F_8 + Z) + C_5 + F_4$, which proves part (i).

To establish (ii), if we choose a triangle followed by a couple of edges then the possible subgraphs are either (a) all graphs with 3 blocks: two K_2's and a K_3, or (b) a zig-zag graph, which can clearly be obtained in *two separate ways*; therefore

$$t \times \binom{m-3}{2} \;=\; \big(\text{all graphs with 3 blocks: two } K_2\text{'s and a } K_3\big) \;+\; 2Z$$

i.e. $t \times \binom{m-3}{2} = F_4 + 2Z$, as required.

Finally, F_8 comprises subgraphs consisting of the two blocks: a C_4 and a K_2. Now a four-cycle can be identified with a subgraph of K_4 in three different ways, but can only occur as a subgraph of a zig-zag in a unique way; as a consequence

$$F_8 \;=\; \mathcal{P}(C_4)\cdot(m-4) + 3\times\mathcal{P}(K_4)\cdot(m-6) + \mathcal{P}(Z)\cdot(m-5)$$

and assertion (iii) is proved. □

It follows directly from Lemma 2.1(i)–(iii) that

$$F_1 = \binom{m}{5} - t\times\binom{m-3}{2} - \big(3\mathcal{P}(K_4)(m-6) + \mathcal{P}(Z)(m-5) + \mathcal{P}(C_4)(m-4)\big) - C_5 + Z$$

and substituting this into Equation (2.1), one can rewrite $\mathbf{c}_{n-5}(\mathcal{G})$ into the form

$$\begin{aligned}
&C_5 - Z - \big(F_2 + F_6 + F_9 + F_{11} + F_{13} + F_{14} + F_{15} + F_{16} + F_{22}\big)\\
&+ \big(F_3 + F_5 + F_{10} + F_{12} + F_{17} + F_{18} + F_{19} + F_{20} + F_{21}\big)\\
&+ \big(G^1 + G^3 + G^{56} + G^{58} + G^{62} + G^{64} + G^{66} + G^{68} + G^{70} + G^{72} + G^{75}\\
&\quad + G^{78} + G^{86} + G^{96} + G^{100} + G^{108} + G^{111} + G^{76} + G^{80} + G^{82} + G^{87}\\
&\quad + G^{92} + G^{94} + G^{98} + G^{102} + G^{104} + G^{105} + G^{90}\big)\\
&- \big(G^0 + G^2 + G^{110} + G^{101} + G^{55} + G^{81} + G^{95} + G^{97} + G^{103} + G^{106}\\
&\quad + G^{107} + G^{109} + G^{57} + G^{59} + G^{65} + G^{67} + G^{71} + G^{74} + G^{77} + G^{79}\\
&\quad + G^{83} + G^{89} + G^{91} + G^{93} + G^{99} + G^{63} + G^{69} + G^{73}\big)\\
&-\binom{m}{5} + t\times\binom{m-3}{2} + 3\mathcal{P}(K_4)(m-6) + \mathcal{P}(Z)(m-5) + \mathcal{P}(C_4)(m-4).
\end{aligned}$$

Remark. The method of expressing a graph (or set of graphs) in terms of the induced graphs that they embed into will henceforth be called a *purification step* or *purification process*. As an illustration, the previous identity from Lemma 2.1(iii)

$$F_8 \;=\; 3\mathcal{P}(K_4)\cdot(m-6) + \mathcal{P}(Z)\cdot(m-5) + \mathcal{P}(C_4)\cdot(m-4)$$

is a good example of the purification process applied to a particular family F_8.

2.2 Removing F_2 and F_3

We eliminate the next two families from our formula for $\mathbf{c}_{n-5}(\mathcal{G})$, namely F_2 (a pair of graphs) and F_3 (comprising 16 graphs).

Lemma 2.2 *(i)* $F_2 = \mathcal{P}(K_5)\cdot(m-10)$*;*

(ii) $F_3 = t\times\binom{m-3}{3} - C_4\cup_2 K_3 - \mathcal{P}(K_4)(12m-68) - 2\mathcal{P}(Z)(m-5) - 2\times\binom{t}{2} + 2Z.$

PROOF: The first statement is easy. The graphs consisting of a K_5 and an edge are precisely a K_5 and a separate K_2, or alternatively a K_5 joined to a K_2; we thereby obtain the graph equation

$$\binom{K_5}{1} \times (m-10) \;=\; K_5 \cup_1 K_2 + K_5 \bullet K_2, \quad \text{i.e. } \mathcal{P}(K_5) \times (m-10) = F_2.$$

To establish assertion (ii) is harder. If we count the number of graphs formed by choosing a triangle followed by three edges, a simple counting argument establishes

$$\begin{aligned} t \times \binom{m-3}{3} \\ &= \big(\text{all graphs with 4 blocks: a } K_3 \text{ and three } K_2\text{'s}\big) + C_4 \cup_2 K_3 + 4K_4 \\ &+ 2 \times \Big(K_3 \bullet K_3 + K_3 \cup_2 K_3 + \big(\text{all graphs with 2 blocks: a } Z \text{ and a } K_2\big)\Big) \end{aligned}$$

or equivalently, one can write $F_3 = t \times \binom{m-3}{3} - C_4 \cup_2 K_3 - 4\mathcal{P}(K_4) - 2 \times (F_{23} + F_7)$. However the number of zig-zag graphs together with an edge is given by

$$\big(\text{all graphs with 2 blocks: a } Z \text{ and a } K_2\big) \;=\; 6\mathcal{P}(K_4)(m-6) + \mathcal{P}(Z)(m-5)$$

i.e. $F_7 = 6\mathcal{P}(K_4)(m-6) + \mathcal{P}(Z)(m-5)$, in which case

$$F_3 \;=\; t \times \binom{m-3}{3} - C_4 \cup_2 K_3 - \mathcal{P}(K_4)(12m-68) - 2\mathcal{P}(Z)(m-5) - 2F_{23}.$$

If we count up the graphs obtained by choosing two triangles in succession, then

$$\binom{t}{2} \;=\; \big(\text{all graphs with 2 blocks: } K_3 \text{ and } K_3\big) + Z$$

so that $F_{23} = \binom{t}{2} - Z$, and substituting this into our formula for F_3 proves (ii). $\square$

Applying Lemma 2.2 to our current expression, one deduces that $\mathbf{c}_{n-5}(\mathcal{G})$ equals

$$\begin{aligned} &C_5 + Z - C_4 \cup_2 K_3 - \big(F_6 + F_9 + F_{11} + F_{13} + F_{14} + F_{15} + F_{16} + F_{22}\big) \\ &+\big(F_5 + F_{10} + F_{12} + F_{17} + F_{18} + F_{19} + F_{20} + F_{21}\big) \\ &+\big(G^1 + G^3 + G^{56} + G^{58} + G^{62} + G^{64} + G^{66} + G^{68} + G^{70} + G^{72} + G^{75} + G^{78} \\ &\quad + G^{86} + G^{96} + G^{100} + G^{108} + G^{111} + G^{76} + G^{80} + G^{82} + G^{87} + G^{92} + G^{94} \\ &\quad + G^{98} + G^{102} + G^{104} + G^{105} + G^{90}\big) \\ &-\big(G^0 + G^2 + G^{110} + G^{101} + G^{55} + G^{81} + G^{95} + G^{97} + G^{103} + G^{106} + G^{107} \\ &\quad + G^{109} + G^{57} + G^{59} + G^{65} + G^{67} + G^{71} + G^{74} + G^{77} + G^{79} + G^{83} + G^{89} \\ &\quad + G^{91} + G^{93} + G^{99} + G^{63} + G^{69} + G^{73}\big) \\ &-\binom{m}{5} + t \times \binom{m-2}{3} - 2 \times \binom{t}{2} - \mathcal{P}(K_4)(9m-50) - \mathcal{P}(Z)(m-5) \\ &+ \mathcal{P}(C_4)(m-4) \;-\; \mathcal{P}(K_5)(m-10). \end{aligned}$$

2.3 Removing F_5, F_6, F_9 and F_{10}

We now kill off a further four families by relating them to the possible arrangements of a K_4 combined with another block.

Lemma 2.3 *(i)* $F_5 = \mathcal{P}(K_4) \times \binom{m-6}{2} - K_4 \cup_2 K_3$;

(ii) $F_6 = \mathcal{P}(K_4) \times (t-4) - K_4 \cup_2 K_3$

(iii) One has a pair of subgraph identities:

$$\begin{aligned} F_9 =& 6\mathcal{P}(K_4)\binom{m-6}{2} + \mathcal{P}(Z)\binom{m-5}{2} - 2(W_4 - \text{rim}) \\ &- (W_4 - \text{spoke}) - 3(3 \star K_3), \\ F_{10} =& 3\mathcal{P}(K_4)\binom{m-6}{2} + \mathcal{P}(Z)\binom{m-5}{2} + \mathcal{P}(C_4)\binom{m-4}{2} \\ &- 3 \times \theta_{2,2,2} - C_4 \cup_2 K_3. \end{aligned}$$

Proof: To begin with, the number of ways of combining a K_4 with two edges is described by the equation

$$\mathcal{P}(K_4) \times \binom{m-6}{2} = \big(\text{all graphs with 3 blocks: a } K_4 \text{ and two } K_2\text{'s}\big) + K_4 \cup_2 K_3$$

i.e. $\mathcal{P}(K_4) \times \binom{m-6}{2} = F_5 + K_4 \cup_2 K_3$, which therefore establishes assertion (i) holds. Similarly, if we choose a K_4 followed by a single triangle:

$$\mathcal{P}(K_4) \times \binom{t-4}{1} \;=\; K_4 \cup_2 K_3 + K_4 \cup_1 K_3 + K_4 \bullet K_3$$

so that $\mathcal{P}(K_4) \times (t-4) = K_4 \cup_2 K_3 + F_6$, which implies that (ii) must also hold. Finally to show both identities in (iii), if we count all the possible induced subgraphs obtained from a 'zig-zag graph and two edges' then we deduce that

$$\begin{aligned} &\big(\text{all graphs with 3 blocks: a } Z \text{ and two } K_2\text{'s}\big) + 2 \times (W_4 - \text{rim}) \\ &\quad + (W_4 - \text{spoke}) + 3 \times (3 \star K_3) \;=\; 6\mathcal{P}(K_4)\binom{m-6}{2} + \mathcal{P}(Z)\binom{m-5}{2} \end{aligned}$$

or equivalently,

$$F_9 + 2(W_4 - \text{rim}) + (W_4 - \text{spoke}) + 3(3 \star K_3) \;=\; 6\mathcal{P}(K_4)\binom{m-6}{2} + \mathcal{P}(Z)\binom{m-5}{2}.$$

If we instead apply the same induction process to a 'four-cycle and two edges' then

$$\begin{aligned} &\big(\text{all graphs with 3 blocks: a } C_4 \text{ and two } K_2\text{'s}\big) + 3 \times \theta_{2,2,2} + C_4 \cup_2 K_3 \\ &\quad = \; 3\mathcal{P}(K_4)\binom{m-6}{2} + \mathcal{P}(Z)\binom{m-5}{2} + \mathcal{P}(C_4)\binom{m-4}{2}, \end{aligned}$$

in which case

$$F_{10} + 3 \times \theta_{2,2,2} + C_4 \cup_2 K_3 = 3\mathcal{P}(K_4)\binom{m-6}{2} + \mathcal{P}(Z)\binom{m-5}{2} + \mathcal{P}(C_4)\binom{m-4}{2}.$$

These block-counting steps together imply the twin identities stated in (iii). □

Applying all three parts of Lemma 2.3 and noting that $m - 4 + \binom{m-4}{2} = \binom{m-3}{2}$, our expression for $\mathbf{c}_{n-5}(\mathcal{G})$ can now be updated to equal

$$\begin{aligned}
&C_5 + Z - 2(C_4 \cup_2 K_3) + 2 \times (W_4 - \text{rim}) + (W_4 - \text{spoke}) + 3(3 \star K_3) - 3\theta_{2,2,2}\\
&-\big(F_{11} + F_{13} + F_{14} + F_{15} + F_{16} + F_{22}\big) + \big(F_{12} + F_{17} + F_{18} + F_{19} + F_{20} + F_{21}\big)\\
&+\big(G^1 + G^3 + G^{56} + G^{58} + G^{62} + G^{64} + G^{66} + G^{68} + G^{70} + G^{72}\\
&\qquad\qquad +G^{75} + G^{78} + G^{86} + G^{96} + G^{100} + G^{108} + G^{111} + G^{76} + G^{80}\\
&\qquad\qquad +G^{82} + G^{87} + G^{92} + G^{94} + G^{98} + G^{102} + G^{104} + G^{105} + G^{90}\big)\\
&-\big(G^0 + G^2 + G^{110} + G^{101} + G^{55} + G^{81} + G^{95} + G^{97} + G^{103} + G^{106}\\
&\quad + G^{107} + G^{109} + G^{57} + G^{59} + G^{65} + G^{67} + G^{71} + G^{74} + G^{77}\\
&\quad + G^{79} + G^{83} + G^{89} + G^{91} + G^{93} + G^{99} + G^{63} + G^{69} + G^{73}\big)\\
&-\binom{m}{5} + t \times \binom{m-2}{3} - 2 \times \binom{t}{2} - \mathcal{P}(K_4)(t-4) - \mathcal{P}(K_4)(9m-50)\\
&-2\mathcal{P}(K_4)\binom{m-6}{2} - \mathcal{P}(Z)(m-5) + \mathcal{P}(C_4)\binom{m-3}{2} - \mathcal{P}(K_5)(m-10).
\end{aligned}$$

2.4 Removing $F_{11}, \ldots, F_{22}$

We must now eliminate the last twelve families.

Lemma 2.4 *(i)* $F_{11} = \left(\binom{t}{2} - 6\mathcal{P}(K_4) - \mathcal{P}(Z)\right) \times (m-6) - (W_4 - \text{rim})$;

$$\begin{aligned}
\textit{(ii)}\ F_{12} =&\, 6\mathcal{P}(K_4)(t-4) + \mathcal{P}(Z)(t-2) - 2(W_4 - \text{rim}) - 3(3 \star K_3) - (K_4 \cup_2 K_3)\\
\textit{(iii)}\ F_{13} =&\, 10\mathcal{P}(K_5)(m-10) + 4\mathcal{P}(K_5 - e)(m-9) + \mathcal{P}(K_4 \cup_2 K_3)(m-8)\\
&+ 2\mathcal{P}(W_4)(m-8) + \big(\mathcal{P}((3 \star K_3)) + \mathcal{P}(W_4 - \text{spoke})\big)(m-7)\\
&+ \mathcal{P}(\theta_{2,2,2})(m-6)\\
\textit{(iv)}\ F_{14} =&\, 15\mathcal{P}(K_5)(m-10) + 3\mathcal{P}(K_5 - e)(m-9) + \mathcal{P}(W_4)(m-8).
\end{aligned}$$

Proof: Enumerating the configurations of 'two triangles and an edge' yields

$$\begin{aligned}
&\big(\text{all graphs with 3 blocks: two } K_3\text{'s and an edge}\big)\\
&\quad + (W_4 - \text{rim}) + \big(6\mathcal{P}(K_4) + \mathcal{P}(Z)\big)(m-6) \;=\; \binom{t}{2} \times (m-6)
\end{aligned}$$

thus $F_{11} + (W_4 - \text{rim}) + \big(6\mathcal{P}(K_4) + \mathcal{P}(Z)\big)(m-6) = \binom{t}{2}(m-6)$, which implies (i). Secondly, counting all induced subgraphs obtained from a 'zig-zag and a triangle' together with their multiplicities produces the identity

$$\begin{aligned}\big(\text{all graphs with 2 blocks: a } Z \text{ and a } K_3\big) &+ 2(W_4 - \text{rim}) \\ + 3(3 \star K_3) + K_4 \cup_2 K_3 \;&=\; 6\mathcal{P}(K_4)\binom{t-4}{1} + \mathcal{P}(Z)\binom{t-2}{1}\end{aligned}$$

so that $F_{12} + 2(W_4 - \text{rim}) + 3(3 \star K_3) + K_4 \cup_2 K_3 = 6\mathcal{P}(K_4)\binom{t-4}{1} + \mathcal{P}(Z)\binom{t-2}{1}$. To show the third statement, we allow '$\theta_{2,2,2}$ with an edge' to undergo purification:

$$\begin{aligned}\big(\text{all graphs with 2 blocks: a } \theta_{2,2,2} \text{ and a } K_2\big) \;&=\; 10\mathcal{P}(K_5)(m-10) \\ + 4\mathcal{P}(K_5 - e)(m-9) + (\mathcal{P}(K_4 \cup_2 K_3) &+ 2\mathcal{P}(W_4))(m-8) + \mathcal{P}((3 \star K_3) \\ &+ \mathcal{P}(W_4 - \text{spoke}))(m-7) + \mathcal{P}(\theta_{2,2,2})(m-6)\end{aligned}$$

and because the left-hand side is equal to F_{13}, assertion (iii) follows immediately. Finally, applying the same purification process to 'W_4 with an edge' implies that

$$\begin{aligned}&\big(\text{all graphs with 2 blocks: a } W_4 \text{ and a } K_2\big) \\ &= 15\mathcal{P}(K_5)(m-10) + 3\mathcal{P}(K_5 - e)(m-9) + \mathcal{P}(W_4)(m-8)\end{aligned}$$

and the left-hand side is F_{14}, which completes the proof of the lemma. □

Lemma 2.5 *One has the following graph identities for $F_{15}, \ldots, F_{18}$:*

(i) $F_{15} = 30\mathcal{P}(K_5)(m-10) + 6\mathcal{P}(K_5 - e)(m-9) + \mathcal{P}(K_4 \cup_2 K_3)(m-8)$;

(ii) $F_{16} = 60\mathcal{P}(K_5)(m-10) + 24\mathcal{P}(K_5 - e)(m-9) + 12\mathcal{P}(W_4)(m-8)$
$+6\mathcal{P}(K_4 \cup_2 K_3)(m-8) + 2\mathcal{P}(W_4 - \text{rim})(m-7)$
$+4\mathcal{P}(W_4 - \text{spoke})(m-7) + \mathcal{P}(C_4 \cup_2 K_2)(m-6)$;

(iii) $F_{17} = 60\mathcal{P}(K_5)(m-10) + 18\mathcal{P}(K_5 - e)(m-9) + 4\mathcal{P}(W_4)(m-8)$
$+4\mathcal{P}(K_4 \cup_2 K_3)(m-8) + \mathcal{P}(W_4 - \text{rim})(m-7)$;

(iv) $F_{18} = 30\mathcal{P}(K_5)(m-10) + 9\mathcal{P}(K_5 - e)(m-9) + \mathcal{P}(K_4 \cup_2 K_3)(m-8)$
$+4\mathcal{P}(W_4)(m-8) + \mathcal{P}(W_4 - \text{spoke})(m-7)$.

Proof: Enumerating the graphs containing $K_4 \cup_2 K_3$ and an edge as blocks, one readily computes that

$$\begin{aligned}&\big(\text{all graphs with 2 blocks: a } K_4 \cup_2 K_3 \text{ and a } K_2\big) \\ &= 30\mathcal{P}(K_5)(m-10) + 6\mathcal{P}(K_5 - e)(m-9) + \mathcal{P}(K_4 \cup_2 K_3)(m-8)\end{aligned}$$

and the left-hand side is F_{15}, which proves (i). Counting $C_4 \cup_2 K_3$ with an edge,

$$\begin{aligned}\big(\text{all graphs with 2 blocks: a } C_4 \cup_2 K_3 \text{ and a } K_2\big) \;&=\; 60\mathcal{P}(K_5)(m-10) \\ +24\mathcal{P}(K_5 - e)(m-9) + 12\mathcal{P}(W_4)(m-8) &+ 6\mathcal{P}(K_4 \cup_2 K_3)(m-8) \\ +2\mathcal{P}(W_4 - \text{rim})(m-7) + 4\mathcal{P}(W_4 - \text{spoke})&(m-7) + \mathcal{P}(C_4 \cup_2 K_2)(m-6)\end{aligned}$$

and the left-hand side is F_{16}, hence (ii) follows. An analogous calculation shows

$$\begin{aligned}&\big(\text{all graphs with 2 blocks: a } (W_4 - \text{rim}) \text{ and a } K_2\big)\\ &= 60\mathcal{P}(K_5)(m-10) + 18\mathcal{P}(K_5 - e)(m-9) + 4\mathcal{P}(W_4)(m-8)\\ &\qquad + 4\mathcal{P}(K_4 \cup_2 K_3)(m-8) + \mathcal{P}(W_4 - \text{rim})(m-7)\end{aligned}$$

which equals F_{17}. Finally to establish assertion (iv), we note that F_{18} equals

$$\begin{aligned}\big(\text{all graphs with 2 blocks: a } (W_4 - \text{spoke}) \text{ and a } K_2\big) &= 30\mathcal{P}(K_5)(m-10)\\ &+ 9\mathcal{P}(K_5 - e)(m-9) + \mathcal{P}(K_4 \cup_2 K_3)(m-8)\\ &+ 4\mathcal{P}(W_4)(m-8) + \mathcal{P}(W_4 - \text{spoke})(m-7).\end{aligned}$$

□

Inputting the data from Lemmas 2.4 and 2.5 into our formula, and after some book-keeping, the previous expression we derived for $\mathbf{c}_{n-5}(\mathcal{G})$ can be rewritten as

$$\begin{aligned}&C_5 + Z - 2(C_4 \cup_2 K_3) + (W_4 - \text{rim}) + (W_4 - \text{spoke}) - 3 \times \theta_{2,2,2} - K_4 \cup_2 K_3\\ &-F_{22} + \big(F_{19} + F_{20} + F_{21}\big)\\ &+\big(G^1 + G^3 + G^{56} + G^{58} + G^{62} + G^{64} + G^{66} + G^{68} + G^{70} + G^{72} + G^{75}\\ &\quad +G^{78} + G^{86} + G^{96} + G^{100} + G^{108} + G^{111} + G^{76} + G^{80} + G^{82} + G^{87}\\ &\quad +G^{92} + G^{94} + G^{98} + G^{102} + G^{104} + G^{105} + G^{90}\big)\\ &-\big(G^0 + G^2 + G^{110} + G^{101} + G^{55} + G^{81} + G^{95} + G^{97} + G^{103} + G^{106} + G^{107}\\ &\quad +G^{109} + G^{57} + G^{59} + G^{65} + G^{67} + G^{71} + G^{74} + G^{77} + G^{79} + G^{83} + G^{89}\\ &\quad +G^{91} + G^{93} + G^{99} + G^{63} + G^{69} + G^{73}\big) - \binom{m}{5} + t\binom{m-2}{3} - \binom{t}{2}(m-4)\\ &+ \mathcal{P}(K_4)(5t - 3m - 6) - 2\mathcal{P}(K_4)\binom{m-6}{2} + \mathcal{P}(Z)(t-3) + \mathcal{P}(C_4)\binom{m-3}{2}\\ &- 26\mathcal{P}(K_5)(m-10) - 10\mathcal{P}(K_5 - e)(m-9) - \mathcal{P}(C_4 \cup_2 K_2)(m-6)\\ &- 7\mathcal{P}(W_4)(m-8) - \mathcal{P}\big((3 \star K_3)\big)(m-7) - 4\mathcal{P}(W_4 - \text{spoke})(m-7)\\ &- \mathcal{P}(W_4 - \text{rim})(m-7) \; - \mathcal{P}(\theta_{2,2,2})(m-6) - 3\mathcal{P}(K_4 \cup_2 K_3)(m-8).\end{aligned}$$

The reader who has not yet fallen into a coma will notice that the only remaining graph families are F_{19}, F_{20}, F_{21} and F_{22}, which we shall now dispose of judiciously.

Lemma 2.6 *(i)* $F_{19} = 10\mathcal{P}(K_5)(m-10) + \mathcal{P}(K_5 - e)(m-9)$

$$\begin{aligned}\textit{(ii)}\ F_{20} =\,&12\mathcal{P}(K_5)(m-10) + 6\mathcal{P}(K_5 - e)(m-9) + 2\mathcal{P}(K_4 \cup_2 K_3)(m-8)\\ &+ 2\mathcal{P}(W_4 - \text{spoke})(m-7) + 4\mathcal{P}(W_4)(m-8)\\ &+ \mathcal{P}(W_4 - \text{rim})(m-7) + \mathcal{P}(C_4 \cup_2 K_3)(m-6) + \mathcal{P}(C_5)(m-5)\\ \textit{(iii)}\ F_{21} =\,&10\mathcal{P}(K_5)(m-10) + 3\mathcal{P}(K_5 - e)(m-9)\\ &+ \mathcal{P}(K_4 \cup K_3)(m-8) + \mathcal{P}\big((3 \star K_3)\big)(m-7)\\ \textit{(iv)}\ F_{22} =\,&3\mathcal{P}(K_4)(t-4) + \mathcal{P}(Z)(t-2) + \mathcal{P}(C_4) \times t - 3(3 \star K_3) - C_4 \cup_2 K_3.\end{aligned}$$

PROOF: The first three assertions are rudimentary block-counting arguments,

$$\begin{aligned} F_{19} &:= \big(\text{all graphs with 2 blocks: a } K_5 - e \text{ and a } K_2\big) \\ &= 10\mathcal{P}(K_5)(m-10) + \mathcal{P}(K_5 - e)(m-9) \qquad \text{which proves (i).} \end{aligned}$$

Secondly, counting up the configurations of a five-cycle and an edge,

$$\begin{aligned} F_{20} &:= \big(\text{all graphs with 2 blocks: a } C_5 \text{ and a } K_2\big) \\ &= 12\mathcal{P}(K_5)(m-10) + 6\mathcal{P}(K_5 - e)(m-9) + 2\mathcal{P}(K_4 \cup_2 K_3)(m-8) \\ &\quad + 4\mathcal{P}(W_4)(m-8) + \mathcal{P}(W_4 - \text{rim})(m-7) + 2\mathcal{P}(W_4 - \text{spoke})(m-7) \\ &\quad + \mathcal{P}(C_4 \cup_2 K_3)(m-6) + \mathcal{P}(C_5)(m-5) \qquad \text{which proves (ii).} \end{aligned}$$

Thirdly, doing the same count for a $3 \star K_3$ and an edge yields

$$\begin{aligned} F_{21} &:= \big(\text{all graphs with 2 blocks: a } 3 \star K_3 \text{ and a } K_2\big) \\ &= 10\mathcal{P}(K_5)(m-10) + 3\mathcal{P}(K_5 - e)(m-9) \\ &\quad + \mathcal{P}(K_4 \cup K_3)(m-8) + \mathcal{P}\big((3 \star K_3)\big)(m-7) \quad \text{which proves (iii).} \end{aligned}$$

Finally, because F_{22} includes all graphs with the two blocks: a C_4 and a triangle, in fact $F_{22} + C_4 \cup_2 K_3 + 3\big(3 \star K_3\big) = 3\mathcal{P}(K_4)(t-4) + \mathcal{P}(Z)(t-2) + \mathcal{P}(C_4) \times t.$ $\square$

Applying Lemma 2.6 to our running formula, and then noting that $G^0 = K_6$ while $K_n = \mathcal{P}(K_n)$ for every n, our expression for the coefficient $\mathbf{c}_{n-5}(\mathcal{G})$ becomes

$$\begin{aligned} & C_5 + Z - C_4 \cup_2 K_3 + (W_4 - \text{rim}) + (W_4 - \text{spoke}) - 3 \times \theta_{2,2,2} + 3(3 \star K_3) \\ & -K_4 \cup_2 K_3 + \big(G^1 + G^3 + G^{56} + G^{58} + G^{62} + G^{64} + G^{66} + G^{68} + G^{70} + G^{72} \\ & \qquad + G^{75} + G^{78} + G^{86} + G^{96} + G^{100} + G^{108} + G^{111} + G^{76} + G^{80} \\ & \qquad + G^{82} + G^{87} + G^{92} + G^{94} + G^{98} + G^{102} + G^{104} + G^{105} + G^{90}\big) \\ & - \big(G^2 + G^{110} + G^{101} + G^{55} + G^{81} + G^{95} + G^{97} + G^{103} + G^{106} + G^{107} + G^{109} \\ & \qquad + G^{57} + G^{59} + G^{65} + G^{67} + G^{71} + G^{74} + G^{77} + G^{79} \\ & \qquad + G^{83} + G^{89} + G^{91} + G^{93} + G^{99} + G^{63} + G^{69} + G^{73}\big) \\ & -\binom{m}{5} + t\binom{m-2}{3} - \binom{t}{2}(m-4) + \mathcal{P}(K_4)(2t - 3m + 6) - 2\mathcal{P}(K_4)\binom{m-6}{2} \\ & - \mathcal{P}(Z) + \mathcal{P}(C_4)\binom{m-3}{2} - \mathcal{P}(C_4) \times t + 6\mathcal{P}(K_5)(m-10) - 3\mathcal{P}(W_4)(m-8) \\ & - 2\mathcal{P}(W_4 - \text{spoke})(m-7) - \mathcal{P}(\theta_{2,2,2})(m-6) + \mathcal{P}(C_5)(m-5) - \mathcal{P}(K_6). \end{aligned}$$

2.5 Purification processes for the remaining terms

The remaining graphs are connected hence we can no longer use any of our earlier block-counting tricks. Instead we purify them individually with the aid of a subgraph counting program (see Table 2).

Graph	Corresponding Purified Decomposition
Z	$6\mathcal{P}(K_4) + \mathcal{P}(Z)$
C_5	$12\mathcal{P}(K_5) + 6\mathcal{P}(K_5 - e) + 2\mathcal{P}(K_4 \cup_2 K_3) + 4\mathcal{P}(W_4)$ $+\mathcal{P}\big((W_4 - \text{rim})\big) + 2\mathcal{P}\big((W_4 - \text{spoke})\big) + \mathcal{P}(C_4 \cup_2 K_3) + \mathcal{P}(C_5)$
$\theta_{2,2,2}$	$10\mathcal{P}(K_5) + 4\mathcal{P}(K_5 - e) + \mathcal{P}(K_4 \cup_2 K_3) + 2\mathcal{P}(W_4)$ $+\mathcal{P}(W_4 - \text{spoke}) + \mathcal{P}\big((3 \star K_3)\big) + \mathcal{P}(\theta_{2,2,2})$
$W_4 - \text{rim}$	$60\mathcal{P}(K_5) + 18\mathcal{P}(K_5 - e) + 4\mathcal{P}(K_4 \cup_2 K_3) + 4\mathcal{P}(W_4) + \mathcal{P}(W_4 - \text{rim})$
$W_4 -$ spoke	$30\mathcal{P}(K_5) + 9\mathcal{P}(K_5 - e) + \mathcal{P}(K_4 \cup_2 K_3) + 4\mathcal{P}(W_4) + \mathcal{P}(W_4 - \text{spoke})$
$C_4 \cup_2 K_3$	$60\mathcal{P}(K_5) + 24\mathcal{P}(K_5 - e) + 6\mathcal{P}(K_4 \cup_2 K_3) + 12\mathcal{P}(W_4)$ $+2\mathcal{P}(W_4 - \text{rim}) + 4\mathcal{P}(W_4 - \text{spoke}) + \mathcal{P}(C_4 \cup_2 K_3)$
$K_4 \cup_2 K_3$	$30\mathcal{P}(K_5) + 6\mathcal{P}(K_5 - e) + \mathcal{P}(K_4 \cup_2 K_3)$
$3 \star K_3$	$10\mathcal{P}(K_5) + 3\mathcal{P}(K_5 - e) + \mathcal{P}(K_4 \cup_2 K_3) + (3 \star K_3)$

Table 2: Purification decompositions for the remaining graphs of order $\leqslant 5$.

For example, the first row of this table tells us that the number of zig-zags in any given graph is the number of *induced* zig-zags combined with six times the number of K_4's inside the graph (because there are 6 possible mappings of Z into a K_4). Furthermore, the last seven rows taken in combination imply that

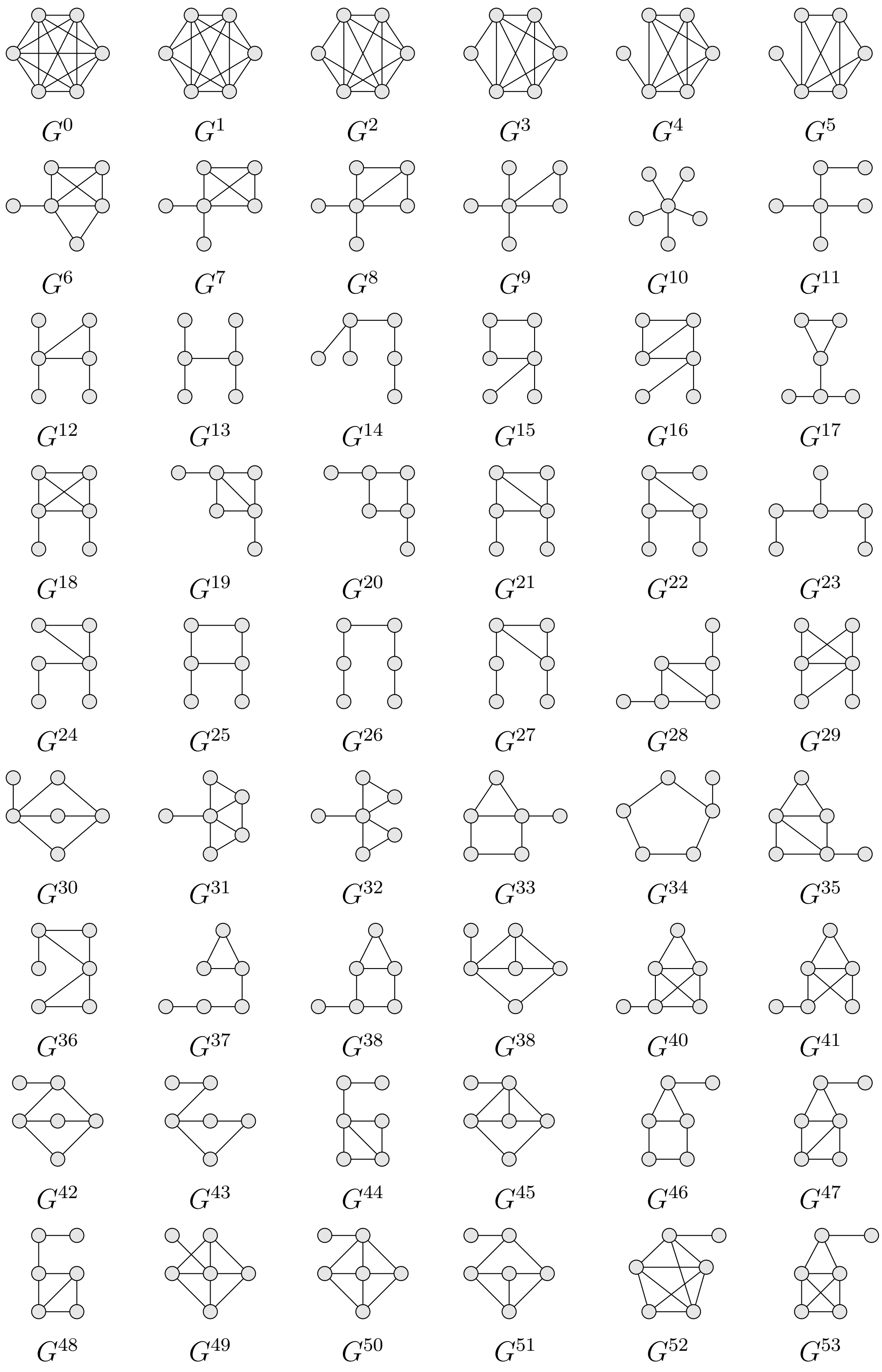
$$\begin{aligned} C_5 - C_4 \cup_2 K_3 + (W_4 - \text{rim}) + (W_4 - \text{spoke}) - 3 \times \theta_{2,2,2} + 3(3 \star K_3) - K_4 \cup_2 K_3 \\ = 12\mathcal{P}(K_5) - 6\mathcal{P}(W_4) - 4\mathcal{P}((W_4 - \text{spoke})) + \mathcal{P}(C_5) - 3\mathcal{P}(\theta_{2,2,2}). \end{aligned}$$

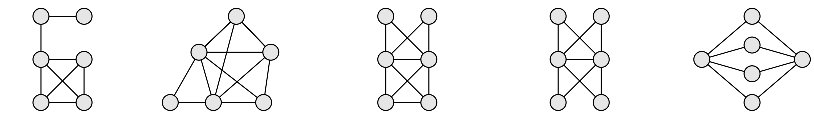
Substituting these equations into our formula, the coefficient $\mathbf{c}_{n-5}(\mathcal{G})$ will equal

$$\begin{aligned} &\big(G^1 + G^3 + G^{56} + G^{58} + G^{62} + G^{64} + G^{66} + G^{68} + G^{70} + G^{72} + G^{75} + G^{78} \\ &\qquad + G^{86} + G^{96} + G^{100} + G^{108} + G^{111} + G^{76} + G^{80} + G^{82} \\ &\qquad + G^{87} + G^{92} + G^{94} + G^{98} + G^{102} + G^{104} + G^{105} + G^{90}\big) \\ &-\big(G^2 + G^{110} + G^{101} + G^{55} + G^{81} + G^{95} + G^{97} + G^{103} + G^{106} + G^{107} + G^{109} \\ &\quad + G^{57} + G^{59} + G^{65} + G^{67} + G^{71} + G^{74} + G^{77} + G^{79} + G^{83} + G^{89} + G^{91} \\ &\quad + G^{93} + G^{99} + G^{63} + G^{69} + G^{73}\big) - \binom{m}{5} + t\binom{m-2}{3} - \binom{t}{2}(m-4) \\ &+ \mathcal{P}(K_4)(2t - 3m + 12) - 2\mathcal{P}(K_4)\binom{m-6}{2} + \mathcal{P}(C_4)\binom{m-3}{2} - \mathcal{P}(C_4) \times t \\ &+ 6\mathcal{P}(K_5)(m-8) - 3\mathcal{P}(W_4)(m-6) - 2\mathcal{P}(W_4 - \text{spoke})(m-5) \\ &- \mathcal{P}(\theta_{2,2,2})(m-3) + \mathcal{P}(C_5)(m-4) - \mathcal{P}(K_6). \end{aligned}$$

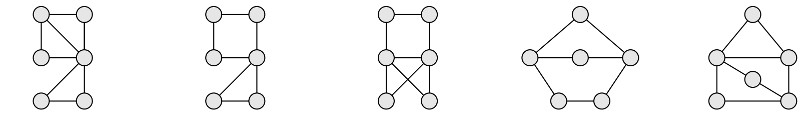
Finally we apply the purification process to the remaining 55 graphs of order 6 – the purification information for each graph has been tabulated in Appendix C. Inputting this data into our existing expression for $\mathbf{c}_{n-5}(\mathcal{G})$ above, and also noting that $G^{58} = \theta_{2,2,2,2}$, $G^{63} = \theta_{2,2,3}$, $G^{70} = C_6$, $G^{99} = K_{3,3}$, $G^{104} = W_5$ and lastly $G^{93} = (W_5 - \text{spoke})$, one obtains exactly the same formula stated in Theorem 1.5.

A Graphs with c vertices and $c-5$ components: $5 < c \leqslant 10$

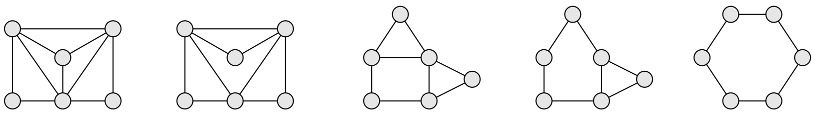




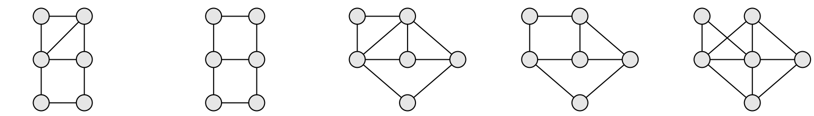
G^{54} G^{55} G^{56} G^{57} G^{58} G^{59}



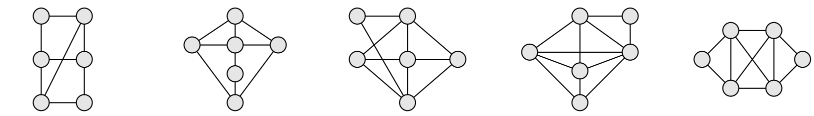
G^{60} G^{61} G^{62} G^{63} G^{64} G^{65}



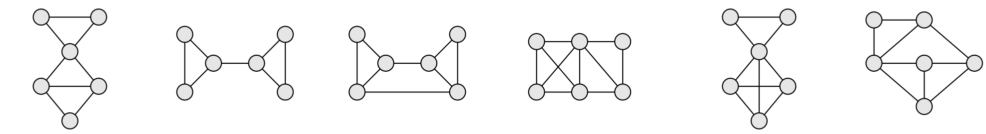
G^{66} G^{67} G^{68} G^{69} G^{70} G^{71}



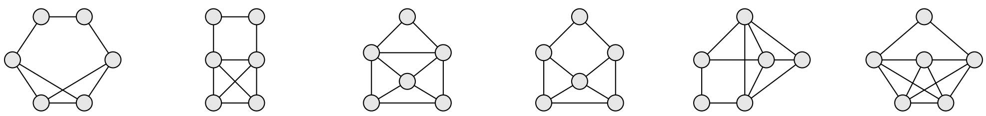
G^{72} G^{73} G^{74} G^{75} G^{76} G^{77}



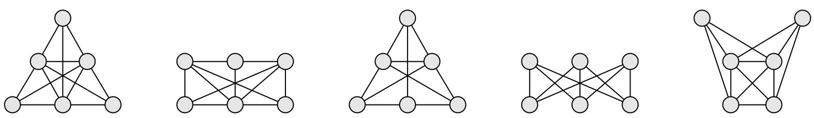
G^{78} G^{79} G^{80} G^{81} G^{82} G^{83}



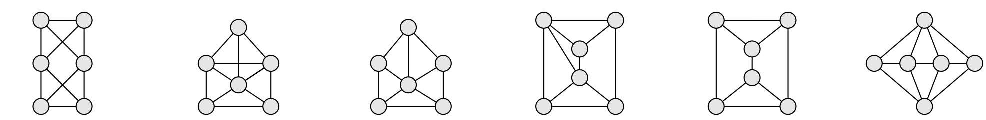
G^{84} G^{85} G^{86} G^{87} G^{88} G^{89}



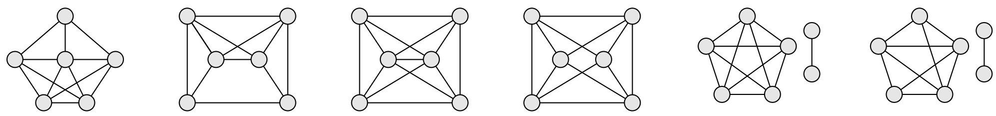
G^{90} G^{91} G^{92} G^{93} G^{94} G^{95}



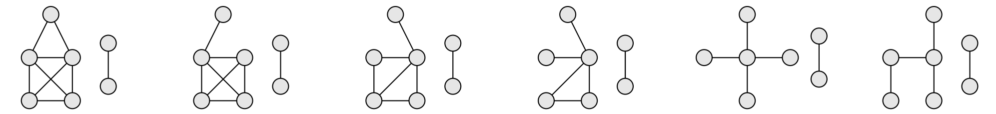
G^{96} G^{97} G^{98} G^{99} G^{100} G^{101}



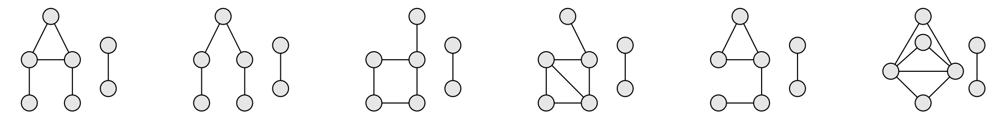
G^{102} G^{103} G^{104} G^{105} G^{106} G^{107}



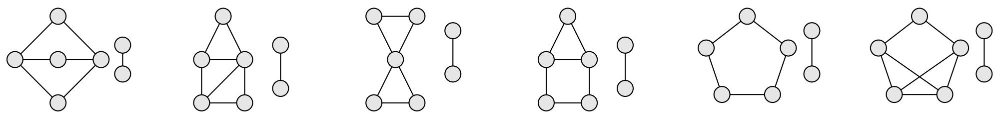
G^{108} G^{109} G^{110} G^{111} G^{112} G^{113}



G^{114} G^{115} G^{116} G^{117} G^{118} G^{119}

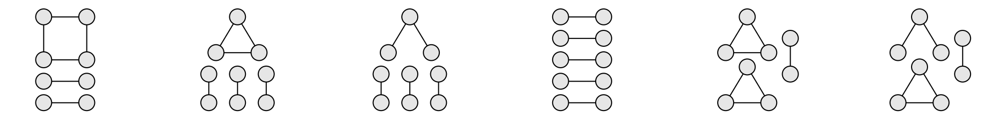


G^{120} G^{121} G^{122} G^{123} G^{124} G^{125}

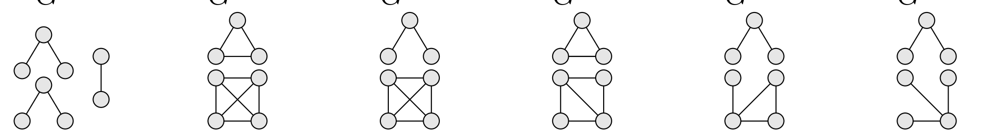


G^{126} G^{127} G^{128} G^{129} G^{130} G^{131}

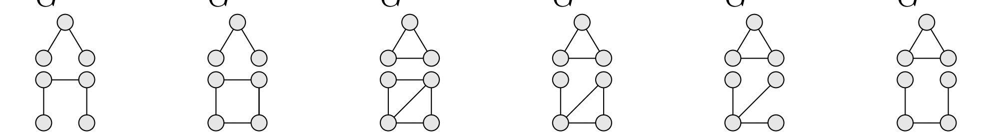
G^{132} G^{133} G^{134} G^{135} G^{136} G^{137}



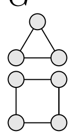
G^{138} G^{139} G^{140} G^{141} G^{142} G^{143}



G^{144} G^{145} G^{146} G^{147} G^{148} G^{149}



G^{150} G^{151} G^{152} G^{153} G^{154} G^{155}



G^{156}

B Splitting the labelled graphs $G^{\bullet}$ into families

Symbol	Family	Graphs
F_1	Forests with 5 edges	G^{10}, G^{11}, G^{13}, G^{14}, G^{23}, G^{26}, G^{118}, G^{119}, G^{121}, G^{136}, G^{137}, G^{140}, G^{141}, G^{144}, G^{149}, G^{150}
F_2	Graphs with 2 blocks: K_5 & K_2	G^{4}, G^{112}
F_3	Graphs with 4 blocks: K_3 & 3 K_2's	G^{9}, G^{12}, G^{17}, G^{22}, G^{24}, G^{27}, G^{37}, G^{117}, G^{120}, G^{124}, G^{135}, G^{139}, G^{143}, G^{148}, G^{154}, G^{155}
F_4	Graphs with 3 blocks: K_3 & 2 K_2's	
F_5	Graphs with 3 blocks: K_4 & 2 K_2's	G^{7}, G^{18}, G^{54}, G^{115}, G^{133}, G^{146}
F_6	Graphs with 2 blocks: K_4 & K_3	G^{88}, G^{145}
F_7	Graphs with 2 blocks: Z & K_2	
F_8	Graphs with 2 blocks: C_4 & K_2	
F_9	Graphs with 3 blocks: 2 K_2's & Z	G^{8}, G^{16}, G^{19}, G^{21}, G^{28}, G^{44}, G^{48}, G^{116}, G^{123}, G^{134}, G^{147}
F_{10}	Graphs with 3 blocks: C_4 & 2 K_2's	G^{15}, G^{20}, G^{25}, G^{43}, G^{122}, G^{138}, G^{151}
F_{11}	Graphs with 3 blocks: 2 K_3's & K_2	G^{32}, G^{36}, G^{85}, G^{128}, G^{142}, G^{153}
F_{12}	Graphs with 2 blocks: Z & K_3	G^{60}, G^{84}, G^{152}
F_{13}	Graphs with 2 blocks: $\theta_{2,2,2}$ & K_2	G^{30}, G^{42}, G^{126}
F_{14}	Graphs with 2 blocks: W_4 & K_2	G^{49}, G^{50}, G^{132}
F_{15}	Graphs with 2 blocks: $K_4 \cup_2 K_3$ & K_2	G^{6}, G^{40}, G^{53}, G^{114}
F_{16}	Graphs with 2 blocks: $C_4 \cup_2 K_3$ & K_2	G^{33}, G^{38}, G^{46}, G^{129}
F_{17}	Graphs with 2 blocks: W_4-rim & K_2	G^{31}, G^{35}, G^{47}, G^{127}
F_{18}	Graphs with 2 blocks: W_4-spoke & K_2	G^{39}, G^{45}, G^{51}, G^{131}
F_{19}	Graphs with 2 blocks: $(K_5 - e)$ & K_2	G^{5}, G^{52}, G^{113}
F_{20}	Graphs with 2 blocks: C_5 & K_2	G^{34}, G^{130}
F_{21}	Graphs with 2 blocks: $(3 \star K_3)$ & K_2	G^{29}, G^{41}, G^{125}
F_{22}	Graphs with 2 blocks: K_3 & C_4	G^{61}, G^{156}
F_{23}	Graphs with 2 blocks: K_3 & K_3	
F_{24}	Graphs with 2 blocks: K_4 & K_3	

C Purification tables for the remaining 55 graphs

Graph	Corresponding Purified Decomposition
G^1	$15\mathcal{P}(G^0) + \mathcal{P}(G^1)$
G^2	$60\mathcal{P}(G^0) + 8\mathcal{P}(G^1) + \mathcal{P}(G^2)$
G^3	$60\mathcal{P}(G^0) + 12\mathcal{P}(G^1) + 3\mathcal{P}(G^2) + \mathcal{P}(G^3)$
G^{55}	$180\mathcal{P}(G^0)+48\mathcal{P}(G^1)+15\mathcal{P}(G^2)+3\mathcal{P}(G^3)+\mathcal{P}(G^{55})+9\mathcal{P}(G^{96})+2\mathcal{P}(G^{100}) + 4\mathcal{P}(G^{110})$
G^{56}	$90\mathcal{P}(G^0)+30\mathcal{P}(G^1)+12\mathcal{P}(G^2)+3\mathcal{P}(G^3)+2\mathcal{P}(G^{55})+\mathcal{P}(G^{56})+9\mathcal{P}(G^{96}) + 3\mathcal{P}(G^{100}) + 2\mathcal{P}(G^{101}) + 4\mathcal{P}(G^{110})$
G^{57}	$15\mathcal{P}(G^0) + 6\mathcal{P}(G^1) + 3\mathcal{P}(G^2) + \mathcal{P}(G^3) + \mathcal{P}(G^{55}) + \mathcal{P}(G^{56}) + \mathcal{P}(G^{57}) + 3\mathcal{P}(G^{96}) + \mathcal{P}(G^{100}) + \mathcal{P}(G^{101}) + \mathcal{P}(G^{110})$
G^{58}	$15\mathcal{P}(G^0) + 7\mathcal{P}(G^1) + 3\mathcal{P}(G^2) + \mathcal{P}(G^3) + \mathcal{P}(G^{55}) + \mathcal{P}(G^{56}) + \mathcal{P}(G^{57}) + \mathcal{P}(G^{58}) + \mathcal{P}(G^{65}) + \mathcal{P}(G^{80}) + \mathcal{P}(G^{95}) + 3\mathcal{P}(G^{96}) + \mathcal{P}(G^{97}) + \mathcal{P}(G^{100}) + \mathcal{P}(G^{101}) + \mathcal{P}(G^{102}) + \mathcal{P}(G^{107}) + \mathcal{P}(G^{108}) + 3\mathcal{P}(G^{110}) + 3\mathcal{P}(G^{111})$
G^{59}	$360\mathcal{P}(G^0) + 144\mathcal{P}(G^1) + 60\mathcal{P}(G^2) + 18\mathcal{P}(G^3) + 10\mathcal{P}(G^{55}) + 4\mathcal{P}(G^{56}) + \mathcal{P}(G^{59}) + 2\mathcal{P}(G^{66}) + 2\mathcal{P}(G^{76}) + 4\mathcal{P}(G^{81}) + \mathcal{P}(G^{87}) + 36\mathcal{P}(G^{96}) + 6\mathcal{P}(G^{97}) + 20\mathcal{P}(G^{100}) + 8\mathcal{P}(G^{101}) + 4\mathcal{P}(G^{103}) + 10\mathcal{P}(G^{108}) + 40\mathcal{P}(G^{110})$
G^{62}	$180\mathcal{P}(G^0) + 84\mathcal{P}(G^1) + 36\mathcal{P}(G^2) + 12\mathcal{P}(G^3) + 6\mathcal{P}(G^{55}) + 2\mathcal{P}(G^{56}) + \mathcal{P}(G^{59}) + \mathcal{P}(G^{62}) + 2\mathcal{P}(G^{66}) + \mathcal{P}(G^{74}) + 2\mathcal{P}(G^{76}) + 2\mathcal{P}(G^{80}) + 4\mathcal{P}(G^{81}) + \mathcal{P}(G^{87}) + \mathcal{P}(G^{91}) + \mathcal{P}(G^{92}) + 2\mathcal{P}(G^{94}) + 6\mathcal{P}(G^{95}) + 18\mathcal{P}(G^{96}) + 6\mathcal{P}(G^{97}) + 14\mathcal{P}(G^{100}) + 4\mathcal{P}(G^{101}) + 5\mathcal{P}(G^{103}) + \mathcal{P}(G^{105}) + 4\mathcal{P}(G^{107}) + 14\mathcal{P}(G^{108}) + 4\mathcal{P}(G^{109}) + 36\mathcal{P}(G^{110}) + 12\mathcal{P}(G^{111})$
G^{63}	$180\mathcal{P}(G^0) + 96\mathcal{P}(G^1) + 42\mathcal{P}(G^2) + 12\mathcal{P}(G^3) + 6\mathcal{P}(G^{55}) + 2\mathcal{P}(G^{56}) + \mathcal{P}(G^{59}) + \mathcal{P}(G^{62}) + \mathcal{P}(G^{63}) + \mathcal{P}(G^{64}) + 2\mathcal{P}(G^{65}) + 2\mathcal{P}(G^{66})+2\mathcal{P}(G^{74})+2\mathcal{P}(G^{75})+3\mathcal{P}(G^{76})+3\mathcal{P}(G^{79})+6\mathcal{P}(G^{80})+5\mathcal{P}(G^{81}) + \mathcal{P}(G^{87}) + \mathcal{P}(G^{89}) + \mathcal{P}(G^{90}) + \mathcal{P}(G^{91}) + 3\mathcal{P}(G^{92}) + 3\mathcal{P}(G^{93}) + 5\mathcal{P}(G^{94}) + 12\mathcal{P}(G^{95}) + 18\mathcal{P}(G^{96}) + 12\mathcal{P}(G^{97}) + 6\mathcal{P}(G^{98}) + 19\mathcal{P}(G^{100}) + 4\mathcal{P}(G^{101}) + 4\mathcal{P}(G^{102}) + 11\mathcal{P}(G^{103}) + 5\mathcal{P}(G^{104}) + 8\mathcal{P}(G^{105}) + 6\mathcal{P}(G^{106}) + 15\mathcal{P}(G^{107}) + 28\mathcal{P}(G^{108}) + 16\mathcal{P}(G^{109}) + 56\mathcal{P}(G^{110}) + 36\mathcal{P}(G^{111})$

Graph	Corresponding Purified Decomposition
G^{64}	$360\mathcal{P}(G^{0}) + 168\mathcal{P}(G^{1}) + 66\mathcal{P}(G^{2}) + 18\mathcal{P}(G^{3}) + 10\mathcal{P}(G^{55}) + 4\mathcal{P}(G^{56}) + \mathcal{P}(G^{59}) + \mathcal{P}(G^{64}) + 4\mathcal{P}(G^{65}) + 2\mathcal{P}(G^{66}) + \mathcal{P}(G^{74}) + 3\mathcal{P}(G^{76}) + 2\mathcal{P}(G^{79}) + 8\mathcal{P}(G^{80}) + 5\mathcal{P}(G^{81}) + \mathcal{P}(G^{87}) + \mathcal{P}(G^{89}) + 2\mathcal{P}(G^{92}) + 2\mathcal{P}(G^{94}) + 12\mathcal{P}(G^{95}) + 36\mathcal{P}(G^{96}) + 18\mathcal{P}(G^{97}) + 6\mathcal{P}(G^{98}) + 24\mathcal{P}(G^{100}) + 8\mathcal{P}(G^{101}) + 8\mathcal{P}(G^{102}) + 8\mathcal{P}(G^{103}) + 4\mathcal{P}(G^{105}) + 16\mathcal{P}(G^{107}) + 32\mathcal{P}(G^{108}) + 12\mathcal{P}(G^{109}) + 80\mathcal{P}(G^{110}) + 48\mathcal{P}(G^{111})$
G^{65}	$90\mathcal{P}(G^{0}) + 36\mathcal{P}(G^{1}) + 12\mathcal{P}(G^{2}) + 3\mathcal{P}(G^{3}) + 2\mathcal{P}(G^{55}) + \mathcal{P}(G^{56}) + \mathcal{P}(G^{65}) + 2\mathcal{P}(G^{80}) + 3\mathcal{P}(G^{95}) + 9\mathcal{P}(G^{96}) + 3\mathcal{P}(G^{97}) + 3\mathcal{P}(G^{100}) + 2\mathcal{P}(G^{101}) + 2\mathcal{P}(G^{102}) + 3\mathcal{P}(G^{107}) + 4\mathcal{P}(G^{108}) + 14\mathcal{P}(G^{110}) + 12\mathcal{P}(G^{111})$
G^{66}	$360\mathcal{P}(G^{0}) + 120\mathcal{P}(G^{1}) + 42\mathcal{P}(G^{2}) + 12\mathcal{P}(G^{3}) + 4\mathcal{P}(G^{55}) + \mathcal{P}(G^{66}) + 2\mathcal{P}(G^{81}) + 18\mathcal{P}(G^{96}) + 10\mathcal{P}(G^{100}) + \mathcal{P}(G^{103}) + 4\mathcal{P}(G^{108}) + 24\mathcal{P}(G^{110})$
G^{67}	$120\mathcal{P}(G^{0}) + 48\mathcal{P}(G^{1}) + 18\mathcal{P}(G^{2}) + 6\mathcal{P}(G^{3}) + 2\mathcal{P}(G^{55}) + \mathcal{P}(G^{66}) + \mathcal{P}(G^{67}) + 2\mathcal{P}(G^{81}) + \mathcal{P}(G^{92}) + 6\mathcal{P}(G^{96}) + 6\mathcal{P}(G^{100}) + \mathcal{P}(G^{103}) + 2\mathcal{P}(G^{107}) + 4\mathcal{P}(G^{108}) + 16\mathcal{P}(G^{110}) + 8\mathcal{P}(G^{111})$
G^{68}	$360\mathcal{P}(G^{0}) + 168\mathcal{P}(G^{1}) + 66\mathcal{P}(G^{2}) + 18\mathcal{P}(G^{3}) + 6\mathcal{P}(G^{55}) + 3\mathcal{P}(G^{66}) + 3\mathcal{P}(G^{67}) + \mathcal{P}(G^{68}) + \mathcal{P}(G^{71}) + \mathcal{P}(G^{74}) + 2\mathcal{P}(G^{76}) + 2\mathcal{P}(G^{80}) + 8\mathcal{P}(G^{81}) + 2\mathcal{P}(G^{87}) + 2\mathcal{P}(G^{89}) + 5\mathcal{P}(G^{92}) + \mathcal{P}(G^{93}) + 2\mathcal{P}(G^{94}) + 6\mathcal{P}(G^{95}) + 18\mathcal{P}(G^{96}) + 6\mathcal{P}(G^{97}) + 28\mathcal{P}(G^{100}) + 8\mathcal{P}(G^{101}) + 8\mathcal{P}(G^{102}) + 11\mathcal{P}(G^{103}) + 5\mathcal{P}(G^{104}) + 4\mathcal{P}(G^{105}) + 16\mathcal{P}(G^{107}) + 30\mathcal{P}(G^{108}) + 8\mathcal{P}(G^{109}) + 80\mathcal{P}(G^{110}) + 48\mathcal{P}(G^{111})$
G^{69}	$360\mathcal{P}(G^{0}) + 192\mathcal{P}(G^{1}) + 84\mathcal{P}(G^{2}) + 24\mathcal{P}(G^{3}) + 6\mathcal{P}(G^{55}) + 3\mathcal{P}(G^{66}) + 3\mathcal{P}(G^{67}) + 2\mathcal{P}(G^{68}) + \mathcal{P}(G^{69}) + 2\mathcal{P}(G^{71}) + \mathcal{P}(G^{72}) + 2\mathcal{P}(G^{74}) + \mathcal{P}(G^{75}) + 4\mathcal{P}(G^{76}) + 2\mathcal{P}(G^{77}) + 2\mathcal{P}(G^{79}) + 4\mathcal{P}(G^{80}) + 14\mathcal{P}(G^{81}) + 4\mathcal{P}(G^{82}) + 2\mathcal{P}(G^{83}) + 2\mathcal{P}(G^{86}) + 6\mathcal{P}(G^{87}) + 6\mathcal{P}(G^{89}) + 4\mathcal{P}(G^{90}) + 4\mathcal{P}(G^{91}) + 10\mathcal{P}(G^{92}) + 7\mathcal{P}(G^{93}) + 10\mathcal{P}(G^{94}) + 18\mathcal{P}(G^{95}) + 18\mathcal{P}(G^{96}) + 12\mathcal{P}(G^{97}) + 6\mathcal{P}(G^{98}) + 42\mathcal{P}(G^{100}) + 4\mathcal{P}(G^{90}) + 4\mathcal{P}(G^{91}) + 16\mathcal{P}(G^{101}) + 16\mathcal{P}(G^{102}) + 24\mathcal{P}(G^{103}) + 15\mathcal{P}(G^{104}) + 14\mathcal{P}(G^{105}) + 6\mathcal{P}(G^{106}) + 30\mathcal{P}(G^{107}) + 54\mathcal{P}(G^{108}) + 4\mathcal{P}(G^{90}) + 4\mathcal{P}(G^{91}) + 28\mathcal{P}(G^{109}) + 112\mathcal{P}(G^{110}) + 72\mathcal{P}(G^{111})$

Graph	Corresponding Purified Decomposition
G^{70}	$60\mathcal{P}(G^0)+36\mathcal{P}(G^1)+18\mathcal{P}(G^2)+6\mathcal{P}(G^3)+2\mathcal{P}(G^{55})+\mathcal{P}(G^{66})+\mathcal{P}(G^{67})+\mathcal{P}(G^{68})+\mathcal{P}(G^{69})+\mathcal{P}(G^{70})+\mathcal{P}(G^{71})+\mathcal{P}(G^{72})+\mathcal{P}(G^{73})+\mathcal{P}(G^{74})+\mathcal{P}(G^{75})+2\mathcal{P}(G^{76})+2\mathcal{P}(G^{77})+2\mathcal{P}(G^{78})+2\mathcal{P}(G^{79})+2\mathcal{P}(G^{80})+4\mathcal{P}(G^{81})+2\mathcal{P}(G^{82})+\mathcal{P}(G^{83})+\mathcal{P}(G^{86})+2\mathcal{P}(G^{87})+2\mathcal{P}(G^{89})+2\mathcal{P}(G^{90})+2\mathcal{P}(G^{91})+3\mathcal{P}(G^{92})+3\mathcal{P}(G^{93})+4\mathcal{P}(G^{94})+6\mathcal{P}(G^{95})+6\mathcal{P}(G^{96})+6\mathcal{P}(G^{97})+6\mathcal{P}(G^{98})+6\mathcal{P}(G^{99})+10\mathcal{P}(G^{100})+4\mathcal{P}(G^{101})+4\mathcal{P}(G^{102})+7\mathcal{P}(G^{103})+5\mathcal{P}(G^{104})+5\mathcal{P}(G^{105})+3\mathcal{P}(G^{106})+8\mathcal{P}(G^{107})+14\mathcal{P}(G^{108})+10\mathcal{P}(G^{109})+24\mathcal{P}(G^{110})+16\mathcal{P}(G^{111})$
G^{71}	$360\mathcal{P}(G^0)+144\mathcal{P}(G^1)+54\mathcal{P}(G^2)+12\mathcal{P}(G^3)+4\mathcal{P}(G^{55})+\mathcal{P}(G^{66})+\mathcal{P}(G^{71})+2\mathcal{P}(G^{76})+4\mathcal{P}(G^{81})+2\mathcal{P}(G^{87})+18\mathcal{P}(G^{96})+6\mathcal{P}(G^{97})+20\mathcal{P}(G^{100})+8\mathcal{P}(G^{101})+7\mathcal{P}(G^{103})+5\mathcal{P}(G^{104})+14\mathcal{P}(G^{108})+48\mathcal{P}(G^{110})$
G^{72}	$720\mathcal{P}(G^0)+336\mathcal{P}(G^1)+132\mathcal{P}(G^2)+36\mathcal{P}(G^3)+8\mathcal{P}(G^{55})+2\mathcal{P}(G^{66})+2\mathcal{P}(G^{71})+\mathcal{P}(G^{72})+\mathcal{P}(G^{74})+6\mathcal{P}(G^{76})+4\mathcal{P}(G^{77})+2\mathcal{P}(G^{79})+4\mathcal{P}(G^{80})+16\mathcal{P}(G^{81})+8\mathcal{P}(G^{82})+2\mathcal{P}(G^{83})+6\mathcal{P}(G^{87})+4\mathcal{P}(G^{91})+4\mathcal{P}(G^{92})+2\mathcal{P}(G^{93})+10\mathcal{P}(G^{94})+24\mathcal{P}(G^{95})+36\mathcal{P}(G^{96})+24\mathcal{P}(G^{97})+12\mathcal{P}(G^{98})+52\mathcal{P}(G^{100})+16\mathcal{P}(G^{101})+24\mathcal{P}(G^{103})+10\mathcal{P}(G^{104})+6\mathcal{P}(G^{105})+16\mathcal{P}(G^{107})+68\mathcal{P}(G^{108})+32\mathcal{P}(G^{109})+160\mathcal{P}(G^{110})+48\mathcal{P}(G^{111})$
G^{73}	$180\mathcal{P}(G^0)+96\mathcal{P}(G^1)+42\mathcal{P}(G^2)+12\mathcal{P}(G^3)+4\mathcal{P}(G^{55})+\mathcal{P}(G^{66})+\mathcal{P}(G^{71})+\mathcal{P}(G^{72})+\mathcal{P}(G^{73})+\mathcal{P}(G^{74})+\mathcal{P}(G^{75})+4\mathcal{P}(G^{76})+4\mathcal{P}(G^{77})+4\mathcal{P}(G^{78})+4\mathcal{P}(G^{79})+4\mathcal{P}(G^{80})+6\mathcal{P}(G^{81})+4\mathcal{P}(G^{82})+\mathcal{P}(G^{83})+2\mathcal{P}(G^{87})+2\mathcal{P}(G^{91})+2\mathcal{P}(G^{92})+2\mathcal{P}(G^{93})+6\mathcal{P}(G^{94})+12\mathcal{P}(G^{95})+18\mathcal{P}(G^{96})+18\mathcal{P}(G^{97})+18\mathcal{P}(G^{98})+18\mathcal{P}(G^{99})+18\mathcal{P}(G^{100})+4\mathcal{P}(G^{101})+11\mathcal{P}(G^{103})+5\mathcal{P}(G^{104})+6\mathcal{P}(G^{105})+3\mathcal{P}(G^{106})+10\mathcal{P}(G^{107})+30\mathcal{P}(G^{108})+22\mathcal{P}(G^{109})+56\mathcal{P}(G^{110})+24\mathcal{P}(G^{111})$
G^{74}	$720\mathcal{P}(G^0)+288\mathcal{P}(G^1)+96\mathcal{P}(G^2)+24\mathcal{P}(G^3)+8\mathcal{P}(G^{55})+2\mathcal{P}(G^{66})+\mathcal{P}(G^{74})+2\mathcal{P}(G^{76})+4\mathcal{P}(G^{80})+6\mathcal{P}(G^{81})+2\mathcal{P}(G^{92})+2\mathcal{P}(G^{94})+12\mathcal{P}(G^{95})+36\mathcal{P}(G^{96})+12\mathcal{P}(G^{97})+28\mathcal{P}(G^{100})+8\mathcal{P}(G^{103})+2\mathcal{P}(G^{105})+12\mathcal{P}(G^{107})+36\mathcal{P}(G^{108})+8\mathcal{P}(G^{109})+112\mathcal{P}(G^{110})+48\mathcal{P}(G^{111})$

Graph	Corresponding Purified Decomposition
G^{75}	$360\mathcal{P}(G^0) + 168\mathcal{P}(G^1) + 60\mathcal{P}(G^2) + 12\mathcal{P}(G^3) + 4\mathcal{P}(G^{55}) + \mathcal{P}(G^{66}) + \mathcal{P}(G^{74}) + \mathcal{P}(G^{75}) + 2\mathcal{P}(G^{76}) + 2\mathcal{P}(G^{79}) + 4\mathcal{P}(G^{80}) + 4\mathcal{P}(G^{81}) + 2\mathcal{P}(G^{92}) + 2\mathcal{P}(G^{93}) + 4\mathcal{P}(G^{94}) + 12\mathcal{P}(G^{95}) + 18\mathcal{P}(G^{96}) + 12\mathcal{P}(G^{97}) + 6\mathcal{P}(G^{98}) + 22\mathcal{P}(G^{100}) + 12\mathcal{P}(G^{103}) + 5\mathcal{P}(G^{104}) + 8\mathcal{P}(G^{105}) + 6\mathcal{P}(G^{106}) + 16\mathcal{P}(G^{107}) + 38\mathcal{P}(G^{108}) + 20\mathcal{P}(G^{109}) + 88\mathcal{P}(G^{110}) + 48\mathcal{P}(G^{111})$
G^{76}	$360\mathcal{P}(G^0) + 120\mathcal{P}(G^1) + 36\mathcal{P}(G^2) + 6\mathcal{P}(G^3) + 2\mathcal{P}(G^{55}) + \mathcal{P}(G^{76}) + \mathcal{P}(G^{81}) + 18\mathcal{P}(G^{96}) + 6\mathcal{P}(G^{97}) + 8\mathcal{P}(G^{100}) + 2\mathcal{P}(G^{103}) + 8\mathcal{P}(G^{108}) + 32\mathcal{P}(G^{110})$
G^{77}	$180\mathcal{P}(G^0) + 72\mathcal{P}(G^1) + 24\mathcal{P}(G^2) + 6\mathcal{P}(G^3) + \mathcal{P}(G^{55}) + \mathcal{P}(G^{76}) + \mathcal{P}(G^{77}) + 2\mathcal{P}(G^{81}) + 2\mathcal{P}(G^{82}) + \mathcal{P}(G^{94}) + 3\mathcal{P}(G^{95}) + 9\mathcal{P}(G^{96}) + 6\mathcal{P}(G^{97}) + 3\mathcal{P}(G^{98}) + 6\mathcal{P}(G^{100}) + 2\mathcal{P}(G^{103}) + 11\mathcal{P}(G^{108}) + 6\mathcal{P}(G^{109}) + 28\mathcal{P}(G^{110})$
G^{78}	$90\mathcal{P}(G^0) + 42\mathcal{P}(G^1) + 15\mathcal{P}(G^2) + 3\mathcal{P}(G^3) + \mathcal{P}(G^{55}) + \mathcal{P}(G^{76}) + \mathcal{P}(G^{77}) + \mathcal{P}(G^{78}) + \mathcal{P}(G^{79}) + \mathcal{P}(G^{80}) + \mathcal{P}(G^{81}) + \mathcal{P}(G^{82}) + \mathcal{P}(G^{94}) + 3\mathcal{P}(G^{95}) + 9\mathcal{P}(G^{96}) + 9\mathcal{P}(G^{97}) + 9\mathcal{P}(G^{98}) + 9\mathcal{P}(G^{99}) + 4\mathcal{P}(G^{100}) + 2\mathcal{P}(G^{103}) + \mathcal{P}(G^{105}) + 2\mathcal{P}(G^{107}) + 11\mathcal{P}(G^{108}) + 9\mathcal{P}(G^{109}) + 22\mathcal{P}(G^{110}) + 6\mathcal{P}(G^{111})$
G^{79}	$360\mathcal{P}(G^0) + 144\mathcal{P}(G^1) + 42\mathcal{P}(G^2) + 6\mathcal{P}(G^3) + 2\mathcal{P}(G^{55}) + \mathcal{P}(G^{76}) + \mathcal{P}(G^{79}) + 2\mathcal{P}(G^{80}) + \mathcal{P}(G^{81}) + \mathcal{P}(G^{94}) + 6\mathcal{P}(G^{95}) + 18\mathcal{P}(G^{96}) + 12\mathcal{P}(G^{97}) + 6\mathcal{P}(G^{98}) + 10\mathcal{P}(G^{100}) + 4\mathcal{P}(G^{103}) + 2\mathcal{P}(G^{105}) + 6\mathcal{P}(G^{107}) + 24\mathcal{P}(G^{108}) + 12\mathcal{P}(G^{109}) + 64\mathcal{P}(G^{110}) + 24\mathcal{P}(G^{111})$
G^{80}	$180\mathcal{P}(G^0) + 60\mathcal{P}(G^1) + 15\mathcal{P}(G^2) + 3\mathcal{P}(G^3) + \mathcal{P}(G^{55}) + \mathcal{P}(G^{80}) + 3\mathcal{P}(G^{95}) + 9\mathcal{P}(G^{96}) + 3\mathcal{P}(G^{97}) + 2\mathcal{P}(G^{100}) + 2\mathcal{P}(G^{107}) + 5\mathcal{P}(G^{108}) + 20\mathcal{P}(G^{110}) + 12\mathcal{P}(G^{111})$
G^{81}	$360\mathcal{P}(G^0) + 96\mathcal{P}(G^1) + 24\mathcal{P}(G^2) + 6\mathcal{P}(G^3) + \mathcal{P}(G^{81}) + 4\mathcal{P}(G^{100}) + 2\mathcal{P}(G^{108}) + 16\mathcal{P}(G^{110})$
G^{82}	$90\mathcal{P}(G^0) + 30\mathcal{P}(G^1) + 9\mathcal{P}(G^2) + 3\mathcal{P}(G^3) + \mathcal{P}(G^{81}) + \mathcal{P}(G^{82}) + 2\mathcal{P}(G^{100}) + 2\mathcal{P}(G^{108}) + \mathcal{P}(G^{109}) + 8\mathcal{P}(G^{110})$
G^{83}	$360\mathcal{P}(G^0) + 144\mathcal{P}(G^1) + 48\mathcal{P}(G^2) + 12\mathcal{P}(G^3) + 6\mathcal{P}(G^{81}) + 4\mathcal{P}(G^{82}) + \mathcal{P}(G^{83}) + 2\mathcal{P}(G^{87}) + 2\mathcal{P}(G^{92}) + 18\mathcal{P}(G^{100}) + 8\mathcal{P}(G^{101}) + 4\mathcal{P}(G^{103}) + \mathcal{P}(G^{105}) + 6\mathcal{P}(G^{107}) + 16\mathcal{P}(G^{108}) + 4\mathcal{P}(G^{109}) + 56\mathcal{P}(G^{110}) + 24\mathcal{P}(G^{111})$

Graph	Corresponding Purified Decomposition
G^{86}	$180\mathcal{P}(G^0)+84\mathcal{P}(G^1)+30\mathcal{P}(G^2)+6\mathcal{P}(G^3)+4\mathcal{P}(G^{81})+2\mathcal{P}(G^{82})+\mathcal{P}(G^{83})+\mathcal{P}(G^{86})+2\mathcal{P}(G^{87})+2\mathcal{P}(G^{89})+3\mathcal{P}(G^{92})+14\mathcal{P}(G^{100})+8\mathcal{P}(G^{101})+8\mathcal{P}(G^{102})+5\mathcal{P}(G^{103})+4\mathcal{P}(G^{105})+3\mathcal{P}(G^{106})+12\mathcal{P}(G^{107})+16\mathcal{P}(G^{108})+6\mathcal{P}(G^{109})+44\mathcal{P}(G^{110})+36\mathcal{P}(G^{111})$
G^{87}	$360\mathcal{P}(G^0)+120\mathcal{P}(G^1)+36\mathcal{P}(G^2)+6\mathcal{P}(G^3)+2\mathcal{P}(G^{81})+\mathcal{P}(G^{87})+12\mathcal{P}(G^{100})+8\mathcal{P}(G^{101}))+2\mathcal{P}(G^{103}+6\mathcal{P}(G^{108})+32\mathcal{P}(G^{110})$
G^{89}	$360\mathcal{P}(G^0)+144\mathcal{P}(G^1)+42\mathcal{P}(G^2)+6\mathcal{P}(G^3)+3\mathcal{P}(G^{81})+\mathcal{P}(G^{87})+\mathcal{P}(G^{89})+2\mathcal{P}(G^{92})+16\mathcal{P}(G^{100})+8\mathcal{P}(G^{101})+8\mathcal{P}(G^{102})+4\mathcal{P}(G^{103})+2\mathcal{P}(G^{105})+12\mathcal{P}(G^{107})+18\mathcal{P}(G^{108})+4\mathcal{P}(G^{109})+64\mathcal{P}(G^{110})+48\mathcal{P}(G^{111})$
G^{90}	$180\mathcal{P}(G^0)+84\mathcal{P}(G^1)+30\mathcal{P}(G^2)+6\mathcal{P}(G^3)+3\mathcal{P}(G^{81})+\mathcal{P}(G^{87})+\mathcal{P}(G^{89})+\mathcal{P}(G^{90})+\mathcal{P}(G^{91})+2\mathcal{P}(G^{92})+2\mathcal{P}(G^{93})+3\mathcal{P}(G^{94})+6\mathcal{P}(G^{95})+13\mathcal{P}(G^{100})+4\mathcal{P}(G^{101})+4\mathcal{P}(G^{102})+7\mathcal{P}(G^{103})+5\mathcal{P}(G^{104})+3\mathcal{P}(G^{105})+8\mathcal{P}(G^{107})+18\mathcal{P}(G^{108})+8\mathcal{P}(G^{109})+44\mathcal{P}(G^{110})+24\mathcal{P}(G^{111})$
G^{91}	$180\mathcal{P}(G^0)+72\mathcal{P}(G^1)+24\mathcal{P}(G^2)+6\mathcal{P}(G^3)+2\mathcal{P}(G^{81})+\mathcal{P}(G^{87})+\mathcal{P}(G^{91})+2\mathcal{P}(G^{94})+6\mathcal{P}(G^{95})+8\mathcal{P}(G^{100})+4\mathcal{P}(G^{101})+3\mathcal{P}(G^{103})+10\mathcal{P}(G^{108})+4\mathcal{P}(G^{109})+28\mathcal{P}(G^{110})$
G^{92}	$360\mathcal{P}(G^0)+120\mathcal{P}(G^1)+30\mathcal{P}(G^2)+6\mathcal{P}(G^3)+2\mathcal{P}(G^{81})+\mathcal{P}(G^{92})+8\mathcal{P}(G^{100})+\mathcal{P}(G^{103})+4\mathcal{P}(G^{107})+8\mathcal{P}(G^{108})+40\mathcal{P}(G^{110})+24\mathcal{P}(G^{111})$
G^{93}	$360\mathcal{P}(G^0)+144\mathcal{P}(G^1)+42\mathcal{P}(G^2)+6\mathcal{P}(G^3)+2\mathcal{P}(G^{81})+\mathcal{P}(G^{92})+\mathcal{P}(G^{93})+2\mathcal{P}(G^{94})+6\mathcal{P}(G^{95})+14\mathcal{P}(G^{100})+7\mathcal{P}(G^{103})+5\mathcal{P}(G^{104})+2\mathcal{P}(G^{105})+6\mathcal{P}(G^{107})+22\mathcal{P}(G^{108})+8\mathcal{P}(G^{109})+64\mathcal{P}(G^{110})+24\mathcal{P}(G^{111})$
G^{94}	$360\mathcal{P}(G^0)+120\mathcal{P}(G^1)+30\mathcal{P}(G^2)+6\mathcal{P}(G^3)+\mathcal{P}(G^{81})+\mathcal{P}(G^{94})+6\mathcal{P}(G^{95})+6\mathcal{P}(G^{100})+2\mathcal{P}(G^{103})+12\mathcal{P}(G^{108})+4\mathcal{P}(G^{109})+40\mathcal{P}(G^{110})$
G^{95}	$60\mathcal{P}(G^0)+16\mathcal{P}(G^1)+3\mathcal{P}(G^2)+\mathcal{P}(G^3)+\mathcal{P}(G^{95})+\mathcal{P}(G^{108})+4\mathcal{P}(G^{110})$
G^{96}	$20\mathcal{P}(G^0)+4\mathcal{P}(G^1)+\mathcal{P}(G^2)+\mathcal{P}(G^{96})$
G^{97}	$60\mathcal{P}(G^0)+16\mathcal{P}(G^1)+3\mathcal{P}(G^2)+3\mathcal{P}(G^{96})+\mathcal{P}(G^{97})+\mathcal{P}(G^{108})+4\mathcal{P}(G^{110})$
G^{98}	$60\mathcal{P}(G^0)+20\mathcal{P}(G^1)+4\mathcal{P}(G^2)+3\mathcal{P}(G^{96})+2\mathcal{P}(G^{97})+\mathcal{P}(G^{98})+3\mathcal{P}(G^{108})+2\mathcal{P}(G^{109})+8\mathcal{P}(G^{110})$

Graph	Corresponding Purified Decomposition
G^{99}	$10\mathcal{P}(G^0) + 4\mathcal{P}(G^1) + \mathcal{P}(G^2) + \mathcal{P}(G^{96}) + \mathcal{P}(G^{97}) + \mathcal{P}(G^{98}) + \mathcal{P}(G^{99}) + \mathcal{P}(G^{108}) + \mathcal{P}(G^{109}) + 2\mathcal{P}(G^{110})$
G^{100}	$180\mathcal{P}(G^0) + 36\mathcal{P}(G^1) + 6\mathcal{P}(G^2) + \mathcal{P}(G^{100}) + 4\mathcal{P}(G^{110})$
G^{101}	$45\mathcal{P}(G^0) + 12\mathcal{P}(G^1) + 3\mathcal{P}(G^2) + \mathcal{P}(G^{100}) + \mathcal{P}(G^{101}) + 2\mathcal{P}(G^{110})$
G^{102}	$45\mathcal{P}(G^0) + 15\mathcal{P}(G^1) + 3\mathcal{P}(G^2) + \mathcal{P}(G^{100}) + \mathcal{P}(G^{101}) + \mathcal{P}(G^{102}) + \mathcal{P}(G^{107}) + \mathcal{P}(G^{108}) + 6\mathcal{P}(G^{110}) + 6\mathcal{P}(G^{111})$
G^{103}	$360\mathcal{P}(G^0) + 96\mathcal{P}(G^1) + 18\mathcal{P}(G^2) + 4\mathcal{P}(G^{100}) + \mathcal{P}(G^{103}) + 4\mathcal{P}(G^{108}) + 24\mathcal{P}(G^{110})$
G^{104}	$72\mathcal{P}(G^0) + 24\mathcal{P}(G^1) + 6\mathcal{P}(G^2) + 2\mathcal{P}(G^{100}) + \mathcal{P}(G^{103}) + \mathcal{P}(G^{104}) + 2\mathcal{P}(G^{108}) + 8\mathcal{P}(G^{110})$
G^{105}	$360\mathcal{P}(G^0) + 120\mathcal{P}(G^1) + 24\mathcal{P}(G^2) + 6\mathcal{P}(G^{100}) + 2\mathcal{P}(G^{103}) + \mathcal{P}(G^{105}) + 4\mathcal{P}(G^{107}) + 12\mathcal{P}(G^{108}) + 4\mathcal{P}(G^{109}) + 48\mathcal{P}(G^{110}) + 24\mathcal{P}(G^{111})$
G^{106}	$60\mathcal{P}(G^0) + 24\mathcal{P}(G^1) + 6\mathcal{P}(G^2) + 2\mathcal{P}(G^{100}) + \mathcal{P}(G^{103}) + \mathcal{P}(G^{105}) + \mathcal{P}(G^{106}) + 2\mathcal{P}(G^{107}) + 4\mathcal{P}(G^{108}) + 2\mathcal{P}(G^{109}) + 12\mathcal{P}(G^{110}) + 8\mathcal{P}(G^{111})$
G^{107}	$180\mathcal{P}(G^0) + 48\mathcal{P}(G^1) + 6\mathcal{P}(G^2) + \mathcal{P}(G^{100}) + \mathcal{P}(G^{107}) + 2\mathcal{P}(G^{108}) + 16\mathcal{P}(G^{110}) + 12\mathcal{P}(G^{111})$
G^{108}	$180\mathcal{P}(G^0) + 36\mathcal{P}(G^1) + 3\mathcal{P}(G^2) + \mathcal{P}(G^{108}) + 8\mathcal{P}(G^{110})$
G^{109}	$90\mathcal{P}(G^0) + 24\mathcal{P}(G^1) + 3\mathcal{P}(G^2) + 2\mathcal{P}(G^{108}) + \mathcal{P}(G^{109}) + 8\mathcal{P}(G^{110})$
G^{110}	$45\mathcal{P}(G^0) + 6\mathcal{P}(G^1) + \mathcal{P}(G^{110})$
G^{111}	$15\mathcal{P}(G^0) + 3\mathcal{P}(G^1) + \mathcal{P}(G^{110}) + \mathcal{P}(G^{111})$

D SageMath code that was used in the proof

```
G=graphs.RandomGNP(n,0.6)
M=graphs.CompleteGraph(2)
m=G.subgraph_search_count(M, True)/M.automorphism_group(return_group=False,order=True)
T=graphs.CompleteGraph(3)
t=G.subgraph_search_count(T, True)/T.automorphism_group(return_group=False,order=True)
K4=graphs.CompleteGraph(4)
k4=G.subgraph_search_count(K4, True)/K4.automorphism_group(return_group=False,order=True)
K5=graphs.CompleteGraph(5)
k5=G.subgraph_search_count(K5, True)/K5.automorphism_group(return_group=False,order=True)
K6=graphs.CompleteGraph(6)
k6=G.subgraph_search_count(K6, True)/K6.automorphism_group(return_group=False,order=True)
C4=graphs.CycleGraph(4)
c4=G.subgraph_search_count(C4, True)/C4.automorphism_group(return_group=False,order=True)
C5=graphs.CycleGraph(5)
c5=G.subgraph_search_count(C5, True)/C5.automorphism_group(return_group=False,order=True)
C6=graphs.CycleGraph(6)
c6=G.subgraph_search_count(C6, True)/C6.automorphism_group(return_group=False,order=True)
W4=graphs.WheelGraph(5)
w4=G.subgraph_search_count(W4, True)/W4.automorphism_group(return_group=False,order=True)
W5=graphs.WheelGraph(6)
w5=G.subgraph_search_count(W5, True)/W5.automorphism_group(return_group=False,order=True)
G105=Graph({0:[1,2,3,4], 1:[2,5], 2: [3], 3:[4,5], 4:[5]})
g105=G.subgraph_search_count(G105, True)/G105.automorphism_group(return_group=False,order=True)
W4s=Graph({0:[1,3,4], 1:[2,4], 2:[3,4]})
w4s=G.subgraph_search_count(W4s, True)/W4s.automorphism_group(return_group=False,order=True)
W5s=Graph({0:[1,5,4], 1:[2,5], 2:[3,5], 3:[4,5]})
w5s=G.subgraph_search_count(W5s, True)/W5s.automorphism_group(return_group=False,order=True)
T222=Graph({0:[1,2,3], 1:[4], 2:[4], 3:[4]})
```

```
t222=G.subgraph_search_count(T222, True)/T222.automorphism_group(return_group=False,order=True)
T223=Graph({0:[1,2,3], 1:[4], 2:[4], 3:[5], 5:[4]})
t223=G.subgraph_search_count(T223, True)/T223.automorphism_group(return_group=False,order=True)
T2222=Graph({0:[1,2,3,5], 1:[4], 2:[4], 3:[4],5:[4]})
t2222=G.subgraph_search_count(T2222, True)/T2222.automorphism_group(return_group=False,order=True)
K33=graphs.CompleteBipartiteGraph(3,3)
k33=G.subgraph_search_count(K33, True)/K33.automorphism_group(return_group=False,order=True)
G65=Graph({0:[1,2,4], 1:[2,4], 2:[3,5], 3:[4],4:[5]})
g65=G.subgraph_search_count(G65, True)/G65.automorphism_group(return_group=False,order=True)
G80=Graph({0:[1,3,4], 1:[2,4], 2:[3,4], 3:[4], 5:[0,2]})
g80=G.subgraph_search_count(G80, True)/G80.automorphism_group(return_group=False,order=True)
G95=Graph({0:[1,2,3,5], 1:[2,3,4], 2:[3,4], 3:[4],4:[5]})
g95=G.subgraph_search_count(G95, True)/G95.automorphism_group(return_group=False,order=True)
G97=Graph({0:[1,3,4,5], 1:[2,4], 2:[3,4,5], 3:[4], 4:[5] })
g97=G.subgraph_search_count(G97, True)/G97.automorphism_group(return_group=False,order=True)
G102=Graph({0:[1,2,3], 1:[2,3], 2:[4,5], 3:[4,5], 4:[5]})
g102=G.subgraph_search_count(G102, True)/G102.automorphism_group(return_group=False,order=True)
G107=Graph({0:[1,2,3,4], 1:[2,5], 2:[3,5], 3:[4,5],4:[5]})
g107=G.subgraph_search_count(G107, True)/G107.automorphism_group(return_group=False,order=True)
G108=Graph({0:[1,4,5], 1:[2,3,5], 2:[3,4,5], 3:[4,5], 4:[5]})
g108=G.subgraph_search_count(G108, True)/G108.automorphism_group(return_group=False,order=True)
G109=Graph({0:[1,2,3,4], 1:[2,3,5],2:[3,4], 3:[5], 4:[5]})
g109=G.subgraph_search_count(G109, True)/G109.automorphism_group(return_group=False,order=True)
G110=Graph({0:[1,2,3,4], 1:[2,3,5],2:[3,4,5], 3:[4,5], 4:[5]})
g110=G.subgraph_search_count(G110, True)/G110.automorphism_group(return_group=False,order=True)
G111=Graph({0:[1,2,3,4], 1:[2,3,5],2:[4,5], 3:[4,5], 4:[5]})
g111=G.subgraph_search_count(G111, True)/G111.automorphism_group(return_group=False,order=True)
G75=Graph({0:[1,2], 1:[3,4], 2:[3,5], 3:[4], 4:[5]})
g75=G.subgraph_search_count(G75, True)/G75.automorphism_group(return_group=False,order=True)
G78=Graph({0:[1,3,5], 1:[2,4], 2:[5], 3:[4], 4:[5]})
g78=G.subgraph_search_count(G78, True)/G78.automorphism_group(return_group=False,order=True)
```

```
G79=Graph({0:[1,3,4], 1:[2,4], 2:[3,4], 3:[5], 5:[4]})
g79=G.subgraph_search_count(G79, True)/G79.automorphism_group(return_group=False,order=True)
G90=Graph({0:[1,5], 1:[2], 2:[3,4], 3:[4], 4:[5],5:[3]})
g90=G.subgraph_search_count(G90, True)/G90.automorphism_group(return_group=False,order=True)
G106=Graph({0:[1,2,4], 1:[2,5], 2:[3], 3:[4,5], 4:[5]})
g106=G.subgraph_search_count(G106, True)/G106.automorphism_group(return_group=False,order=True)
G.subgraph_search_count(K4, True)/K4.automorphism_group(return_group=False,order=True)
ans=binomial(m-2,3)*t-binomial(m,5)-binomial(t,2)*(m-4)+(2*t-3*m+12-2*binomial(m-6,2))*k4+6\
*(m-8)*k5-24*k6+(binomial(m-3,2)-t)*c4+(m-4)*c5+c6-3*(m-6)*w4-2*(m-5)*w4s-4*w5-g105-(m-3)*t222\
-t223+t2222-4*k33+2*g65+3*g80+6*g95+4*g97+4*g102+5*g107+8*g108+4*g109+12*g110+16*g111-2*g75\
-g78-g79-2*g90-3*w5s-4*g106
print(ans)
print(G.chromatic_polynomial())
f=G.chromatic_polynomial()
print(f[n-5])
```

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