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The optimization of signed trees

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Abstract

A signed graph G is a graph where each edge is assigned a $+$ (positive edge) or a $-$ (negative edge). The signed degree of a vertex v in a signed graph, denoted by $\mathrm{sdeg}(v)$, is the number of positive edges incident to v subtracted by the number of negative edges incident to v. Finally, we say G realizes the set D if $D = \{\mathrm{sdeg}(v) : v \in V(G)\}$. The topic of signed degree sets and signed degree sequences has been studied from many directions. In this paper, we study properties needed for signed trees to have a given signed degree set. We start by proving that D is the signed degree set of a tree if and only if $1 \in D$ or $-1 \in D$. Further, for every valid set D, we find the smallest diameter that a tree must have to realize D. Lastly, for valid sets D with nonnegative numbers, we find the smallest order that a tree must have to realize D.

1 Introduction

Every graph in this paper is finite, simple and undirected.

A signed graph is an ordered pair (G, s) where G is a graph and $s : E(G) \to \{-, +\}$ is a function that assigns either a positive $(+)$ or a negative $(-)$ sign to every edge. Further, we refer to an edge e as positive or negative if $s(e) = +$ or $s(e) = -$ respectively. The notion of signed graphs was introduced by Harary [5], and since then they have been widely studied (the interested reader is referred to Zaslavsky [8]). This paper focuses in particular on signed degrees and signed degree sets. The signed degree of a vertex v in a signed graph, denoted by $\mathrm{sdeg}(v)$, is the

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number of positive edges incident to v subtracted by the number of negative edges incident to v. For example, Figure 1 has a vertex v_1 with signed degree $+1$. Moreover, the signed degree set of a signed graph (G, s) is the set $D = \{\text{sdeg}(v) : v \in V(G)\}$; equivalently, we say that (G, s) realizes D. The graph in Figure 1, for example, has signed degree set $\{1, -1\}$.

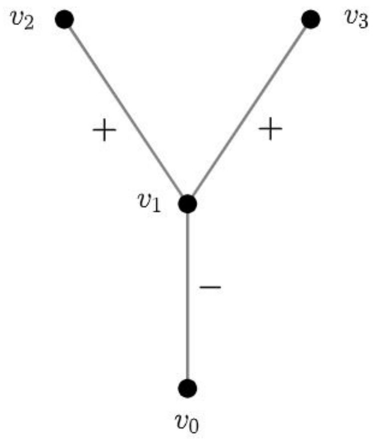

Figure 1: An example of a signed graph

The topic of signed degree sets stems from the study of signed degree sequences. In [1], Chang et al. found necessary and sufficient conditions needed for an integral sequence to be the signed degree sequence of a signed graph, while noting that a polynomial time algorithm that verifies this condition could be designed. In the same paper, the authors repeat the result but restricted to the case of signed trees. In [7], Naikoo and Pizada characterized the signed degree sequences of $k-$partite graphs. In the same paper, the authors prove that every set of integers is the signed degree set of some $k-$partite graph. Another related subject is studying net regular graphs: those graphs G for which there exists a σ such that (G, σ) realizes a set D with $|D| = 1$. This concept has been studied [3, 4, 6] from various perspectives, including its relationship to the spectrum of graphs, regularity degree-wise, and a characterization of net-regular trees.

In this paper, we investigate, for a given degree set D, the minimum diameter that a tree T must have to be the underlying graph of a signed tree (T, s) that realizes D. We also study the same question but in regards to the order of T. We start the paper in Section 2 by determining which sets are valid, i.e. the sets that can be realized by a signed tree. In Section 3, we identify the minimum diameter needed when given an arbitrary valid signed degree set. Finally, in Section 4, we repeat this process but for order. Although we have limited results in Section 4, we were able to find the minimum order of signed degree sets that consist of non-negative numbers. In Section 5 we conclude the paper with a conjecture.

Throughout the paper we refer to vertices with degree 1 as pendant vertices. To simplify our notation on signed degree sets, we will always use the variables $x_1, x_2, x_3, \dots$ and so on to refer to positive integers greater than 1. Similarly, we use the variables $y_1, y_2, y_3, \ldots$ and so on to refer to negative integers less than -1. Finally, we use $z_1, z_2, z_3, \ldots$ and so on to refer to any integer but 1 and -1.

2 Valid Sets

Before we discuss optimization of diameter and order, we must first determine which sets can be considered. An important property of trees is that they have at least two pendant vertices. Since pendant vertices have degree one, they will be incident to either a negative or positive edge, which leads us to the following observation.

Observation. If $1 \notin D$ and $-1 \notin D$, then D cannot be the set of signed degrees of a signed tree.

We can take it a step further and consider the following lemma. For it, we will use the notation $CT(n_1, \ldots, n_m)$, where $n_i > 0$, to denote the caterpillar CT where any longest path $P : v_0, \ldots, v_{m+1}$ has $\deg(v_i) = n_i$ for $i \neq 0$ and $i \neq m+1$, and every other vertex in CT has degree 1.

Lemma 2.1. *There exists a signed tree (T, s) that realizes D if and only if either $-1 \in D$ or $1 \in D$.*

Proof. As noted before, if a signed tree (T, s) realizes D, then $-1 \in D$ or $1 \in D$. For the other direction, assume that $-1 \in D$ or $1 \in D$. Proving that there exists a tree (T, s) that realizes D implies that there exists a tree that realizes $\{-n : n \in D\}$ (we simply invert every sign in (T, s)). Thus, it suffices to prove the theorem when $1 \in D$ and $-1 \in D$, and when $1 \in D$ but $-1 \notin D$. We split the proof into these two cases.

Case 1: Assume $1, -1 \in D$, and assume $0 \notin D$ (we deal with 0 at the end of the case). Let D contain n positive integers and m negative integers, none equal to 1 or -1. The cases where $n = 0$ or $m = 0$ follow from the case where $n > 0$ and $m > 0$ as the signed degree of a vertex v can be changed to 1 by adding sufficient pendant vertices to v and assigning the appropriate signs to the new edges. Let $D = \{1, -1, x_1, \ldots, x_n, y_1, \ldots, y_m\}$. We will construct a signed tree that realizes this set.

Let $CT := CT(x_1, \ldots, x_n, |y_1| + 4, \ldots, |y_m| + 4)$ be a caterpillar where $P = v_0, \ldots, v_{n+m+1}$ is a longest path. Without loss of generality, let $\deg(v_i) = x_i$ for $1 \leq i \leq n$, and let $\deg(v_{n+i}) = |y_i| + 4$ for $1 \leq i \leq m$. We will define a function $s : E(CT) \to \{+, -\}$ such that (CT, s) realizes D. For an edge e in CT, define s as follows.

$$s(e) = \begin{cases} + & \text{if } e \text{ is contained in } P\text{, or } e \text{ is incident to a vertex } v_i \text{ where } 1 \leq i \leq n, \\ - & \text{otherwise.} \end{cases}$$

Hence $\text{sdeg}(v_i) = x_i$ for $1 \leq i \leq n$, and $\text{sdeg}(v_i) = 2-(2+|y_i|) = y_i$ for $n+1 \leq i \leq m$. Since every other vertex has degree 1 or -1, (CT, s) realizes D. To account for 0, let $D' = D \cup \{0\}$. Let v be a pendant vertex in CT. Attach a new vertex v' to v, and extend s so that $s(vv') = -$. Now (CT, s) realizes D', finishing this case.

Case 2: Assume $1 \in D$ but $-1 \notin D$. Let (T, s) be a tree that realizes $D \cup \{-1\}$ (whose existence follows from Case 1). Let T' be the tree constructed from T such that for every vertex $v \in T$ where $\mathrm{sdeg}(v) = -1$, we attach two new vertices v_1, v_2 to v in T', and let s' be the extension of s so that $s(vv_i) = +$ for $i \in \{1, 2\}$. Consequently, no vertex in (T', s') has signed degree -1, and every new vertex has signed degree 1. It follows that (T', s') realizes D. □

As a consequence of this lemma we know which degree sets are of interest to us: the ones with -1 or 1. We thus define these sets as **valid sets**. Having established this fact, we are able to delve into optimization.

3 Minimum Diameter for a Tree

There are infinitely many signed trees that realize a given valid degree set D. From these signed trees, we start by studying the ones with optimal diameter.

Definition 3.1. Let D be a valid set. Define its diameter, denoted by $\mathrm{diam}(D)$, as the smallest diameter that the underlying tree of a signed tree must have so it realizes D, or equivalently:

$$\mathrm{diam}(D) = \min\{\mathrm{diam}(T) : \text{the signed tree } (T, s) \text{ realizes } D\}.$$

Further, a signed tree (T, s) is optimal if $\mathrm{diam}(T) = \mathrm{diam}(D)$.

Definition 3.2. We denote the bipartite graph $K_{1,n}$ as $ST(n)$. It follows that $ST(n)$ is a tree, and $(ST(n), s)$ is a signed tree of diameter 2.

We invite the reader to verify that $\mathrm{diam}(\{1, 0\}) = 3$ and that $\mathrm{diam}(\{1, x\}) = 2$ for $x > 1$.

We start by considering degree sets that are composed of only positive values.

Theorem 3.3. *If $D = \{1, x_1, \dots, x_n\}$, then*

$$\mathrm{diam}(D) = \begin{cases} 2 & \text{if } n = 1 \\ 3 & \text{if } n = 2 \\ 4 & \text{if } n > 2. \end{cases}$$

Proof. Throughout this proof, always assume s maps every edge to a $+$. When $n = 1$, the signed graph $(ST(x_1), s)$ realizes D and has diameter 2, and evidently no signed tree with diameter 1 can realize D. For $n = 2$, attach $x_2 - 1$ vertices to a pendant vertex in $ST(x_1)$. The resulting signed tree will realize D and have diameter 3. Diameter 2 for this set is not possible because trees of diameter 2 are stars, and stars only have one vertex with degree higher than 1. Similarly, trees with diameter 3 have exactly 2 vertices whose degree is not equal to 1, so no signed tree of diameter 3 can realize a set D when $n > 2$. In other words, when $n > 2$, it must be that $\text{diam}(D) \geq 4$. It remains to show that $\text{diam}(D) \leq 4$. Assume that $x_1 < x_2 < \cdots < x_n$, and notice that since $x_1 > 1$, it must be that $x_n > n$. Starting with $ST(x_n)$, attach x_i vertices to a pendant vertex of $ST(x_n)$ for each $i \in [n-1]$. Since $x_n > n$, there will be enough pendant vertices of $ST(x_n)$ to do this. Call this resulting tree T. Observe that (T, s) realizes D and has diameter 4, thus proving that $\text{diam}(D) \leq 4$. □

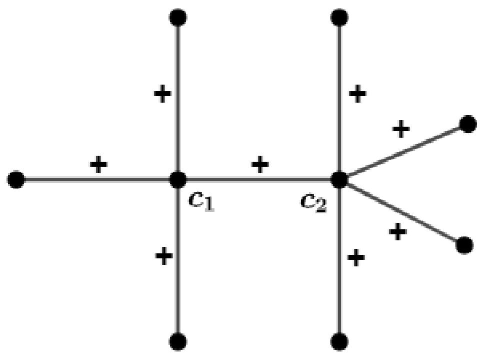

Figure 2: An optimal signed tree by Theorem 3.3 given the set $D = \{1, 4, 5\}$

Building off this theorem, we consider what changes must be made to a tree described in Theorem 3.3 (and that can be seen in Figure 2) in order for it to accommodate a vertex of signed degree 0. As previously noted, we know that $\text{diam}(\{1, 0\}) = 3$, which we now know is higher than the diameter of $\{1, x\}$ for $x > 1$. So, when adding additional positive values to the degree set of $\{1, 0\}$, the behavior of optimal diameter closely matches that of the previous theorem. In particular, the diameter of $\{1, 0, x_1, \ldots, x_n\}$ is the same as the diameter of $\{1, x_1, \ldots, x_{n+1}\}$ when $n \geq 1$. We invite the reader to verify the following.

Corollary 3.4. *If* $D = \{1, 0, x_1, \ldots, x_n\}$, *then*

$$\text{diam}(D) = \begin{cases} 3 & \text{if } n = 1 \\ 4 & \text{if } n > 1 \end{cases}$$

This technique of using simpler degree sets to prove results of more intricate ones will repeat many times throughout the paper.

Moving to negative values, we want to prove in general that if $-1 \notin D$ and if (T, s) realizes D, then vertices far away from the center of T cannot have negative signed degrees. The following definition helps us formalize this notion.

Definition 3.5. A vertex v in a tree T is called a limiting pendant vertex if it is a pendant vertex in a longest path of the tree (see Figure 3).

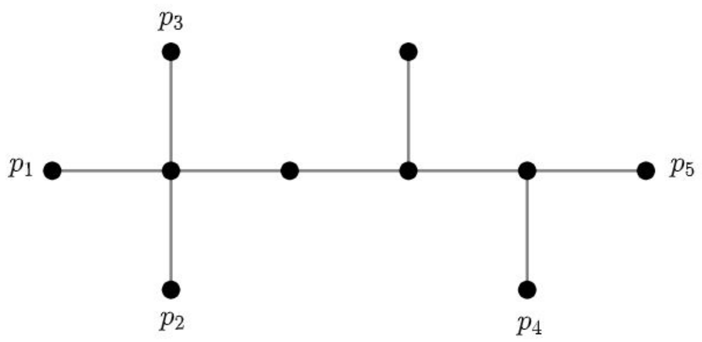

Figure 3: A tree with limiting pendant vertices labelled p_i, $1 \leq i \leq 5$.

Lemma 3.6. *Let (T, s) be a signed tree that realizes a set D where $-1 \notin D$. If v is a vertex in T with negative signed degree, then v is not a pendant vertex nor adjacent to a limiting pendant vertex.*

Proof. We proceed by contradiction. Let v be a vertex such that v is adjacent to limiting pendant vertex p while $\text{sdeg}(v) = y < 0$. Let S be a longest path in T containing p. To realize y, the vertex v must be incident to at least $|y| + 1$ negative edges since $\text{sdeg}(vp) = +$. Note that S may contain at most 1 of these negative edges, so there exists a vertex w such that $s(vw) = -$ and w is not a vertex in S. This, however, implies that w is a pendant vertex by the maximality of S, so $\text{sdeg(w)} = -1$, contradicting that $-1 \notin D$. □

The first case of degree sets D with negative values that we will consider is the one where 1 is the only positive value in D. A signed tree that realizes such a degree set must have pendant vertices with positive signed degree. As the following result confirms, this implies that we need a larger diameter.

Lemma 3.7. *If (T, s) is a signed tree that realizes $D = \{1, y_1, \ldots, y_m\}$, then*

$$\text{diam}(T) \geq \begin{cases} 4 & \text{if } m = 1. \\ 5 & \text{if } m = 2. \\ 6 & \text{if } m > 2. \end{cases}$$

Proof. In a tree of diameter 3, there exists no vertex that is neither a pendant vertex, nor adjacent to a limiting pendant vertex. By Lemma 3.6, a signed tree of diameter 3 that realizes a set without -1 cannot have a vertex of negative signed degree, so if $m = 1$, then $\text{diam}(T, s) \geq 4$. For the case when $m = 2$, notice that a tree of diameter 4 or less there exists at most one vertex that is neither a pendant vertex nor adjacent to a limiting pendant vertex. By Lemma 3.6, a signed tree of diameter 4 that realizes a set without -1 cannot have two vertices of negative signed degree, so $\text{diam}(T, s) \geq 5$. A similar logic can be used to prove the case where $m > 2$. □

Our penultimate case is the one where D may contain any integer with the exception of -1. As the proof of Theorem 3.8 demonstrates, the fact that negatives require such a big diameter allows us to add positive numbers to the signed degree set without changing this parameter.

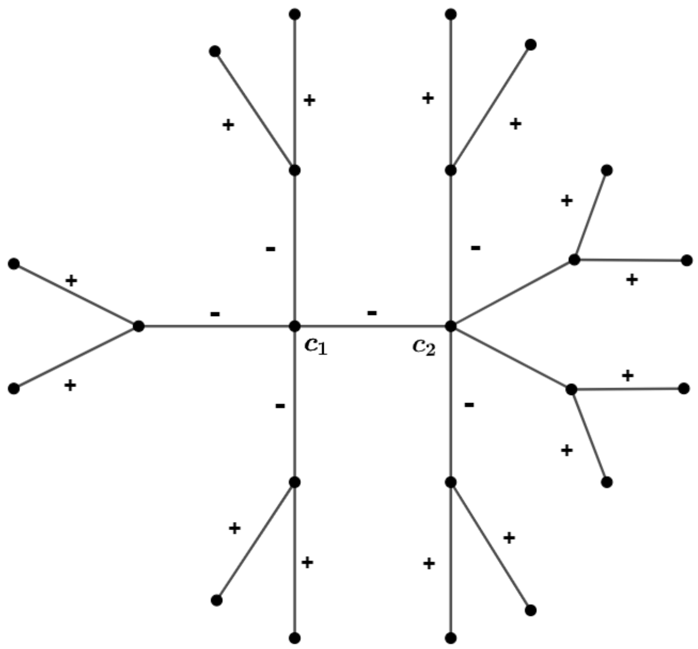

Figure 4: An optimal signed tree that realizes the degree set $D = \{1, -4, -5\}$ constructed from the tree in Figure 2.

Theorem 3.8. *If* $D = \{1, y_1, \ldots, y_m, x_1, \ldots, x_n\}$ *where* $m \geq 1$ *and* $n \geq 0$ *(where if* $n = 0$*, then* $D = \{1, y_1, \ldots, y_m\}$*), then* $\text{diam}(D) = \text{diam}(D \cup \{0\})$*) and*

$$\text{diam}(D) = \begin{cases} 4 & \text{if } m = 1 \\ 5 & \text{if } m = 2 \\ 6 & \text{if } m > 2. \end{cases}$$

Proof. Consider the case when $m = 1$. Let $D' = \{-1, y_1\}$. Let (T, s) be the signed tree where $T = ST(|y_1|)$ and where s assigns a $-$ to every edge. Note that (T, s) realizes D'. Let c refer to the vertex in T such that $\deg(c) = |y_1|$. Let $P = \{p_1, \ldots, p_{|y_1|}\}$ be the set of pendant vertices in T. For each $i \in \{1, \ldots, |y_1|\}$, attach two vertices to p_i, and extend s so it assigns $+$ to the corresponding new edges. This will change every vertex with signed degree of -1 to signed degree of 1, thus making (T, s) now realize $\{1, y_1\}$. For an example of this construction, but applied to the case when $m = 2$, see Figure 4. Note that this process will increase the diameter of T to 4. Finally, to make the tree realize D, we will use a procedure that we first illustrate with x_1. Attach two new vertices u_1 and u_2 to c such that $s(u_1c) = -$ and $s(u_2c) = +$. This way the signed degree of c remains unchanged. Attach $x_1 + 1$ vertices to u_1, and let s assign $+$ to these new edges. This makes the signed degree of u_1 be x_1. In other words, (T, s) now realizes $\{1, x_1, y_1\}$. Observe that the diameter of T will not increase using this procedure. Continuing in this manner for every $x_i \in D$, we see that we can modify (T, s) so that it realizes D while still having diameter 4. This proves that $\text{diam}(D) \leq 4$, and by Lemma 3.7, we have the result for $m = 1$.

We can use a similar technique when $m = 2$. We start with a signed tree (T, s) as described in Theorem 3.3 that realizes $D' = \{-1, y_1, y_2\}$, and we modify it so it realizes $\{1, y_1, y_2\}$ with diameter 5. Further, since the diameter is greater than 4, we can apply the same method as the prior case where we modify the tree so it has a vertex with signed degree x_i while maintaining the diameter. Explicitly, we let c be the center of the tree, and repeat the procedure of the previous paragraph. This gives the desired upper bound while Lemma 3.7 gives the desired lower bound. This technique also works when $m > 2$. □

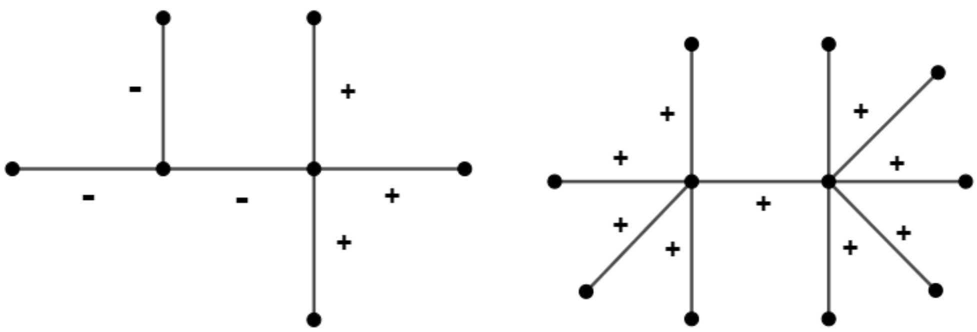

Figure 5: The signed graph on the left realizing $D = \{1, -1, -2, 2\}$, and the signed graph on the right realizing $D = \{1, 5, 6\}$

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Our final and most general case for the optimization of diameter is signed degree sets that contain 1 and -1 along with $z_1, \ldots, z_n$, where the numbers z_i are meant to denote distinct integer not equal to 1 or -1. The crux of this proof lies in the fact that the result matches that of Theorem 3.3. In short, these results are similar because it is possible to build an optimal tree that realizes a degree set $\{1, -1, z_1, \ldots, z_n\}$ by modifying an optimal tree that realizes $\{1, x_1, \ldots, x_n\}$.

Theorem 3.9. *If* $D = \{1, -1, z_1, \ldots, z_n\}$, *then*

$$\text{diam}(D) = \begin{cases} 2 & \text{if } n = 1 \\ 3 & \text{if } n = 2 \\ 4 & \text{if } n > 2. \end{cases}$$

Proof. Let (T, s) be an optimal signed tree that realizes D. Further, set $\Delta = \Delta(T)$, the maximum degree of T, and let $D' = \{1, \Delta + 1, \ldots, \Delta + n\}$. We will show that $\text{diam}(D) = \text{diam}(D')$, which by Theorem 3.3 will give us the result.

To prove that $\operatorname{diam}(D') \leq \operatorname{diam}(D)$, we will show that we can create a tree (T', s') that realizes D' from (T, s) such that $\operatorname{diam}(T) = \operatorname{diam}(T')$. This would suffice because of the fact that $\operatorname{diam}(D') \leq \operatorname{diam}(T') = \operatorname{diam}(T) = \operatorname{diam}(D)$. To construct (T', s'), we start by setting F as a tree such that $F \cong T$. Let $V = \{v_1, \ldots, v_n\}$ be a collection of non-pendant vertices in F. Define F' as that tree that outcomes from attaching $\Delta + i - \deg_F(v_i)$ new vertices to each vertex v_i. Finally, let T' be that tree that outcomes from attaching to every non-pendant vertex $w \notin V$ in F' the quantity of $\Delta + 1 - \deg_{F'}(w)$ new vertices. Notice that every nonpendant vertex of T has been changed when compared to T'. Moreover, for every valid i, note that $\deg_{T'}(v_i) = \Delta + i$, and for every non-pendant vertex $w \notin V$ we have $\deg_{T'}(w) = \Delta + 1$. Thus, if s' assigns $+$ to every edge in T, then (T', s') realizes D'. Since we added vertices only to non-pendant vertices, it must be that $\operatorname{diam}(T) = \operatorname{diam}(T')$. For an illustration of a signed graph (T, s) and (T', s'), see Figure 5.

To prove that $\operatorname{diam}(D) \leq \operatorname{diam}(D')$, we start by letting (H, s) be an optimal signed tree that realizes D'. We will show that we can create a signed tree (H', s') that realizes D such that $\operatorname{diam}(H') = \operatorname{diam}(H)$. Let $p_i = \Delta + i - z_i$, and notice that $p_i > 0$. Set $V = \{v_1, \ldots, v_n\}$ as a set of vertices in H such that $\operatorname{sdeg}(v_i) = \Delta + i$, and let G be that tree that outcomes from attaching p_i new vertices to v_i. Finally, for each non-pendant vertex w in H such that $w \notin V$, let H' be that tree that outcomes from attaching $|\operatorname{sdeg}_H(w)| + 1$ vertices [to w]. It remains to define s'. Set s' as the extension of s where edges in $E(G) \setminus E(H)$ get assigned a $-$, and edges in $E(H') \setminus E(G)$ get assigned a $+$ if $\operatorname{sdeg}_H(w) < 0$ or a $-$ if $\operatorname{sdeg}_H(w) > 0$. Notice then that $\operatorname{sdeg}_{H'}(v_i) = \Delta + i - (\Delta + i - z_i) = z_i$, and for nonpendant vertices $w \neq v_i$ we have that $\operatorname{sdeg}_{H'}(w) = 1$ because of our choice of the sign that s assigns to the new edges incident to w in H'. Every other vertex will be pendant vertices because they were either pendant vertices in H, or they were vertices added in the construction, and since we did not add vertices to new ones, they have remained pendant. Thus, (H', s') realizes D. Since we only attached vertices to non-pendant vertices, $\operatorname{diam}(H') = \operatorname{diam}(H)$. This concludes the proof as $\operatorname{diam}(D) \leq \operatorname{diam}(H') = \operatorname{diam}(H) = \operatorname{diam}(D')$. □

4 Minimum Order for a Tree

We now shift our attention to optimizing order, where the order of a graph G is denoted by $\sigma(G)$. Similar to the previous section, we want to optimize order while realizing a given degree set D.

Definition 4.1. Let D be a valid set. Define its order, denoted by $\sigma(D)$, as the smallest order that the underlying tree of a signed tree must have so that it realizes D, or equivalently:

$$\sigma(D) = \min\{\sigma(T) : \text{the signed tree } (T, s) \text{ realizes } D\}.$$

Further, a signed tree (T, s) is now optimal when $\sigma(T) = \sigma(D)$ (so it no longer refers to diameter).

Unlike our diameter section, we have not found the minimum order for every possible valid set D. Although conjecturing the optimal order is somewhat easy, proving an equation gives optimal order demonstrated being hard. We have solved only two cases, the first case encompassing only positive integers, and the second case of positive integers and zero. The former case, as with diameter, is the simplest one.

Theorem 4.2. *If* $D = \{1, x_1, \ldots, x_n\}$*, then* $\sigma(D) = 2 - n + \sum x_i$.

Proof. We will prove the theorem by induction. For $n = 1$, notice that $(ST(x_1), s)$, where s assigns positives to every edge, is a tree of least order that realizes D, and that $\sigma(ST(x_1)) = x_1 + 1$ which satisfies the formula. Assume that the result holds for $n - 1$. Let (T, s) be a tree that realizes D with optimal order. Further, let p be a limiting pendant vertex, and set v as the vertex adjacent to p. Since every signed degree is positive and the tree is optimal, then every edge must be positive. It follows that $\deg(v) = x_a$ for some a. If we let T' be that tree obtained by removing every pendant vertex adjacent to v, then T' stays connected and v becomes a pendant vertex in T' since p was a limiting pendant vertex. Further, it cannot be the case that $(T', s|_{E(T')})$ realizes D since $\sigma(T') < \sigma(T)$, so it must be that (T', s) realizes $D - \{x_a\}$. By the induction hypothesis, $\sigma(T') = 2 - (n - 1) + \sum x_i - x_a$. This, combined with the fact that $\sigma(T') = \sigma(T) - (x_a - 1)$, completes the inductive step. □

Even though $\sigma(D)$ for positives and zero has a very similar expression to that of just positives (see Theorem 4.7), the proof of Theorem 4.7 is much harder. We first need to establish two results about the structure of every optimal tree that realizes a set with positive values and 0.

The proof technique we will use for these two results has the following structure: to prove that every optimal signed tree of a set D realizes a statement P, we will demonstrate that if there exists an optimal signed tree (T, s) that fails P, then there exists a signed tree (T', s') that realizes D and has the property that $\sigma(T') < \sigma(T)$. This will be enough as it contradicts the fact (T, s) is optimal. Further, the underlying tree T' will be based on T, having changes made through the "transfer" of vertices. We formalize this in the following definition.

Definition 4.3. Let T be a tree with the distinct vertices u, v, and w. If $uv \in E(T)$ and the unique path between u and w includes v, then the graph T' that results from transferring u to w is the graph with the following properties.

- $V(T') = V(T)$, and
- $E(T') = E(T) \setminus \{uv\} \cup \{uw\}$.

Notice that the condition of having v be in the path between u and w guarantees that T' is a tree as well. **Since multiple transfers usually occur in a single proof, we will keep denoting the tree after the transfer also as T for the sake of simplicity.** The proofs and cases are short enough that hopefully the abuse of notation will not cause confusion. In this paper, we use $N(v)$ to refer to the neighborhood of a vertex v.

Lemma 4.4. *If (T, s) is an optimal signed tree that realizes $D = \{1, 0, x_1, \ldots, x_n\}$, and ab is an edge in T such that $s(ab) = -$, then* $\text{sdeg}(a) = \text{sdeg}(b) = 0$.

Proof. Assume for a contradiction that a has signed degree $k > 0$. It follows that there exist $k+1$ vertices $w_1, \ldots, w_{k+1}$ adjacent to a such that $s(aw_i) = +$. Transfer the vertices in the set $N(w_1) - \{a\}$ to a pendant vertex p that realizes the conditions for transfers (i.e. the transfer will not produce a cycle), and modify s so it assigns the same respective sign to these new edges. Notice that the signed degree set of (T, s) did not change. If we repeat this process again with $w_2, w_3, \ldots, w_{k-1}$, and w_k such that when dealing with w_j we have $p \neq w_i$ with $i < j$, then T becomes a tree where $w_1, \ldots, w_k$ are pendant vertices. Further, (T, s) will still realize the degree set D. Finally, if we transfer $w_1, \ldots, w_{k-1}$ of the vertices to a pendant vertex $p' \neq w_k$ and then delete w_k, we notice that $\text{sdeg}(p') = k$ and that $\text{sdeg}(a) = 0$. Thus, T still realizes D but its order has decreased, contradicting the assumption that T was optimal at the beginning of the proof. We conclude that $\text{sdeg}(a) = 0$. The same argument can be applied to b. □

From now on, whenever a transfer happens in a signed tree, we also modify s so it maintains the sign in the new edge unless otherwise stated.

Lemma 4.5. *If (T, s) is an optimal signed tree that realizes $D = \{1, 0, x_1, \ldots, x_n\}$, then there is only one edge e in T such that $s(e) = -$.*

Proof. For a contradiction, assume that there exists an optimal tree T with at least 2 negative edges: u_1v_1 and u_2v_2, where the vertices are labeled such that the unique path from u_1 to u_2 includes both negative edges. By Lemma 4.4, u_1 has signed degree 0, so there must exist a vertex w adjacent to u_1 such that u_1w is positive. Transfer every vertex in $N(u_1) - \{v_1, w\}$ to u_2. This change will not affect the set that (T, s) realizes because the signed degrees of u_1 and u_2 will not change. Transfer the vertices in $N(w) - \{u_1\}$ to a pendant vertex p making sure p satisfies the conditions for transfer. Again, notice that T still realizes D. However, if $T' = T - \{u_1, w\}$, then $(T', s|_{E(T')})$ realizes D since $\text{sdeg}(v_2) = 0$ by Lemma 4.4 and since deleting u_1 only changes the signed degree of v_1 from 0 to 1. This contradicts the assumption that T was of optimal order. Thus, there cannot be two negative edges in T. □

The following corollary follows immediately from Lemma 4.4 and Lemma 4.5

Corollary 4.6. *If uv is the unique negative edge in an optimal signed tree (T, s) that realizes D, then* $\deg(u) = \deg(v) = 2$.

We will denote the negative edge, and surrounding vertices, by G_0, as illustrated in Figure 6. This allows for a very simple proof of our last result. As with diameter, we note that the result of a previous case facilitates the proof of another case.

Theorem 4.7. *If $D = \{1, 0, x_1, \ldots, x_n\}$, then $\sigma(D) = 4 - n + \sum x_i$.*

Figure 6: The graph G_0.

Proof. Let T be an optimal tree that realizes D. As noted, the graph G_0 is an induced subgraph of T, and s assigns the corresponding signs as indicated in Figure 6. There are three cases:

Case 1: $\deg(u_1) = 1$. Consider the graph $T' = T - \{u_1, u\}$. T' realizes $D - \{0\}$. Further, T' is an optimal tree of $D - \{0\}$ because if it was not, then T would not be an optimal tree for D. Thus, $\sigma(T') = \sigma(D - \{0\}) = 2 - n + \sum x_i$. Since $\sigma(T) = \sigma(T') + 2$, the result holds.

Case 2: $\deg(u_2) = 1$. We can apply the same argument as in Case 1.

Case 3: $\deg(u_1) \neq 1$ and $\deg(u_2) \neq 1$. Let T_1 and T_2 be the two components of $T - uv$, and let U and V be the set of signed degrees that $(T_1, s|_{E(T_1)})$ and $(T_2, s|_{E(T_2)})$ realize respectively. Notice that $U \cup V = D - \{0\}$, and that $U - \{1\}$ and $V - \{1\}$ are mutually exclusive since T is optimal. Similarly, $(T_1, s|_{E(T_1)})$ and $(T_2, s|_{E(T_2)})$ must also be optimal for U and V. Thus, by Theorem 4.2,

$$\begin{aligned}\sigma(T) &= \sigma(T_1) + \sigma(T_2)\\ &= 2 - (|U| - 1) + \sum_{x_i \in U} x_i + 2 - (|V| - 1) + \sum_{x_j \in V} x_j\\ &= 6 - (|U| + |V|) + \sum x_i\\ &= 6 - (n + 2) + \sum x_i\\ &= 4 - n + \sum x_i.\end{aligned}$$

□

5 Conclusion

We have studied two ways to optimize a signed tree, by diameter and by order. In this paper, we found the optimal signed trees by diameter for any given valid degree set. In addition, we found the optimal signed trees in regards to order for when the degree set contains positives and when the degree set contains 0 and positives.

Future research directions include optimizing order for degree sets containing negative values. Observe that the trees constructed in this paper to prove results suggest that optimizing diameter requires a large order, and similarly optimizing order requires a large diameter. We conjecture that if $|D| > 2$ and (T, s) realizes D, then it is not possible to have both $\text{diam}(T) = \text{diam}(D)$ and $\sigma(T) = \sigma(D)$.

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