Vertex-edge marking score of certain triangular lattices

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Abstract

The vertex-edge marking game is played between two players on a graph, G, with one player marking vertices and the other marking edges. The players want to minimize/maximize, respectively, the number of marked edges incident to an unmarked vertex. The vertex-edge coloring number for G is the maximum score achievable with perfect play. Brešar et al. [Ann. Comb. 25 (2021), 179–194] give an upper bound of 5 for the vertex-edge coloring number for finite planar graphs. It is not known whether the bound is tight. In this paper, in response to questions in Brešar et al., we show that the vertex-edge coloring number for the infinite regular triangularization of the plane is 4. We also give two general techniques that allow us to calculate the vertex-edge coloring number in many related triangularizations of the plane.

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1   Introduction

Combinatorial questions regarding colorings of maps and graphs go back to the 19th century, where Francis and Frederick Guthrie, under the advisement of De Morgan [6], posed the four-color conjecture. It was not until over a century later, in 1976, that a computer-assisted proof was presented by Appel and Haken [1]. Since that time, many variations of coloring problems have arisen. In particular, in 1981, Brams and Gardner [5, Chapter 16, p. 253] posed a coloring game on maps as a dynamical version of the map-coloring problem. As we neared the 21st century, Bodlaender [3], along with a number of other authors, began investigating similar coloring games. As introduced by Bartnicki et al. [2], the *vertex-edge marking game* is one of the many different coloring games that can be played on graphs.

The vertex-edge marking game involves two competing players: Alice marks vertices and Bob marks edges. Bob seeks to surround any unmarked vertex with as many marked edges as possible. Alice is competing against Bob and tries to limit the number of marked edges incident to any unmarked vertex. Starting with Alice, players alternate turns. On any turn of the game, the *vertex score* at a vertex $v \in V$ is the number of marked edges incident to v if v is unmarked and 0 otherwise. The *final score* of the game is the maximum over all turns and vertices of the vertex score. For a given graph $G = (V, E)$, 1 plus the final score of a game in which Alice and Bob each play optimally is called the *vertex-edge coloring number* of G, denoted $\mathrm{col}_{ve}(G)$.

Brešar et al. [4] investigated many properties of the vertex-edge coloring number, including determining the upper bound $\mathrm{col}_{ve}(G) \leq 5$ for all finite planar graphs. Multiple finite and infinite planar graphs have been found with $\mathrm{col}_{ve}(G) = 4$, but it remains unknown whether any exist with $\mathrm{col}_{ve}(G) = 5$.

Brešar et al. [4] expressed hope that possibly the infinte regular triangular lattice $\mathcal{T}$ might give an example of a graph G with $\mathrm{col}_{ve}(G) = 5$. However, in Corollary 3.3, we show that $\mathrm{col}_{ve}(\mathcal{T}) = 4$. This follows from a more general result to bound $\mathrm{col}_{ve}(G) \leq 4$ in Theorem 3.1 that depends on 2-colorability and angle markings. This result is applied to other triangular tilings of the plane in Corollaries 4.1 and 4.2. A further technique for bootstrapping the calculation of $\mathrm{col}_{ve}(G)$ from a subgraph is given in Theorem 4.3 and applied to graphs in Corollaries 4.4 and 4.5. Section 5 ends with a technique to facilitate establishing higher lower bounds for $\mathrm{col}_{ve}(G)$ and some conjectures.

2   Initial Definitions and Results

The *vertex-edge marking game* is played on a graph $G = (V, E)$ by two players who alternate turns each round of the game. At the beginning of the game, nothing on the graph is marked. The first player, known as Alice, marks vertices on her turn. The second player, known as Bob, marks edges on his turn. Alice's goal is to minimize the maximum number of marked edges adjacent to an unmarked vertex. Bob's goal is to maximize the maximum number of marked edges adjacent to an unmarked vertex.

More precisely, a game, $\mathcal{G}$, of the vertex-edge marking game played on the graph $G = (V, E)$ consists of a series of rounds, starting with round $r = 1$, in which Alice and Bob each take a turn with Alice always going first. Alice always marks one of the remaining unmarked vertices and Bob always marks one of the remaining unmarked edges. At the end of round r, $r \geq 1$, write $\mathrm{MV}(r)$ and $\mathrm{UMV}(r)$ for the set of marked and unmarked vertices of G, respectively, and write $\mathrm{ME}(r)$ for the set of marked edges of G. For finite graphs, the game is played until either player runs out of moves, i.e., until either all vertices or all edges are marked. For an infinite graph, the game continues forever.

For $v \in V$, the *vertex score* of v after round r is

$$\mathrm{score}(v, r) = \begin{cases} 0 & \text{if } v \in \mathrm{MV}(r), \\ |\{e \in \mathrm{ME}(r) : e \text{ is incident to } v\}| & \text{if } v \in \mathrm{UMV}(r). \end{cases}$$

The *r-round score* after round r is

$$\mathrm{score}(r) = \sup\{\mathrm{score}(v, r) : v \in V\}.$$

The *final score* of the game $\mathcal{G}$ is

$$\mathrm{score}(\mathcal{G}) = \sup\{\mathrm{score}(r) : \text{all rounds } r\}.$$

We say that Bob has a *winning strategy* for the score s if, regardless of Alice's moves each turn in a game $\mathcal{G}$, Bob can choose his moves to force $\mathrm{score}(\mathcal{G}) \geq s$. Finally, the *vertex-edge coloring number* of G is

$$\mathrm{col}_{\mathrm{ve}}(G) = \sup\{s : \text{Bob has a winning strategy for the score } s\} + 1.$$

Note that, as long as G has an edge, $\mathrm{col}_{\mathrm{ve}}(G) \geq 2$. Clearly, $\mathrm{col}_{\mathrm{ve}}(G) \leq \Delta(G) + 1$ where $\Delta(G)$ is the maximum degree of a vertex.

If H is a subgraph of G, then [4, Lemma 1]

$$\mathrm{col}_{\mathrm{ve}}(H) \leq \mathrm{col}_{\mathrm{ve}}(G). \tag{2.1}$$

A graph, G, has a *d-bounded orientation* if the edges can be oriented so that each vertex has a maximal out-degree of d. In such a case, it is known [4, Lemma 3] that

$$\mathrm{col}_{\mathrm{ve}}(G) \leq d + 2. \tag{2.2}$$

From this it follows, [4, Proposition 6], that for every finite planar graph, G,

$$\mathrm{col}_{\mathrm{ve}}(G) \leq 5.$$

In a vertex-edge marking game, $\mathcal{G}$, on a graph G, a *free path* is a path P of G with vertex sequence $v_0, \ldots, v_k$ with $k \geq 2$ so that

- the first and last edges of P, $v_0 v_1$ and $v_{k-1} v_k$, are marked,
- the interior vertices, $v_1, \ldots, v_{k-1}$, are not marked, and
- each interior vertex is incident to at least one edge not in P.

If there is a round in which it is Bob's turn and there is a free path in the game, then, [4, Lemma 7], Bob has a strategy to force

$$\mathrm{score}(\mathcal{G}) \geq 3. \tag{2.3}$$

3   The Triangular Lattice

Write $\mathcal{T}$ for the infinite regular triangular lattice in the plane. In [4, Question 3], the question was raised whether $\mathrm{col}_{\mathrm{ve}}(\mathcal{T})$ is 4 or 5. In this section, we show that the answer is $\mathrm{col}_{\mathrm{ve}}(\mathcal{T}) = 4$.

Theorem 3.1. *Let $\mathcal{L}$ be a plane graph so that*

- *the bounded faces of $\mathcal{L}$ are 2-colorable, gray and white,*
- *all gray faces of $\mathcal{L}$ are triangles,*
- *every edge of $\mathcal{L}$ belongs to precisely one gray triangle,*
- *exactly one angle from each gray triangle is marked, and*
- *each vertex of $\mathcal{L}$ is incident to at most two unmarked angles in gray triangles.*

Then

$$\mathrm{col}_{\mathrm{ve}}(\mathcal{L}) \leq 4.$$

Remark 3.2. See Figures 1, 2, and 5 for examples of infinite graphs that satisfy the conditions imposed on $\mathcal{L}$ here. Note that the white faces are not required to be triangles.

Proof. Note that each edge of $\mathcal{L}$ belongs to exactly one gray triangle that we will call the *corresponding triangle*. Therefore, as Bob marks edges, each marked edge is either the first, second, or third marked edge of its corresponding triangle.

To prove this theorem, we construct a strategy for Alice that only allows Bob to obtain a score of at most 3 on any vertex. Alice begins by marking any vertex on her first turn. Alice's subsequent plays are determined by the edge marked by Bob in the previous round and whether it was the first, second, or third marked edge of its corresponding triangle, T. The rules are as follows.

R1: If Bob marked the first edge of T, Alice marks the vertex incident to the marked angle in T, if the vertex is unmarked.

    If that vertex is already marked, Alice marks any other unmarked vertex in T, if one exists.

    Otherwise, Alice marks any unmarked vertex in $\mathcal{L}$.

R2: If Bob marked the second edge of T, Alice marks the vertex incident to both marked edges of T, if it is unmarked.

    If that vertex is already marked, Alice marks any other unmarked vertex in T, if one exists.

    Otherwise, Alice marks any unmarked vertex in $\mathcal{L}$.

R3: If Bob marked the third edge of T, Alice marks the remaining unmarked vertex in T, if one exists.

    Otherwise, Alice marks any unmarked vertex in $\mathcal{L}$.

We now show that this strategy prevents Bob from ever getting a score of 4 or more on a vertex. We do this by showing that every vertex incident to (exactly) three marked edges at the end of Bob's turn is already marked or will be marked by Alice at the start of the next round.

Suppose v is an unmarked vertex incident to three marked edges at the end of a round (after Bob's turn). In the order that Bob played them, label these three edges e_1, e_2, e_3.

Consider first the case where there is an e_i whose corresponding gray triangle, S, has a marked angle incident to v. As v is unmarked, rule R1 implies that $i = 3$, that Bob just played e_3, and that Alice will mark v on her next turn.

We may now assume that all marked edges belong to corresponding gray triangles whose marked vertices are not incident to v. As there are at most two gray triangles with unmarked angles incident to v, at least two of the marked edges belong to the same corresponding triangle, S. If all three edges of S are marked after Bob's turn, then rule R3 implies that e_3 is an edge of S, that Bob just played e_3, and that Alice will mark v on her next turn. However, if the third edge of S is unmarked, then rule R2 implies that e_3 is an edge of S, that Bob just played e_3, and that Alice will mark v on her next turn. $\square$

Corollary 3.3. *Let $\mathcal{T}$ be the infinite regular triangular lattice in the plane. Then*

$$\mathrm{col}_{\mathrm{ve}}(\mathcal{T}) = 4.$$

Proof. Since it is known that $\mathrm{col}_{\mathrm{ve}}(\mathcal{H}) = 4$ for the infinite regular hexagonal lattice $\mathcal{H}$, [4, Theorem 10], the lower bound, $\mathrm{col}_{\mathrm{ve}}(\mathcal{T}) \geq 4$, follows from Inequality (2.1).

We use Theorem 3.1 to obtain the upper bound. Begin by 2-coloring the faces of $\mathcal{T}$ as in Figure 1 so that the gray triangles point to the right and the white triangles point to the left. Mark the unique rightmost angle of each gray triangle, the one pointing to the right. Theorem 3.1 shows that $\mathrm{col}_{\mathrm{ve}}(\mathcal{T}) \leq 4$. $\square$
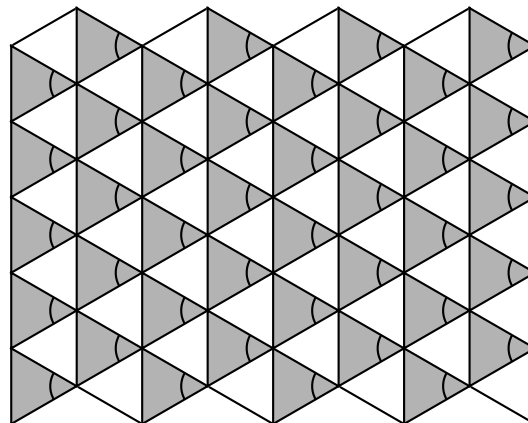

Figure 1: $\mathcal{T}$ with Marked Angles

4   Other Triangularizations

Theorem 3.1 applies to many graphs. For example, write $\mathcal{R}$ for the infinite triangular lattice obtained by adding a vertex to the center of each face of the infinite regular square lattice with added edges between each new vertex and each vertex in the corresponding face. See Figure 2.
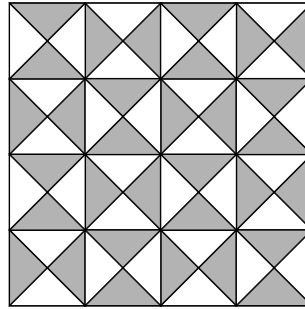

Figure 2: Triangularization $\mathcal{R}$

Corollary 4.1. $\mathrm{col}_{\mathrm{ve}}(\mathcal{R}) = 4$.

Proof. Since the infinite square lattice $\mathcal{S}$ has $\mathrm{col}_{\mathrm{ve}}(\mathcal{S}) = 4$, [4, Proposition 11], the lower bound, $\mathrm{col}_{\mathrm{ve}}(\mathcal{R}) \geq 4$, follows from Inequality (2.1).

Figure 3: Vertical Triangles

Figure 4: Horizontal Triangles

To get the upper bound, begin by 2-coloring the faces of $\mathcal{R}$ as in Figure 2. Notice the gray triangles are either vertical, as in Figure 3, or horizontal, as in Figure 4. Mark the vertical and horizontal gray triangles as pictured in Figures 3 and 4, respectively, with vertical triangles marked to the right and horizontal triangles marked upwards. Theorem 3.1 shows that $\mathrm{col}_{\mathrm{ve}}(\mathcal{R}) \leq 4$.   □

As another example, write $\mathcal{C}$ for the infinite triangular lattice $\mathcal{C}$ obtained from the regular square-octagon lattice by adding a vertex in the center of each face and edges from each new vertex to the vertices of the corresponding face. See Figure 5.
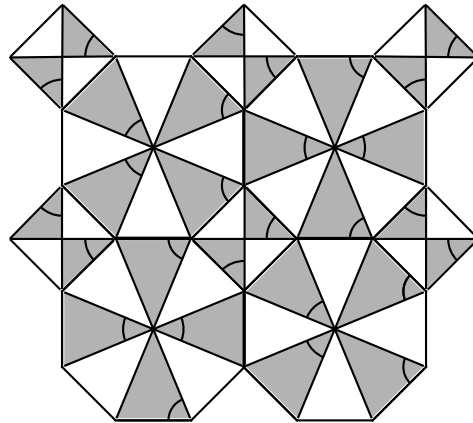
Figure 5: Triangularization $\mathcal{C}$

Corollary 4.2. $\mathrm{col}_{\mathrm{ve}}(\mathcal{C}) = 4$.

Proof. As in Corollary 4.1, the infinite square lattice $\mathcal{S}$ is a subgraph of $\mathcal{C}$, and $\mathrm{col}_{\mathrm{ve}}(\mathcal{C}) \geq 4$ holds. For an upper bound, color the faces of $\mathcal{C}$ and mark angles as in Figure 5. Then Theorem 3.1 shows that $\mathrm{col}_{\mathrm{ve}}(\mathcal{C}) \leq 4$. $\qquad\square$

Certain geometric structures permit the extension of a bound for $\mathrm{col}_{\mathrm{ve}}(G)$ to a larger graph G' if we know enough about Alice's strategy. As a starting illustration, let $\mathcal{T}'$ be $\mathcal{T}$ with a vertex added to the center of each face and edges connecting each new vertex to the vertices of the corresponding face. See Figure 6. Note that the resulting graph is not 2-colorable.
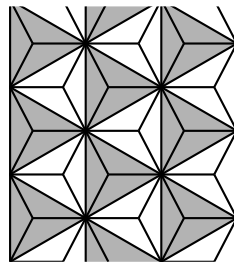

Figure 6: Triangularization $\mathcal{T}'$

Theorem 4.3. *Let $G' = (V', E')$ be a graph with a vertex-induced subgraph $G = (V, E)$ so that*

- *for $v \in V' - V$, $\deg(v) < n$ in G',*
- $\mathrm{col}_{\mathrm{ve}}(G) = n$, *and*
- *Alice has a strategy on G able to leave no unmarked vertices with $n-1$ marked incident edges at the end of her turn.*

Then

$$\mathrm{col}_{\mathrm{ve}}(G') = n.$$

Proof. We begin with a note on the third requirement. To prevent Bob from getting a score of n on G, Alice can ignore vertices in G that only have degree $n-1$. In other words, she only needs a strategy able to mark any vertex with $n-1$ marked incident edges *and at least one other unmarked incident edge* at the end of her turn. Therefore, the third requirement demands more of Alice since she can no longer ignore any vertices of degree $n-1$.

Turning to the proof, as usual, the lower bound follows from Inequality (2.1). For the upper bound, note that vertices in $V' - V$ have degree at most $n-1$. Therefore those vertices cannot help Bob get a score of n or higher. Now in G, Alice has a strategy that allows her to mark any vertex with $n-1$ incident marked edges at the end of her turn.

Alice's strategy for the game on G' is as follows. If Bob marks an edge $e \in E' - E$ with both incident vertices in $V' - V$, Alice is free to mark any vertex of G'. If e has one incident vertex in V and it is unmarked, Alice marks that vertex. Otherwise she is free to mark any other vertex. If Bob marks an edge in E, then Alice uses her strategy on G to play in G, leaving no unmarked vertices with $n-1$ marked incident edges in G at the end of her turn.

To see this strategy forces a final score $< n$, first note that vertices in $V' - V$ cannot help Bob get a score of n. Now if Bob were able to get n edges marked in G' incident to an unmarked vertex $v \in V$, the edges cannot all be in E by Alice's original strategy. Therefore at least one of the edges must be in $E' - E$. But by the strategy in the preceding paragraph, there must be exactly one such edge and it must have been marked last. However, that means that in the previous round, before this edge was marked, Bob had $n-1$ marked edges in E incident to v. But in that case, Alice's original strategy would have marked v already. $\square$

Corollary 4.4. $\mathrm{col}_{\mathrm{ve}}(\mathcal{T}') = 4$.

Proof. This follows by examining the proof in Corollary 3.3 to see that the conditions of Theorem 4.3 are met. $\square$

The same technique gives the following result.

Corollary 4.5. *For a triangular lattice, D, obtained from any of the lattices from Corollaries 3.3, 4.1, and 4.2 by adding a vertex to the center of every triangular face and connecting it to each corner of the triangle, $\mathrm{col}_{\mathrm{ve}}(D) = 4$.*

5   Futher Results and Conjectures

Although this paper has so far focused on bounding $\mathrm{col}_{\mathrm{ve}}(G)$ from above, there is no absolute upper bound for all graphs. For example, it is known that $\mathrm{col}_{\mathrm{ve}}(K_n)$ is unbounded, where K_n is the complete graph on n vertices [4, Theorem 17]. Bounding the vertex-edge coloring number from below often involves free paths and Inequality (2.3). However, this is only useful for obtaining a lower bound of 4 for $\mathrm{col}_{\mathrm{ve}}(G)$. Higher lower bounds can be obtained by generalizing the notion of free paths.

Definition 5.1. In a vertex-edge marking game, $\mathcal{G}$, on a graph G, an *n-free path* is a path P of G with vertex sequence $v_0, \ldots, v_k$ with $k \geq 2$ so that

- the first and last edges of P, $v_0 v_1$ and $v_{k-1} v_k$, are marked,

- the interior vertices, $v_1, \ldots, v_{k-1}$, are not marked, and

- each interior vertex is incident to at least $n + 1$ edges not in P and at least n of those edges are marked.

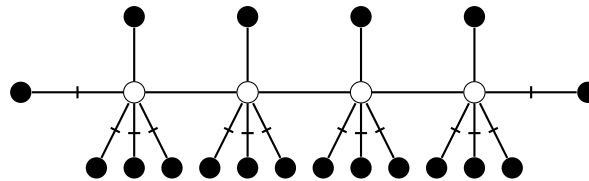Note that a 0-free path coincides with a free path. See Figure 7 for an example of a 3-free path of length 5.

Figure 7: A 3-Free Path of Length 5

Theorem 5.2. *Let G be a graph on which a vertex-edge marking game, $\mathcal{G}$, is being played. If there is a round in which there is an n-free path, P, and it is Bob's turn, then Bob has a strategy to force*

$$\mathrm{score}(\mathcal{G}) \geq n + 3.$$

Proof. The proof follows immediately by induction on the length k of P. The length $k = 2$ case is trivial. For the inductive step, if the edge $v_1 v_2$ is already marked, we have a shorter n-path and Bob wins. If $v_1 v_2$ is unmarked, then Bob marks $v_1 v_2$ in his first move. If Alice does not mark v_1, Bob marks the unmarked edge incident to v_1 and wins. Otherwise, we have a shorter n-path and Bob wins. $\square$

Actually, the existence of an n-free path is both sufficient and necessary for $\mathrm{col}_{\mathrm{ve}}(G) \geq n + 4$.

Corollary 5.3. *Let G be a graph. Then*

$$\mathrm{col}_{\mathrm{ve}}(G) \geq n + 4$$

if and only if Bob has a strategy able to provide him with having an n-free path on one of his turns.

Proof. For the only if part, note that Alice must attempt the strategy to leave no unmarked vertices of degree $> n + 2$ with $n + 2$ marked incident edges at the end of her turn if she wants to avoid the final score $n + 3$. This attempt can only fail if Bob has a strategy that allows him to mark the edge $v_1 v_2$ between two unmarked vertices v_1, v_2 of degree $> n + 2$ which have $n + 1$ marked incident edges each. However, this means that Bob has a strategy to force an n-free path of length 3 on one of his turns.

The converse direction is immediate from Theorem 5.2. $\square$

The focus of this paper was determining the vertex-edge coloring number for specific triangular graphs. We have a number of conjectures for related results. First, the fact that $\mathcal{T}$ was 2-colorable played a major role in the proof of Corollary 3.3. We conjecture that, in fact, this was the key necessary hypothesis.

Conjecture 5.4. *For any face 2-colorable planar graph G,* $\mathrm{col}_{\mathrm{ve}}(G) \leq 4$.

It is natural to try to generalize many of these results to Apollonian networks. Unfortunately, we do not have a sufficiently robust strategy for Alice to permit an inductive application of Theorem 4.3. However, we still believe that the upper bound is 4.

Conjecture 5.5. *For any Apollonian network G,* $\mathrm{col}_{\mathrm{ve}}(G) \leq 4$.

It feels as if adapted techniques from [7] and Inequality (2.2) should give results on Apollonian networks. However, it turns out that Inequality (2.2) is not, by itself, able to prove Conjecture 5.5. For example, note that if (V, E) is an Apollonian network with $|V| = n$ vertices, then the number of edges is $|E| = 3n - 6$. So for large n, $\frac{|E|}{|V|}$ tends to 3 with $\frac{|E|}{|V|} > 2$ for $n > 6$. Therefore, orienting the edges of the graph such that each vertex has at most two outgoing edges is not possible as the average out-degree equals $\frac{|E|}{|V|}$. With that said, switching to the dual graph on an Apollonian network may be a more useful technique.

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