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Classification of 2-symbol orthogonal arrays of even-strength t and $t + 2$ columns up to OD-equivalence

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Abstract

Bulutoglu and Ryan [*Australas. J. Combin.* 70 (2018), 362–385] developed the concept of OD-equivalence of 2-symbol orthogonal arrays (OAs) that captures the symmetries present in the even-strength cases that cannot be captured by array isomorphism. In this paper, we improve upon the classification results up to isomorphism of Stufken and Tang [*Ann. Stat.* 35 (2007), 793–814] by classifying all non-OD-equivalent 2-symbol OAs of even-strength t with $t + 2$ columns and index λ. The classification results up to OD-equivalence that we obtain are significantly simpler than the classification results up to isomorphism of Stufken and Tang in the aforementioned paper.

1 Introduction

Throughout the paper let $[n] = \{1, \dots, n\}$. We first define the concept of an orthogonal array (OA).

Definition 1.1. Let $\lambda \geq 1, s \geq 2$, $k \geq 1, t \geq 1$ be integers, and $t \in [k]$. A $\lambda s^t \times k$ array $\mathbf{D}$ whose entries are symbols from $\{l_1, \dots, l_s\}$ is an orthogonal array of strength t and index λ, denoted by $\mathrm{OA}(\lambda s^t, k, s, t)$, if each of the s^t symbol combinations from $\{l_1, \dots, l_s\}^t$ appears λ times in every $\lambda s^t \times t$ subarray of $\mathbf{D}$.

Each of the $N!k!(s!)^k$ operations that involve permuting rows, columns and the symbols within each column of an s-symbol $N \times k$ array is called an *isomorphism operation.* Two arrays $\mathbf{D}_1$ and $\mathbf{D}_2$ are *isomorphic* if $\mathbf{D}_2$ can be obtained from $\mathbf{D}_1$ by applying a sequence of isomorphism operations. Each isomorphism operation maps an $\mathrm{OA}(\lambda s^t, k, s, t)$ to an $\mathrm{OA}(\lambda s^t, k, s, t)$.

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Classification of OAs up to isomorphism in general is a challenging problem. Recently, there has been a renewed interest in classifying OAs [2, 4, 5]. However, these works make heavy use of computers. On the other hand, Yamamato et al. [8] were the first to analytically classify all $\mathrm{OA}(\lambda 2^t, k, 2, t)$ for $k = t+1, t+2$ up to permutations of columns. Stufken and Tang [7] strengthened the results in [8] by classifying all non-isomorphic $\mathrm{OA}(\lambda 2^t, t+2, 2, t)$ analytically. Their method of classification used J-characteristics for 2-symbol arrays.

For an $N \times k$ array $\mathbf{D} = [\mathbf{d}_1 \cdots \mathbf{d}_k]$ with symbols from $\{-1, 1\}$, Bulutoglu and Ryan [2] defined the column operation R_i on $\mathbf{D}$ by

$$R_i\mathbf{D} = \begin{bmatrix}\mathbf{d}_1 \odot \mathbf{d}_i & \cdots & \mathbf{d}_{i-1} \odot \mathbf{d}_i & \mathbf{d}_i & \mathbf{d}_{i+1} \odot \mathbf{d}_i & \cdots & \mathbf{d}_k \odot \mathbf{d}_i\end{bmatrix}, \tag{1.1}$$

and proved that each R_i maps an $\mathrm{OA}(\lambda 2^t, k, 2, t)$ to an $\mathrm{OA}(\lambda 2^t, k, 2, t)$ if t is even. Each transformation that involves a column operation R_i and/or an isomorphism operation is called an *OD-equivalence operation* [3]. Hence, for even t, each OD-equivalence operation maps an $\mathrm{OA}(\lambda 2^t, k, 2, t)$ to an $\mathrm{OA}(\lambda 2^t, k, 2, t)$.

Two arrays $\mathbf{D}_1$ and $\mathbf{D}_2$ with symbols from $\{-1, 1\}$ are *OD-equivalent* if $\mathbf{D}_2$ can be obtained from $\mathbf{D}_1$ by applying a sequence of OD-equivalence operations [2]. Clearly, if $\mathbf{D}_1$ and $\mathbf{D}_2$ are isomorphic arrays, then $\mathbf{D}_1$ and $\mathbf{D}_2$ are OD-equivalent. However, $\mathbf{D}_1$ and $\mathbf{D}_2$ may be OD-equivalent without being isomorphic [2].

A set of non-OD-equivalent $\mathrm{OA}(N, k, 2, t)$ can be used to generate a set of all non-isomorphic $\mathrm{OA}(N, k, 2, t)$ [2]. In fact, Bulutoglu and Ryan [2] classified all non-isomorphic $\mathrm{OA}(160, k, 2, 4)$ and $\mathrm{OA}(176, k, 2, 4)$ for $k = 5, 6, \ldots, 10$ by first classifying each up to OD-equivalence. Also, it would not have been possible to obtain the classification results up to isomorphism in Bulutoglu and Ryan [2] without first classifying up to OD-equivalence. Furthermore, by applying OD-equivalence with the methods in Geyer et al. [3] we have found 83 non-OD-equivalent $\mathrm{OA}(192, 9, 2, 4)$ after 6 months of CPU time on a 2.1 GHz processor. However, this is not a complete classification of all non-OD-equivalent $\mathrm{OA}(192, 9, 2, 4)$. The $\mathrm{OA}(192, 9, 2, 4)$ is currently the smallest $\mathrm{OA}(N, 9, 2, 4)$ that has not been completely classified yet. Methods in Geyer et al. [3] that make heavy use of OD-equivalence bring a partial classification of non-OD-equivalent $\mathrm{OA}(192, 9, 2, 4)$ within computational reach. Hence, classifying all non-OD-equivalent $\mathrm{OA}(N, k, 2, t)$ is useful in solving the classification problem of $\mathrm{OA}(N, k, 2, t)$ up to isomorphism. In this paper, we improve the results of Stufken and Tang [7] by analytically classifying all non-OD-equivalent $\mathrm{OA}(\lambda 2^t, t+2, 2, t)$ when the strength t is even.

The paper is structured as follows. Section 2 defines J-characteristics of 2-symbol arrays, and describes how OD-equivalence operations act on J-characteristics of such arrays. Section 3 presents the main result. Section 4 discusses future research. In the Appendix, we provide the theorems and lemmas from [7] that we use in Section 3 to establish the main result of the paper.

2 J-characteristics and OD-equivalence

Throughout this section $\mathbf{D}$ will denote an $N \times k$ array with symbols from $\{-1, 1\}$.

For $\ell \subseteq [k]$, let

$$\mathbf{r}_\ell = [r_{\ell 1}, \ldots, r_{\ell k}],$$

where

$$r_{\ell j} = \begin{cases} -1 & \text{if } j \in \ell, \\ 1 & \text{otherwise.} \end{cases}$$

Given an array $\mathbf{D}$, let x_ℓ be the number of times $\mathbf{r}_\ell$ appears as a row of $\mathbf{D}$. The *frequency vector* $\mathbf{x}$ of $\mathbf{D}$ is defined by

$$\mathbf{x} = [x_\emptyset, x_1, x_2, x_{12}, x_3, \ldots, x_{1\ldots k}]^\top \tag{2.1}$$

where $x_{i_1\ldots i_p}$ is used for $x_{\{i_1,\ldots,i_p\}}$.

We now define the J-characteristics.

Definition 2.1. Let $\mathbf{D} = [d_{ij}]$ be an array. For $\ell \subseteq [k]$, let

$$J_\ell(\mathbf{D}) = \sum_{i=1}^{N} \prod_{j\in\ell} d_{ij}.$$

(For $\ell = \emptyset$, $J_\ell(\mathbf{D}) := N$.)

The $J_\ell(\mathbf{D})$ are called the *J-characteristics* of $\mathbf{D}$. Let $J_{i_1\ldots i_r}(\mathbf{D})$ denote $J_{\{i_1,\ldots,i_r\}}(\mathbf{D})$; then the *$J$-vector* of $\mathbf{D}$ is defined by

$$\mathbf{J} = [J_\emptyset(\mathbf{D}), J_1(\mathbf{D}), J_2(\mathbf{D}), J_{12}(\mathbf{D}), J_3(\mathbf{D}), \ldots, J_{1\ldots k}(\mathbf{D})]^\top. \tag{2.2}$$

We now establish the connection between the frequency vector and J-vector of an array. A 2^k full factorial array, with Yates ordering, is expressed by the $2^k \times k$ matrix

$$\mathbf{F} = \left[\mathbf{r}_\emptyset^\top, \mathbf{r}_1^\top, \mathbf{r}_2^\top, \mathbf{r}_{12}^\top, \mathbf{r}_3^\top, \ldots, \mathbf{r}_{1\ldots k}^\top\right]^\top,$$

where $\mathbf{r}_{i_1\ldots i_p}$ is the shorthand notation for $\mathbf{r}_{\{i_1,\ldots,i_p\}}$. For $j \in [k]$, let $\mathbf{h}_j$ denote the jth column of $\mathbf{F}$. Then

$$\mathbf{F} = [\mathbf{h}_1, \ldots, \mathbf{h}_k].$$

The Hadamard product of $\mathbf{z}$ and $\mathbf{v}$ is

$$\mathbf{z} \odot \mathbf{v} = [z_1 v_1, \ldots, z_n v_n]^\top$$

for $\mathbf{z}, \mathbf{v} \in \{-1, 1\}^n$. For $\ell = \{i_1, \ldots, i_p\} \subseteq [k]$, let

$$\mathbf{h}_\ell = \mathbf{h}_{i_1} \odot \cdots \odot \mathbf{h}_{i_p}.$$

Let

$$\mathbf{H} = [\mathbf{h}_\emptyset, \mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_{12}, \mathbf{h}_3, \ldots, \mathbf{h}_{1\ldots k}], \tag{2.3}$$

where $\mathbf{h}_{i_1\ldots i_p}$ is used for for $\mathbf{h}_{\{i_1,\ldots,i_p\}}$. Then $\mathbf{H}$ is the $2^k \times 2^k$ *Sylvester Hadamard matrix* [6]. For $\ell \subseteq [k]$, we have

$$J_\ell(\mathbf{D}) = \sum_{i=1}^{N} \prod_{j\in\ell} d_{ij} = \sum_{u\subseteq[k]} \prod_{j\in\ell} r_{uj} x_u = \sum_{u\subseteq[k]} (\mathbf{h}_\ell)_u x_u = \mathbf{h}_\ell^\top \mathbf{x}.$$

This implies $\mathbf{J} = \mathbf{H}^\top \mathbf{x}$. Since $\mathbf{H}\mathbf{H}^\top = 2^k \mathbf{I}_{2^k}$, where $\mathbf{I}_{2^k}$ is the $2^k \times 2^k$ identity matrix, we have the following fundamental result.

Lemma 2.2. *Let $\mathbf{x}, \mathbf{J}$, and $\mathbf{H}$ be as in equations (2.1), (2.2), and (2.3). Then*

$$\mathbf{x} = 2^{-k}\mathbf{H}\mathbf{J}.$$

By Lemma 2.2, the J-vector of an array uniquely determines its frequency vector. The following lemma determines all $\mathrm{OA}(\lambda 2^t, k, 2, t)$ in terms of their J-characteristics.

Lemma 2.3 (Stufken and Tang [7])**.** *An array $\mathbf{D}$ is an $OA(\lambda 2^t, k, 2, t)$ if and only if $J_\ell(\mathbf{D}) = 0$ for all $\ell \subseteq [k]$ such that $|\ell| \in [t]$.*

The following result is from Stufken and Tang [7] and its generalization in Bulutoglu and Kaziska [1].

Lemma 2.4. *Let $\mathbf{D}$ be an $OA(\lambda 2^t, k, 2, t)$ with $k \geq t+2$. Then the following hold.*

(i) For any $\ell \subseteq [k]$, $J_\ell(\mathbf{D}) = u_\ell 2^t$ for some integer u_ℓ.

(ii) For any $\ell \subseteq [k]$ and index λ, we have $u_\ell \equiv \lambda\binom{|\ell|-1}{t} \pmod 2$.

For isomorphism operations we have the following lemma from Geyer et al. [3].

Lemma 2.5. *Let $\ell \subseteq [k]$ be such that $|\ell| > 0$. Let g be an isomorphism operation and $g\mathbf{D}$ be the array obtained after g is applied to $\mathbf{D}$. Then*

$$J_\ell(g\mathbf{D}) = \pm J_{\ell'}(\mathbf{D}),$$

where $|\ell'| = |\ell|$.

The operations R_i act on the J-characteristics as follows, as shown in Geyer et al. [3].

Lemma 2.6. *Let $\ell \subseteq [k]$ be such that $|\ell| > 0$. Let R_i be an OD-equivalence operation as defined in equation (1.1), $i \in [k]$. Then*

$$J_\ell(R_i\mathbf{D}) = \begin{cases} J_\ell(\mathbf{D}) & \text{if } |\ell| \text{ is even and } i \notin \ell, \\ J_{\ell\setminus\{i\}}(\mathbf{D}) & \text{if } |\ell| \text{ is even and } i \in \ell, \\ J_{\ell\cup\{i\}}(\mathbf{D}) & \text{if } |\ell| \text{ is odd and } i \notin \ell, \\ J_\ell(\mathbf{D}) & \text{if } |\ell| \text{ is odd and } i \in \ell. \end{cases}$$

Unlike isomorphism operations, the R_i operations allow J-characteristics indexed by ℓ to be mapped to J-characteristics indexed by ℓ' with $|\ell| \neq |\ell'|$. The R_i operations are key to improving the results of Stufken and Tang [7].

Lemmas 2.5 and 2.6 from Geyer et al. [3] characterize the action of OD-equivalence operations on the J-characteristics.

Lemma 2.7. *Let $\ell \subseteq [k]$ be such that $|\ell| > 0$. Let g be an OD-equivalence operation and $g\mathbf{D}$ be the array obtained after g is applied to $\mathbf{D}$. Then*

$$J_\ell(g\mathbf{D}) = \pm J_{\ell'}(\mathbf{D})$$

for some $\ell' \subseteq [k]$, where

$$|\ell'| = \begin{cases} |\ell| \text{ or } |\ell|+1 & \text{if } |\ell| \text{ is odd,} \\ |\ell| \text{ or } |\ell|-1 & \text{otherwise.} \end{cases}$$

By using Lemma 2.7, Bulutoglu and Ryan [2] showed the following.

Theorem 2.8. *Let $\mathbf{D}_1$ be an $OA(\lambda 2^t, k, 2, t)$ with $t \geq 1$. Then $\mathbf{D}_2$ is OD-equivalent to $\mathbf{D}_1$ if and only if there exists an OD-equivalence operation g such that $\mathbf{D}_2 = g\mathbf{D}_1$ up to permutation of rows. Moreover, if $\mathbf{D}_2$ is OD-equivalent to $\mathbf{D}_1$, then $\mathbf{D}_2$ is an $OA(\lambda 2^t, k, 2, 2\lfloor t/2 \rfloor)$.*

By Theorem 2.8, if $\mathbf{D}$ is an OA$(\lambda 2^t, k, 2, t)$ with even t, then any array OD-equivalent to $\mathbf{D}$ is an OA$(\lambda 2^t, k, 2, t)$.

3 Classification of even strength OA$(\lambda 2^t, t+2, 2, t)$ up to OD-equivalence

Let $\mathbf{D}$ be an OA$(\lambda 2^t, t+2, 2, t)$. Since $k = t+2$, by Lemma 2.3, we need to consider only $k+1$ coordinates of the J-vector of $\mathbf{D}$. Let $\ell_j = [k] \setminus \{k+1-j\}$ for $j \in [k]$ and $\ell_{k+1} = [k]$.

The following proposition was used to classify all non-isomorphic OA$(\lambda 2^t, t+2, 2, t)$ for even t.

Proposition 3.1 (Stufken and Tang [7])**.** *When $k = t+2$ is even, every OD-equivalence class of $OA(\lambda 2^t, t+2, 2, t)$ contains a unique array $\mathbf{D}$ whose J-vector satisfies either of the following conditions:*

$$J_{\ell_1}(\mathbf{D}) \leq \cdots \leq J_{\ell_k}(\mathbf{D}) \leq -|J_{\ell_{k+1}}(\mathbf{D})|, \tag{3.1}$$

$$J_{\ell_1}(\mathbf{D}) \leq \cdots \leq J_{\ell_{k-1}}(\mathbf{D}) \leq -|J_{\ell_k}(\mathbf{D})|, \quad J_{\ell_{k+1}}(\mathbf{D}) < -|J_{\ell_k}(\mathbf{D})|. \tag{3.2}$$

The following is our main lemma.

Lemma 3.2. *When $k = t+2$ is even, every OD-equivalence class of $OA(\lambda 2^t, t+2, 2, t)$ contains a unique array $\mathbf{D}$ whose J-vector satisfies*

$$J_{\ell_1}(\mathbf{D}) \leq \cdots \leq J_{\ell_k}(\mathbf{D}) \leq -|J_{\ell_{k+1}}(\mathbf{D})|. \tag{3.3}$$

Proof. Suppose that $\mathbf{D}$ is the array whose J-vector satisfies inequalities (3.2). We show there exists a unique OD-equivalent array to $\mathbf{D}$ whose J-vector satisfies inequalities (3.1). Let R_1 be as defined in equation (1.1) and let $\mathbf{D}' = R_1\mathbf{D}$. By Theorem 2.8, $\mathbf{D}'$ is an $\mathrm{OA}(\lambda 2^t, t+2, 2, t)$ that is OD-equivalent to $\mathbf{D}$. Furthermore, by Lemma 2.6

$$J_{\ell_{k+1}}(\mathbf{D}') = J_{\ell_k}(\mathbf{D}),\ J_{\ell_k}(\mathbf{D}') = J_{\ell_{k+1}}(\mathbf{D}),\text{ and } J_{\ell_j}(\mathbf{D}') = J_{\ell_j}(\mathbf{D})$$

for $j \in [k-1]$. Then

$$J_{\ell_1}(\mathbf{D}') \leq \cdots \leq J_{\ell_{k-1}}(\mathbf{D}') \leq -|J_{\ell_{k+1}}(\mathbf{D}')|, \quad J_{\ell_k}(\mathbf{D}') < -|J_{\ell_{k+1}}(\mathbf{D}')|.$$

Hence, after applying an appropriate permutation to the columns of $\mathbf{D}'$, we obtain an OD-equivalent array $\mathbf{D}''$ whose J-vector satisfies inequalities (3.1). Then, by Proposition 3.1, any J-vector of an $\mathrm{OA}(\lambda 2^t, t+2, 2, t)$ satisfying inequalities (3.3) is unique and therefore the corresponding $\mathrm{OA}(\lambda 2^t, t+2, 2, t)$ is unique. □

Lemma 3.2 allows the classification of non-OD-equivalent $\mathrm{OA}(\lambda 2^t, k, 2, t)$ by finding solutions in only one case, namely under inequalities (3.3), whereas the classification of non-isomorphic $\mathrm{OA}(\lambda 2^t, k, 2, t)$ requires finding all solutions in two mutually exclusive cases, namely under either inequalities (3.1) or inequalities (3.2). This reduction in the number of cases that need to be searched significantly simplifies the $\mathrm{OA}(\lambda 2^t, t+2, 2, t)$ classification problem.

Suppose that $\mathbf{D}$ is an $\mathrm{OA}(\lambda 2^t, k, 2, t)$ whose J-vector satisfies inequalities (3.3). By Lemma 2.4, $J_{\ell_j}(\mathbf{D}) = u_j 2^t$, $j \in [k+1]$. Then

$$u_1 \leq \cdots \leq u_k \leq -|u_{k+1}|. \tag{3.4}$$

Lemma 3.3. *Suppose that $k = t+2$, t is even, and λ is odd. Let*

$$\lambda + u_1 + \cdots + u_{k+1} = 4p, \tag{3.5}$$

with $p \in \mathbb{Z}_{\geq 0}$, $u_j \in 2\mathbb{Z}+1$ such that $|u_j| \leq \lambda - 2$ for $j \in [k+1]$. Then the following hold.

(i) Each solution $(u_1, \ldots, u_{k+1}, p)$ to equation (3.5) under inequalities (3.4) determines an $OA(\lambda 2^t, t+2, 2, t)$ with J-vector given by $J_{\ell_j} = 2^t u_j$ for $j \in [k+1]$.

(ii) A complete set of non-OD-equivalent $OA(\lambda 2^t, t+2, 2, t)$ is given by collecting the arrays obtained in (i) over all solutions to equation (3.5).

Proof. The proof follows from Lemma 3.2 and Theorem 1 in Stufken and Tang [7]; see the Appendix. □

Lemma 3.4. *Suppose that $k = t+2$, t is even, and $\lambda = 2\lambda^*$ is even. Let*

$$\lambda^* + u_1 + \cdots + u_{k+1} = 2p, \tag{3.6}$$

with $p \in \mathbb{Z}_{\geq 0}$, $u_j \in \mathbb{Z}$ such that $|u_j| \leq \lambda^*$ for $j \in [k+1]$. Then the following hold.

(i) *Each solution* $(u_1, \ldots, u_{k+1}, p)$ *to equation (3.6) under inequalities (3.4) determines an* $OA(\lambda 2^t, t+2, 2, t)$ *with J-vector given by* $J_{\ell_j} = 2^{t+1} u_j$ *for* $j \in [k+1]$.

(ii) *A complete set of non-OD-equivalent* $OA(\lambda 2^t, t+2, 2, t)$ *is given by collecting the arrays obtained in (i) over all solutions to equation (3.6).*

Proof. The proof follows from Lemma 3.2 and Theorem 2 in Stufken and Tang [7]; see the Appendix. □

Let $Z[a,b]$ and $O[a,b]$ denote the set of integers and odd integers x such that $a \le x \le b$, respectively.

Theorem 3.5. *For even* t, *odd* λ, *and* $k = t+2$, *if* $\lambda \le t-1$, *then equation (3.5) has no* $OA(\lambda 2^t, k, 2, t)$ *solution under inequalities (3.4); if* $\lambda \ge t+1$, *then equation (3.5) has at least one* $OA(\lambda 2^t, k, 2, t)$ *solution under inequalities (3.4), and the complete set* S_1 *of non-OD-equivalent* $OA(\lambda 2^t, k, 2, t)$ *solutions is given by*

$$\begin{aligned}
p &\in Z\left[0, \frac{\lambda - t - 1}{4}\right],\\
u_{k+1} &\in O\left[-\frac{\lambda - 4p}{k+1}, \frac{\lambda - 4p}{k-1}\right],\\
u_k &\in O\left[-\frac{\lambda - 4p + u_{k+1}}{k}, -|u_{k+1}|\right],\\
u_j &\in O\left[-\frac{\lambda - 4p + u_{j+1} + \cdots + u_{k+1}}{j}, u_{j+1}\right], \; j = k-1, k-2, \ldots, 2,\\
u_1 &= -(\lambda - 4p + u_2 + \cdots + u_{k+1}).
\end{aligned}$$

Proof. The proof follows from Lemma 3.3 and Lemma 7 in Stufken and Tang [7]; see the Appendix. □

Theorem 3.6. *For even* t, *even* $\lambda = 2\lambda^*$, *and* $k = t+2$, *the complete set* S_2 *of non-OD-equivalent* $OA(\lambda 2^t, k, 2, t)$ *as solutions to equation (3.6) under inequalities (3.4) is given by*

$$\begin{aligned}
p &\in Z\left[0, \frac{\lambda^*}{2}\right],\\
u_{k+1} &\in Z\left[-\frac{\lambda^* - 2p}{k+1}, \frac{\lambda^* - 2p}{k-1}\right],\\
u_k &\in Z\left[-\frac{\lambda^* - 2p + u_{k+1}}{k}, -|u_{k+1}|\right],\\
u_j &\in Z\left[-\frac{\lambda^* - 2p + u_{j+1} + \cdots + u_{k+1}}{j}, u_{j+1}\right], \; j = k-1, k-2, \ldots, 2,\\
u_1 &= -(\lambda^* - 2p + u_2 + \cdots + u_{k+1}).
\end{aligned}$$

Proof. The proof follows from Lemma 3.4 and Lemma 9 in Stufken and Tang [7]; see the Appendix. □

For even t and $s = 2$, OD-equivalence reduces the solution set to S_1 for odd λ, and S_2 for even λ non-OD-equivalent $\mathrm{OA}(\lambda 2^t, t+2, 2, t)$. The sizes of S_1 and S_2 are smaller than the corresponding sizes obtained for non-isomorphic $\mathrm{OA}(\lambda 2^t, t+2, 2, t)$.

Theorems 3.5 and 3.6 were validated by comparing to the classifications obtained by using the methods of Geyer et al. [3] for $\mathrm{OA}(4\lambda, 4, 2, 2)$ for $\lambda \in [51]$, $\mathrm{OA}(16\lambda, 6, 2, 4)$ for $\lambda \in [30]$, $\mathrm{OA}(64\lambda, 8, 2, 6)$ for $\lambda \in [30]$, $\mathrm{OA}(256\lambda, 10, 2, 8)$ for $\lambda = 1, 3, 5$, and $\mathrm{OA}(1024\lambda, 12, 2, 10)$ for $\lambda = 1, 3$.

4 Conclusion

We used OD-equivalence operations, a larger set of operations than isomorphism operations, to analytically classify all non-OD-equivalent $\mathrm{OA}(\lambda 2^t, t+2, 2, t)$ when t is even. Future research will involve classifying $\mathrm{OA}(\lambda 2^t, t+3, 2, t)$ up to OD-equivalence for even t. We anticipate that classifying $\mathrm{OA}(\lambda 2^t, t+3, 2, t)$ up to OD-equivalence for even t is more tangible than classifying up to isomorphism.

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Appendix

In this section we provide the theoretical results from Stufken and Tang [7] that are used in the paper. The numbering of the theorems and lemmas matches that in [7].

Let

$$\lambda + u_1 + \cdots + u_{k+1} = 4p, \tag{4.1}$$

where $p \in \mathbb{Z}_{\geq 0}$ and $u_j \in 2\mathbb{Z}+1$ are such that $|u_j| \leq \lambda - 2$ for $j = 1, \ldots, k+1$. Furthermore, let

$$u_1 \leq \cdots \leq u_k \leq -|u_{k+1}|, \tag{4.2}$$

$$u_1 \leq \cdots \leq u_{k-1} \leq -|u_k|, \quad u_{k+1} \leq -|u_k| - 2. \tag{4.3}$$

Theorem 1. *Suppose that t is even and λ is odd. We then have that: (i) each solution $(u_1, \ldots, u_{k+1}, p)$ to equation (4.1) under either (4.2) or (4.3) determines an $OA(\lambda 2^t, t+2, 2, t)$ with J-vector given by $J_{\ell_j} = 2^t u_j$ for $j = 1, \ldots, k+1$; (ii) the complete set of non-isomorphic $OA(\lambda 2^t, t+2, 2, t)$s is given by collecting the arrays obtained in (i) over all the solutions to equation (4.1).*

Let $\lambda = 2\lambda^*$, where $\lambda^* \in \mathbb{Z}_{\geq 0}$. Let

$$\lambda^* + u_1 + \cdots + u_{k+1} = 2p, \tag{4.4}$$

where $p \in \mathbb{Z}_{\geq 0}$ and $u_j \in \mathbb{Z}$ are such that $|u_j| \leq \lambda^*$ for $j = 1, \ldots, k+1$. Furthermore, let

$$u_1 \leq \cdots \leq u_k \leq -|u_{k+1}|, \tag{4.5}$$

$$u_1 \leq \cdots \leq u_{k-1} \leq -|u_k|, \quad u_{k+1} \leq -|u_k| - 1. \tag{4.6}$$

Theorem 2. *Suppose that t is even and $\lambda = 2\lambda^*$ is also even. We then have that: (i) each solution $(u_1, \ldots, u_{k+1}, p)$ to equation (4.4) under either (4.5) or (4.6) determines an $OA(\lambda 2^t, t+2, 2, t)$ with J-vector given by $J_{\ell_j} = 2^{t+1}u_j$ for $j = 1, \ldots, k+1$; (ii) the complete set of non-isomorphic $OA(\lambda 2^t, t+2, 2, t)$s for even t and even λ is given by collecting the arrays obtained in (i) over all the solutions to equation (4.4).*

Lemma 7. *For even t and odd λ, if $\lambda \leq t-1$, then equation (4.1) has no solution under inequalities (4.2). If $\lambda \geq t+1$, then equation (4.1) has at least one solution under inequalities (4.2) and the complete set S_1 of solutions is given by*

$$\begin{aligned}
p &\in Z\left[0, \frac{\lambda - t - 1}{4}\right], \\
u_{k+1} &\in O\left[-\frac{\lambda - 4p}{k+1}, \frac{\lambda - 4p}{k-1}\right], \\
u_k &\in O\left[-\frac{\lambda - 4p + u_k + 1}{k}, -|u_{k+1}|\right], \\
u_j &\in O\left[-\frac{\lambda - 4p + u_{j+1} + \cdots + u_{k+1}}{j}, u_{j+1}\right], \ j = k-1, k-2, \ldots, 2, \\
u_1 &= -(\lambda - 4p + u_2 + \cdots + u_{k+1}).
\end{aligned}$$

Lemma 9. *For even t, even $\lambda = 2\lambda^*$, the complete set S_2 of solutions to (4.4) under inequalities (4.5) is given by*

$$\begin{aligned}
p &\in Z\left[0, \frac{\lambda^*}{2}\right], \\
u_{k+1} &\in Z\left[-\frac{\lambda^* - 2p}{k+1}, \frac{\lambda^* - 2p}{k-1}\right], \\
u_k &\in Z\left[-\frac{\lambda^* - 2p + u_{k+1}}{k}, -|u_{k+1}|\right], \\
u_j &\in Z\left[-\frac{\lambda^* - 2p + u_{j+1} + \cdots + u_{k+1}}{j}, u_{j+1}\right], \ j = k-1, k-2, \ldots, 2, \\
u_1 &= -(\lambda^* - 2p + u_2 + \cdots + u_{k+1}).
\end{aligned}$$

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