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The minimal volume of a lattice polytope

Ichiro Sainose

Hiroshima Municipal Motomachi Senior High School
Hiroshima 730-0005, Japan
sainose23517257@nifty.com

Ginji Hamano

School of Science and Engineering, Tokyo Denki University
Saitama 350-0394, Japan
18hz002@ms.dendai.ac.jp

Tatsuo Emura   Takayuki Hibi

Department of Pure and Applied Mathematics
Graduate School of Information Science and Technolog
Osaka University, Osaka 565-0871, Japan
skmj50023@ares.eonet.ne.jp   hibi@math.sci.osaka-u.ac.jp

Abstract

Let $\mathcal{P} \subset \mathbb{R}^d$ be a lattice polytope of dimension d. Let b denote the number of lattice points belonging to the boundary of $\mathcal{P}$ and c those in the interior of $\mathcal{P}$. It follows from a lower bound theorem of Ehrhart polynomials that, when $c > 0$, the volume of $\mathcal{P}$ is bigger than or equal to $(dc + (d-1)b - d^2 + 2)/d!$. In the present paper, via triangulations, a short and elementary proof of the minimal volume formula is given.

1 Introduction

Let $\mathcal{P} \subset \mathbb{R}^d$ be a *lattice polytope* of dimension d. In other words, $\mathcal{P}$ is a convex polytope of dimension d each of whose vertices belongs to $\mathbb{Z}^d$. A *lattice point* of $\mathbb{R}^d$ is a point belonging to $\mathbb{Z}^d$. Let $b = b(\mathcal{P})$ denote the number of lattice points belonging to the boundary $\partial\mathcal{P}$ of $\mathcal{P}$ and $c = c(\mathcal{P})$ those in the interior of $\mathcal{P}$. It follows from the lower bound theorem of Ehrhart polynomials [2] that, when $c > 0$,

$$\mathrm{vol}(\mathcal{P}) \geq (d \cdot c(\mathcal{P}) + (d-1) \cdot b(\mathcal{P}) - d^2 + 2)/d!, \tag{1}$$

where $\mathrm{vol}(\mathcal{P})$ is the (Lebesgue) volume of $\mathcal{P}$. However, the argument in [2] is rather complicated with deep techniques on polytopes. In the present paper a short and

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elementary proof of the minimal volume formula (1) will be given. Pick's formula guarantees that, when $d = 2$, the inequality (1) is an equality [6].

A lattice polytope $\mathcal{P} \subset \mathbb{R}^d$ of dimension d is called *Castelnuovo* [4] if the equality holds in (1). A few remarks on Castelnuovo polytopes will be also stated.

2 Minimal volume formula

In general, let $\mathcal{P} \subset \mathbb{R}^d$ be a convex polytope of dimension d and $V \subset \mathcal{P}$ a finite set to which each of the vertices of $\mathcal{P}$ belongs. A *triangulation* of $\mathcal{P}$ on V is a collection Γ of d-simplices (simplices of dimension d) for which

- each vertex of each d-simplex $F \in \Gamma$ belongs to V;
- each $x \in V$ is a vertex of a d-simplex $F \in \Gamma$;
- if $F \in \Gamma$ and $G \in \Gamma$, then $F \cap G$ is a face of F and of G;
- $\mathcal{P} = \bigcup_{F \in \Gamma} F$.

The existence of a triangulation of $\mathcal{P}$ on V is guaranteed by [5, Lemma 1.1]. Thus in particular, if $\mathcal{P}$ is a lattice polytope, then a triangulation of $\mathcal{P}$ on $\mathcal{P} \cap \mathbb{Z}^d$ exists.

Lemma 2.1 *Let $\mathcal{P} \subset \mathbb{R}^d$ be a convex polytope of dimension d and $V \subset \mathcal{P}$ a finite set to which each of the vertices of $\mathcal{P}$ belongs. Let $b(\mathcal{P}) = |V \cap \partial\mathcal{P}|$, where $\partial\mathcal{P}$ is the boundary of $\mathcal{P}$, and $c(\mathcal{P}) = |V \cap (\mathcal{P} \setminus \partial\mathcal{P})|$, where $\mathcal{P} \setminus \partial\mathcal{P}$ is the interior of $\mathcal{P}$. Suppose that $c(\mathcal{P}) > 0$. Then there exists a triangulation $\Gamma_{\mathcal{P}}$ of $\mathcal{P}$ on V with*

$$|\Gamma_{\mathcal{P}}| \geq d \cdot c(\mathcal{P}) + (d-1) \cdot b(\mathcal{P}) - d^2 + 2.$$

Proof. We construct the required triangulation $\Gamma_{\mathcal{P}}$ by induction on d. Let $d \geq 3$. Let Γ be a triangulation of $\mathcal{P}$ on V. Let Δ denote the set of those $F \cap \partial\mathcal{P}$ with $F \in \Gamma$ for which $F \cap \partial\mathcal{P}$ is a $(d-1)$-simplex. Fix $G_0 \in \Delta$. Remove $G_0 \setminus \partial G_0$ from $\partial\mathcal{P}$, and one can assume that $\mathcal{P}' = \partial\mathcal{P} \setminus (G_0 \setminus \partial G_0)$ is a simplex in $\mathbb{R}^{d-1}$ of dimension $d-1$ via a one-point compactification. Furthermore, the number of points in V belonging to the boundary of $\mathcal{P}'$ is $b(\mathcal{P}') = d$ and that to the interior of $\mathcal{P}'$ is $c(\mathcal{P}') = b(\mathcal{P}) - d$. Since $b(\mathcal{P}) > d$, it follows that $c(\mathcal{P}') > 0$. The induction hypothesis yields a triangulation Δ' of $\mathcal{P}'$ on $\mathcal{P}' \cap V$ for which

$$|\Delta'| \geq (d-1) \cdot (b(\mathcal{P}) - d) + (d-2) \cdot d - (d-1)^2 + 2.$$

Let $\Gamma^{(0)} = \Delta' \cup \{G_0\}$. Then $\partial\mathcal{P} = \bigcup_{G \in \Gamma^{(0)}} G$.

Let $x_1, \ldots, x_c$ denote the points in V belonging to the interior of $\mathcal{P}$. Now, set

$$\Gamma^{(1)} = \{\mathrm{conv}(G \cup \{x_1\}) : G \in \Gamma^{(0)}\},$$

where $\mathrm{conv}(G \cup \{x_1\})$ is the convex hull of $G \cup \{x_1\}$ in $\mathbb{R}^d$, and $\Gamma^{(1)}$ is a triangulation of $\mathcal{P}$ on $V^{(1)} = (\partial\mathcal{P} \cap V) \cup \{x_1\}$. Since $|\Gamma^{(1)}| = |\Gamma^{(0)}| = |\Delta'| + 1$, it follows that

$$\begin{aligned} |\Gamma^{(1)}| &\geq (d-1) \cdot (b(\mathcal{P}) - d) + (d-2) \cdot d - (d-1)^2 + 3 \\ &= d + (d-1) \cdot b(\mathcal{P}) - d^2 + 2. \end{aligned}$$

Let $c \geq 2$ and $x_2 \in F$ with $F \in \Gamma^{(1)}$. Let F_0 be the smallest face of F with $x_2 \in F_0$. Then x_2 belongs to the interior of F_0. Let $e = \dim F_0$ and $y_0, y_1, \ldots, y_e$ the vertices of F_0. Thus $1 \leq e \leq d$. Let $\{G_1, \ldots, G_q\}$ denote the set of those $G \in \Gamma^{(1)}$ for which F_0 is a face of G and, for each $1 \leq i \leq q$, write W_i for the set of vertices of G_i. It follows that, for each $1 \leq i \leq q$ and for each $0 \leq j \leq e$,

$$G_i^{(j)} = \mathrm{conv}((W_i \setminus \{y_j\}) \cup \{x_2\})$$

is a d-simplex. Now, it then turns out that

$$\Gamma^{(2)} = (\Gamma^{(1)} \setminus \{G_1, \ldots, G_q\}) \bigcup \left(\bigcup_{1 \leq i \leq q,\, 0 \leq j \leq e} \{G_i^{(j)}\} \right)$$

is a triangulation of $\mathcal{P}$ on $V^{(2)} = (\partial\mathcal{P} \cap V) \cup \{x_1, x_2\}$. Since $F_0 \not\subset \partial\mathcal{P}$, one can regard

$$\bigcup_{i=1}^{q} \mathrm{conv}(\{W_i \setminus \{y_0, \ldots, y_e\}\})$$

as a boundary of a convex polytope of dimension $d - e$. In particular $q \geq d - e + 1$. Hence

$$\begin{aligned} |\Gamma^{(2)}| &\geq d + (d-1) \cdot b(\mathcal{P}) - d^2 + 2 + (d - e + 1)e \\ &\geq 2 \cdot d + (d-1) \cdot b(\mathcal{P}) - d^2 + 2. \end{aligned}$$

Continuing the procedure yields a triangulation $\Gamma^{(c)}$ of $\mathcal{P}$ on

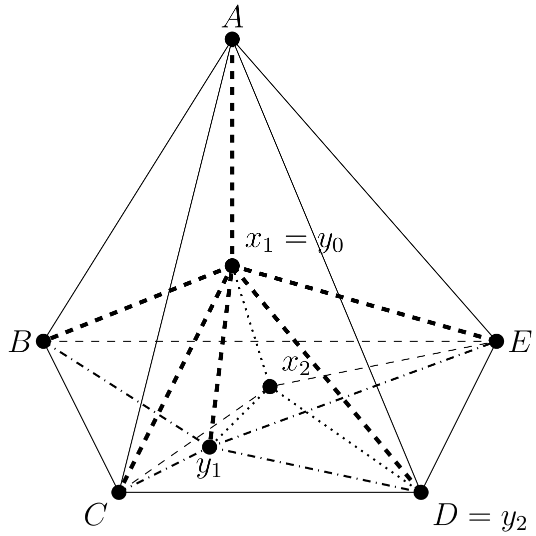
$$V^{(c)} = (\partial\mathcal{P} \cap V) \cup \{x_1, \ldots, x_c\}$$

with

$$|\Gamma^{(c)}| \geq d \cdot c(\mathcal{P}) + (d-1) \cdot b(\mathcal{P}) - d^2 + 2,$$

as desired. □

Example 2.2 The picture drawn below demonstrates the procedure of constructing the triangulation $\Gamma_{\mathcal{P}}$ in the proof of Lemma 2.1. Let $\mathcal{P} = ABCDE$ denote the pyramid over the quadrangle $BCDE$. Let $V = \{A, B, C, D, E, y_1, x_1, x_2\}$ where y_1 belongs to the boundary of $\mathcal{P}$ and where each of x_1 and x_2 belongs to the interior of $\mathcal{P}$. Combining y_1 with each of B, C, D, E yields the triangulation $\Gamma^{(0)}$ of the boundary $\partial\mathcal{P}$ of $\mathcal{P}$. Combining $x_1 \in \mathcal{P} \setminus \partial\mathcal{P}$ with each of A, B, C, D, E and y_1 yields the triangulation $\Gamma^{(1)}$ of $\mathcal{P}$ on $V^{(1)} = \{A, B, C, D, E, y_1, x_1\}$. Let x_2 belong to the interior of the triangle F_0 with the vertices $x_1 = y_0$, y_1, $D = y_2$. Combining x_2 with each of y_0, y_1, y_2 yields the triangulation of F_0 on $\{x_2, y_0, y_1, y_2\}$. Finally, combining x_2 with each of C and E yields the triangulation $\Gamma^{(2)}$ of $\mathcal{P}$ on V.



We now come to the minimal volume formula (1).

Theorem 2.3 *Let $\mathcal{P} \subset \mathbb{R}^d$ be a lattice polytope of dimension d. Let $b(\mathcal{P})$ denote the number of lattice points belonging to the boundary $\partial\mathcal{P}$ of $\mathcal{P}$ and $c(\mathcal{P})$ that number in the interior of $\mathcal{P}$. Suppose that $c(\mathcal{P}) > 0$. Then one has*

$$\mathrm{vol}(\mathcal{P}) \geq (d \cdot c(\mathcal{P}) + (d-1) \cdot b(\mathcal{P}) - d^2 + 2)/d!, \tag{2}$$

where $\mathrm{vol}(\mathcal{P})$ *is the (Lebesgue) volume of $\mathcal{P}$.*

Proof. Lemma 2.1 guarantees the existence of a triangulation $\Gamma_{\mathcal{P}}$ of $\mathcal{P}$ on $\mathcal{P} \cap \mathbb{Z}^d$ with

$$|\Gamma_{\mathcal{P}}| \geq d \cdot c(\mathcal{P}) + (d-1) \cdot b(\mathcal{P}) - d^2 + 2. \tag{3}$$

Since the volume of a lattice d-simplex of $\mathbb{R}^d$ is a multiple of $1/d!$, the minimal volume formula (2) follows from the inequality (3). ☐

3 Castelnuovo polytopes

As before, let $\mathcal{P} \subset \mathbb{R}^d$ be a lattice polytope of dimension d. Following [4], we say that $\mathcal{P}$ is *Castelnuovo* if $\mathcal{P}$ satisfies the equality of (1). When $\mathcal{P}$ is Castelnuovo and when $V = \mathcal{P} \cap \mathbb{Z}^d$, the triangulation $\Gamma_{\mathcal{P}}$ constructed in the proof of Lemma 2.1 is unimodular. (Recall that a triangulation $\Gamma_{\mathcal{P}}$ on $\mathcal{P} \cap \mathbb{Z}^d$ of a lattice polytope $\mathcal{P} \subset \mathbb{R}^d$ of dimension d is called *unimodular* if the volume of each of the d-simplices of $\mathbb{R}^d$ belonging to $\Gamma_{\mathcal{P}}$ is $1/d!$.) Furthermore, the triangulation $\Gamma_{\mathcal{P}}$ constructed in the proof of Lemma 2.1 is *regular*. We refer the reader to [1] for fundamental materials on regular triangulations. It then follows that every Castelnuovo polytope possesses a regular unimodular triangulation.

It is reasonable to find all possible sequences (d, b, c) of integers with $d \geq 3$, $b \geq d+1$, $c \geq 1$ for which there exists a Castelnuovo polytope $\mathcal{P} \subset \mathbb{R}^d$ of dimension d with $b = b(\mathcal{P})$ and $c = c(\mathcal{P})$.

It follows from [3] that, given integers d and c with $d \geq 3$ and $c \geq 1$, there exists a Castelnuovo polytope (in fact, simplex) $\mathcal{P} \subset \mathbb{R}^d$ of dimension d with $b(\mathcal{P}) = d + 1$ and $c = c(\mathcal{P})$.

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