AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **85(2)** (2023), Pages 195–219

Odd-sum colorings of graphs

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Abstract

An assignment $\varphi : V(G) \to \mathbb{Z}$ is said to be an *odd-sum coloring* of a given graph G if no two adjacent vertices receive the same color (i.e., the coloring is proper) and for every vertex $v \in V(G)$ the sum $\sum_{u \in N[v]} \varphi(u)$ of all colors (with repetition) used in the closed neighborhood $N[v]$ is odd. The minimum number of colors required for an odd-sum coloring of G is called the odd-sum chromatic number of G, denoted $\chi_{\text{os}}(G)$. In this paper, we prove that the odd-sum chromatic number always exists and determine the value of this new graph parameter for several basic graph classes including trees, cycles, subdivisions of complete graphs and prisms. We give a tight upper bound on $\chi_{\text{os}}(G)$ in terms of the maximum vertex degree by establishing the inequality $\chi_{\text{os}}(G) \leq 2\Delta(G)$ for non-empty graphs, and also in terms of the (ordinary) chromatic number by showing that $\chi_{\text{os}}(G) \leq 2\chi(G)$. In regard to the classes of planar, triangle-free planar, outerplanar, and bipartite planar graphs, respectively, we determine the tight upper bounds of $8, 6, 6$, and 4 (colors) for the odd-sum chromatic number. The paper concludes with a few open problems.

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1 Introduction

All graphs considered in this paper are simple, finite and undirected. We follow [2] for terminology and notation not defined here. A k-(vertex-)coloring of a graph G is a (not necessarily surjective) mapping $\varphi : V(G) \to S$, where S is a k-set (i.e., a set of size k) which is referred to as the *color set* of φ; thus each $s \in S$ is called a *color*. A coloring φ is said to be *proper* if every color class $\varphi^{-1}(s)$ is an independent subset of the vertex set of G, i.e., intersects no edge in more than one endpoint.

This work is about proper colorings of graphs with additional constraints, obtained by considering the accompanying closed neighborhood-hypergraph. A *hypergraph* (or *set system*) $\mathcal{H} = (V(\mathcal{H}), \mathcal{E}(\mathcal{H}))$ is a generalization of a graph, its (hyper-)edges being subsets of $V(\mathcal{H})$ of arbitrary positive size.

Given a graph $G = (V, E)$, the set of neighbors of a vertex v, denoted $N(v)$, is the *open neighborhood* of v. If we include the vertex v in its set of neighbors, we obtain the *closed neighborhood* of v, denoted $N[v]$. Let $N(G) = \{N(v) : v \in V\}$ and $N[G] = \{N[v] : v \in V\}$. The *open neighborhood-hypergraph* of G is the hypergraph with vertex set V and edge set $N(G)\backslash\{\varnothing\}$, i.e., every non-empty open neighborhood in G corresponds to an edge of the hypergraph. Similarly, the *closed neighborhood-hypergraph* of G is the hypergraph with vertex set V and edge set $N[G]$; this time every closed neighborhood in G corresponds to a (hyper-)edge. Any (vertex) coloring of the open or closed neighborhood-hypergraph of G is said to be a 'neighborhood-constrained' coloring of G.

There are various notions of (vertex-)colorings of hypergraphs, which when restricted to graphs coincide with proper graph colorings. Introduced by Cheilaris et al. [10], a coloring of hypergraph $\mathcal{H}$ is *weak-odd (WO)* if for every edge $e \in \mathcal{E}(\mathcal{H})$ there is a color c with an odd number of vertices of e colored by c. The particular aspect of this notion in regard to the open neighborhood-hypergraph has been recently introduced by Petruševski and Škrekovski [23] (under the name 'odd coloring') and its basic features have been established in Caro, Petruševski and Škrekovski [9]. An *odd coloring* of a graph G is a proper coloring such that for every non-isolated vertex $v \in V(G)$ at least one color c occurs an odd number of times on $N(v)$. This neighborhood-constrained coloring concept immediately spurred considerable interest in the graph theory community (see e.g. [11, 13, 15, 19, 22, 26]). Note in passing that the analogous notion regarding the closed neighborhood-hypergraph brings nothing new as it does not impose any additional constraint to the assumed properness of the coloring.

The principal aim of this paper is to introduce a related hypergraph coloring concept, and to consider it for closed neighborhood-hypergraphs. Namely, given a hypergraph $\mathcal{H}$, an assignment $\varphi : V(\mathcal{H}) \to \mathbb{Z}$ is said to be an *odd-sum (OS)* coloring if for every edge $e \in \mathcal{E}(\mathcal{H})$ the sum $\sum_{u \in e} \varphi(u)$ is odd. So, restricting to the realm of closed neighborhood-hypergraphs, we arrive at the following notion. A proper coloring φ of a graph G is an *odd-sum* coloring (in regard to closed neighborhoods) if for every vertex $v \in V(G)$ the sum $\sum_{u \in N[v]} \varphi(u)$ is odd.

Under any odd-sum coloring φ of a graph G (with respect to closed neighborhoods), for short an *OS coloring*, the vertex set $V(G)$ splits into independent subsets according to the values under the (proper) coloring φ. Let us introduce notation $V_{\text{odd}} = \{v \in V(G) : \varphi(v) \text{ is odd}\}$ and $V_{\text{even}} = \{v \in V(G) : \varphi(v) \text{ is even}\}$. Observe that the independent classes belonging to V_{odd} (respectively V_{even}) may exchange the assigned colors (values) between them, which in turn can be chosen to start from 1 (respectively 2) and to form an interval of consecutive odd (respectively even) positive integers.

The minimum number of colors in any OS coloring of a graph G is its *OS chromatic number*, denoted $\chi_{\text{os}}(G)$. The obvious inequality $\chi(G) \le \chi_{\text{os}}(G)$ may be strict: e.g., $\chi(P_4) = 2$ whereas $\chi_{\text{os}}(P_4) = 3$. Later we shall show that the ratio $\chi_{\text{os}}(G)/\chi(G)$ is at most 2.

Note in passing several fundamental differences between the ordinary chromatic number and the OS chromatic number. To begin with, the former graph parameter is monotonic in regard to the 'subgraph relation', that is, if $H \subseteq G$ then $\chi(H) \le \chi(G)$. However, this nice monotonicity feature does not hold for the OS chromatic number in general; e.g., P_4 is a subgraph of C_4, but nevertheless it holds that $\chi_{\text{os}}(P_4) = 3 > 2 = \chi_{\text{os}}(C_4)$. These two graphs also illustrate the non-monotonicity aspect of χ_{os} even with respect to edge addition. As for the non-monotonicity of χ_{os} regarding vertex addition, observe that for $G = K_3 \,\square\, K_2$ we have $\chi_{\text{os}}(G) = 6$ and $\chi_{\text{os}}(G \vee K_1) = 4$. Another distinction from ordinary (proper) coloring is the behavior of the OS chromatic number under taking disjoint union of graphs: namely, $\chi_{\text{os}}(F \,\dot\cup\, H)$ is not necessarily equal to $\max\{\chi_{\text{os}}(F), \chi_{\text{os}}(H)\}$. For example, $\chi_{\text{os}}(P_4 \,\dot\cup\, C_4) = 4$.

The article is organized as follows. The next section collects several preliminary results of relevance. Section 3 establishes a basic relation between the OS chromatic number of a graph G and its odd-domination sets. It also provides characterizations of several basic graph classes in terms of the value of the OS chromatic number. This is followed by a section on extremal problems concerning the value of χ_{os}. In Section 5 we discuss upper bounds for χ_{os} in planar, outerplanar, triangle-free planar, and bipartite planar graphs. At the end, we briefly convey some of our thoughts for possible further work on the subject of proper odd-sum colorings.

We bring this introductory section to a completion by giving a few comments regarding (proper) odd-sum coloring of graphs with respect to open neighborhoods. It is another natural notion; moreover, it gives an odd coloring with an extra condition. Unfortunately, it is not always possible (see e.g. [17, 12]). There are graphs which do not admit any odd-sum coloring in regard to open neighborhoods: namely, every odd cycle is an obvious 'negative example'. Yet another uncolorable graph, which can be of arbitrarily large minimum degree, is $G = K_{2m} \vee \overline{K_n}$: the join of a complete graph of even order $2m$ and an empty graph on n vertices. To demonstrate the uncolorability of G, we argue by contradiction. Suppose there exists an odd-sum coloring of G with respect to open neighborhoods and let $A = V(K_{2m}) \cap V_{\text{odd}}$, $B = V(\overline{K_n}) \cap V_{\text{odd}}$. Since $N(u) = V(K_{2m})$ for every $u \in V(\overline{K_n})$, the set A is odd-sized. So there exist vertices $v' \in A$ and $v'' \in V(K_{2m})\backslash A$. As $N(v') = V(G)\backslash\{v'\}$, the set B is odd-sized.

Consequently, V_{odd} is even-sized. However, $V_{\mathrm{odd}} \subseteq N(v'')$, implying that the odd-sum condition fails for the vertex v''.

The detected inconsistency concerning odd-sum colorability with respect to open neighborhoods is the main reason why our focus throughout this paper is on odd-sum colorings with respect to closed neighborhoods. It turns out that every graph admits such an OS coloring (a non-obvious fact which shall be proven later on).

2 Preliminaries

Given a graph G on vertex set V, for any $x, y \in V$ we have: $x \in N[y]$ if and only if $y \in N[x]$. Consequently, with $\oplus$ denoting the symmetric difference between sets, the following equivalence holds for any subset $S \subseteq V$:

$$v \in \oplus\{N[s] : s \in S\} \Leftrightarrow |N[v] \cap S| \equiv 1 \,(\mathrm{mod}\, 2)\,. \tag{1}$$

As defined in [25] (under the name 'odd-parity cover'), an *odd-dominating* set D of a graph G is a subset of V such that $N[v] \cap D$ is odd-sized for each $v \in V$. Equivalently, in view of (1), the requirement is that $\oplus\{N[v] : v \in D\} = V$. The following basic result in domination theory was first shown by Sutner [25] in the context of cellular automata (see also [18]). A linear algebraic proof of the same result can be found in Caro [5]. For completeness, we give a relatively short graph-theoretic proof.

Proposition 2.1. *Every graph has an odd-dominating set.*

Proof. Consider a minimal counterexample G. For each vertex $v \in V$ there exists a subset $D_v \subseteq V\backslash\{v\}$ such that $\oplus\{N[w] : w \in D_v\} = V\backslash\{v\}$. Namely, take D_v to be an odd-dominating set of $G - v$ (such a D_v exists by the choice of G). Then $\oplus\{N[w]\backslash\{v\} : w \in D_v\} = V\backslash\{v\}$, implying that either $\oplus\{N[w] : w \in D_v\} = V\backslash\{v\}$ or $\oplus\{N[w] : w \in D_v\} = V$. Since the latter would mean D_v is an odd-dominating set of G, it must be that

$$\oplus\,\{N[w] : w \in D_v\} = V\backslash\{v\}\,. \tag{2}$$

For any subset $S \subseteq V$, let $D_S = \oplus\{D_v : v \in S\}$. From (2) we have

$$\oplus\,\{N[w] : w \in D_S\} = \oplus\{V\backslash\{v\} : v \in S\}\,. \tag{3}$$

Namely, LHS $= \oplus\{N[w] : w \oplus \{D_v : v \in S\}\} = \oplus\{\oplus\{N[w] : w \in D_v\} : v \in S\} \overset{(2)}{=}$ RHS. Consequently, every even-sized subset $S \subseteq V$ satisfies the following

$$\oplus\,\{N[w] : w \in D_S\} = S\,. \tag{4}$$

From (4) we conclude that V is odd-sized, otherwise D_V would be an odd-dominating set of G. Take a vertex $u \in V$ with even degree $\deg(u)$. The set $V\backslash N[u]$ is even-sized, and can be used as S in (4) in order to deduce that

$$\oplus \{N[w] : w \in D_{V\setminus N[u]} \cup \{u\}\} = (V\setminus N[u]) \oplus N[u] = V\,. \tag{5}$$

However, (5) means that $D_{V\setminus N[u]}\cup\{u\}$ is an odd-dominating set of G, a contradiction. □

Note in passing the linear algebraic interpretation of Proposition 2.1. Letting A denote the adjacency matrix of graph G and I the identity square matrix of the corresponding order, the existence of an odd-dominating set of G amounts to the solubility over GF(2) of the equation $(A+I)X = 1$, where 1 is the all-ones column-vector. In the light of this, a graph has a unique odd-dominating set if and only if $A+I$ is non-singular (over GF(2)), i.e., $\det(A+I) = 1$.

There are various papers about minimum cardinality of odd-dominating sets (see e.g. [6, 7, 8]). We also point out the absence of monotonicity concerning odd-dominating sets: if D is an (ordinary) dominating set and B contains D, then B is also a dominating set; however, this is no longer true in the context of odd-domination.

In the next section we shall establish a basic connection between odd-sum colorings and odd-dominating sets D in any graph G. In doing so we shall demonstrate that the odd-sum chromatic number $\chi_{\text{os}}(G)$ relates to the minimum sum of (ordinary) chromatic numbers $\chi(G[D])$ and $\chi(G-D)$. For that reason we complete this short section by mentioning a classical coloring result [4] which gives an upper bound on a graph's chromatic number in terms of the maximum degree Δ and clique number ω.

Proposition 2.2. *(Brooks) Every graph with $\Delta \geq 3$ satisfies $\chi \leq \max\{\Delta, \omega\}$.*

3 Basic results

Here we show that every graph admits an OS coloring via establishing a basic relation that connects this coloring notion with that of an odd-dominating set. Afterwards we determine the OS chromatic number of some basic graph classes and study the behavior of this graph parameter under certain standard graph constructions. Since this is a new graph parameter, we believe that some of these observations will be useful in future study.

3.1 Existence of χ_{os} and a fundamental equality

Let us begin by noting the connection between the notions 'odd-sum coloring' and 'odd-dominating set'. Consider a coloring φ of G such that all colors are integers. Recall the notation $V_{\text{odd}} = \{v \in V(G) : \varphi(v) \text{ is odd}\}$ and $V_{\text{even}} = \{v \in V(G) : \varphi(v) \text{ is even}\}$. Denote $\sigma(v) = \sum_{u\in N[v]} \varphi(u)$. For any vertex $v \in V_{\text{odd}}$, the sum $\sigma(v)$ is odd if and only if v has an even number of neighbors (possibly zero) inside V_{odd}. Contrarily, for any vertex $v \in V_{\text{even}}$, the sum $\sigma(v)$ is odd if and only if v has an odd number of neighbors within V_{odd}. Therefore, φ is an odd-sum coloring if and only if V_{odd} is an odd-dominating set of G. As already mentioned in Section 2, every graph G has an odd-dominating set D. Consequently, we have the following.

Proposition 3.1. *Every graph G admits an odd-sum coloring.*

Proof. Take an odd-dominating set D of G, by Proposition 2.1. Combine a proper coloring of $G[D]$ with color set $2\mathbb{Z}+1$ and a proper coloring of $G-D$ with color set $2\mathbb{Z}$. The result is an OS coloring of G. □

If the proper colorings of $G[D]$ and $G-D$ that were used in the previous proof are optimal in the sense of minimum number of assigned colors, then the resulting OS coloring of G uses exactly $\chi(G[D])+\chi(G-D)$ colors. Hence for a given odd-dominating set D of G the minimum number of colors required for an OS coloring φ of G such that $V_{\text{odd}}=D$ is exactly $\chi(G[D])+\chi(G-D)$. This yields the promised fundamental equality.

Theorem 3.2. *For every graph G it holds that*

$$\chi_{\text{os}}(G)=\min\{\chi(G[D])+\chi(G-D): D \textit{ is an odd-dominating set of } G\}. \tag{6}$$

A straightforward consequence of Theorem 3.2 is the following.

Corollary 3.3. *Every graph G admits an OS coloring with color set $\{1,2,\ldots,2\chi(G)\}$.*

Proof. Take an odd-dominating set D of G. Since $\chi([D]),\chi(G-D)\le\chi(G)$, the induced subgraph $G[D]$ admits a proper coloring with color set $\{1,3,5,\ldots,2\chi(G)-1\}$ and the subgraph $G-D$ admits a proper coloring with color set $\{2,4,6,\ldots,2\chi(G)\}$. The union of these colorings is the required OS coloring of G. □

From Proposition 3.1 and Corollary 3.3 we deduce the following.

Corollary 3.4. *For every graph G it holds that*

$$\chi_{\text{os}}(G)\le\min\{|G|,2\chi(G)\}. \tag{7}$$

Proof. Since $\chi_{\text{os}}(G)$ exists, the highest value it can acquire is if all vertices must be assigned with distinct colors; hence $|G|$ colors always suffice for an OS coloring of G. That $2\chi(G)$ colors also suffice follows immediately from Corollary 3.3. □

In the next subsection it will become clear that the inequality (7) is sharp in the sense that either of the equalities $\chi_{\text{os}}(G)=|G|$ and $\chi_{\text{os}}(G)=2\chi(G)$ is achievable. We end this subsection by observing that, similar to the ordinary proper colorings, for every value $k\in\{\chi_{\text{os}}(G),\ldots,|G|\}$ there exists an OS coloring of G which uses precisely k colors. Let us also note that the established basic connection between the odd-sum requirement and an odd-dominating set makes the improper variant of odd-sum colorings not really interesting: indeed, every graph is 2-colorable in that sense since it admits such a coloring with color set $\{1,2\}$.

3.2 OS chromatic number of several graph classes

Here we determine the proper odd-sum chromatic number of some basic graph classes. For each of the considered graphs G we also discuss whether it admits an OS coloring with color set $\{1,2,\ldots,2\Delta\}$, where Δ is the maximum vertex degree of G.

Complete graphs and their complete subdivisions. First we study the complete graphs in regard to OS colorings. Denoting $V = V(K_n)$, a subset $D \subseteq V$ is an odd-dominating set of K_n if and only if D is odd-sized. Indeed, for every vertex $v \in V$ it holds that $N[v] = V$. As every subset of V induces a clique, we have $\chi(K_n[D]) = |D|$ and $\chi(K_n - D) = |V| - |D|$. This reasoning combined with (6) yields the following.

Observation 3.5. *For every $n \geq 1$ it holds that $\chi_{\rm os}(K_n) = n$.*

Moreover, for every odd value $x \in \{1,2,\ldots,n\}$, K_n admits an OS coloring with color set $\{1,3,5,\ldots,2x-1\} \cup \{2,4,6,\ldots,2n-2x\}$. In particular, by taking $x = 1$, K_n admits an OS coloring with color set $\{1\} \cup \{2,4,\ldots,2n-2\}$. Hence, with the obvious exception of K_1, every other K_n admits an OS coloring with color set $\{1,2,\ldots,2\Delta\}$.

Let us now consider the complete subdivision of a complete graph. Recall that $S(G)$, the *complete subdivision* of graph G, is obtained from G by subdividing every edge in $E(G)$ exactly once. If $G = K_n$ we denote $S(G)$ by SK_n.

Observation 3.6. *For every $n \geq 2$ it holds that $\chi_{\rm os}(\mathrm{SK}_n) = 2$.*

Proof. We distinguish between the cases of even n and of odd n. If n is even, then the set of 2-vertices comprising $V(\mathrm{SK}_n)\backslash V(K_n)$ is an odd-dominating set D such that both D and $V(\mathrm{SK}_n)\backslash D$ are independent. So there exists an OS coloring of SK_n with color set $\{1,2\}$. Contrarily, if n is odd, then $D = V(\mathrm{SK}_n)$ is an odd-dominating set such that $\chi(\mathrm{SK}_n[D]) = 2$ and $\chi(\mathrm{SK}_n) - D) = 0$. Hence SK_n admits an OS coloring with color set $\{1,3\}$.

Since $\chi_{\rm os}(\mathrm{SK}_n) \geq \chi(\mathrm{SK}_n) = 2$, we conclude that $\chi_{\rm os}(\mathrm{SK}_n) = 2$. □

Trees. Both for complete graphs and for their complete subdivisions, the OS chromatic number equals the ordinary chromatic number. Concerning trees, things are slightly different. We start with a consideration of all paths. Recall that a path P_n is a path on n vertices.

Observation 3.7. *For every $n \geq 1$, the path P_n admits an OS coloring with color set $\{1,2,4\}$. Both colors 2 and 4 are required if and only if $n \geq 4$. Moreover, it holds that*

$$\chi_{\rm os}(P_n) = \begin{cases} 1 & \text{if } n = 1\,; \\ 2 & \text{if } 2 \leq n \leq 3\,; \\ 3 & \text{if } n \geq 4\,. \end{cases} \tag{8}$$

Proof. We may assume that $n \geq 3$, since it is obvious that $\chi_{\text{os}}(P_n) = \chi(P_n)$ for $n \leq 2$ and each odd-dominating set is a singleton. Note that a subset $D \subseteq V(P_n)$ is an odd-dominating set of $P_n : v_1 v_2 \cdots v_n$ if and only if the following two requirements are fulfilled:

(i) $|D \cap \{v_1, v_2\}| = |D \cap \{v_{n-1}, v_n\}| = 1$;

(ii) from any triplet of consecutive vertices along P_n, exactly one belongs in D.

Consequently, if $3 \nmid (n-2)$ then P_n has a unique odd-dominating set D. This unique D is independent, and moreover $\chi(P_n - D) = 2$ unless $n = 3$. Contrarily, if $3 \mid (n-2)$ then P_n has precisely two odd-dominating sets D, each of which is independent and once again $\chi(P_n - D) = 2$ unless $n = 3$. So, P_n always admits an OS coloring with color set $\{1, 2, 4\}$. Moreover, both colors 2 and 4 required if and only if $n \geq 4$. □

Let us obtain an analogous result for trees in general.

Theorem 3.8. *Every tree T admits an OS coloring with color set $\{1, 2, 4\}$. If T is non-trivial then $\chi_{\text{os}}(T) \in \{2, 3\}$. Furthermore, letting $A \cup B = V(T)$ be the bipartition of T, the equality $\chi_{\text{os}}(T) = 2$ is true if and only if either A or B is an odd-dominating set of T.*

Proof. Let D be an odd-dominating set of T. Clearly D induces a subgraph with all degrees even. Consequently, D is an independent set. Indeed, for otherwise a component of $T[D]$ is not an isolated vertex implying that this component contains a cycle which is impossible as T is a tree. Hence $\chi(T[D]) = 1$. Since T is bipartite, we have $\chi(T - D) \leq 2$. Therefore T admits an OS coloring with color set $\{1, 2, 4\}$. It follows that $\chi_{\text{os}}(T) \in \{2, 3\}$ unless T is trivial. Moreover, $\chi_{\text{os}}(T) = 2$ if and only if there exists an odd-dominating set D such that the set $V \backslash T$ is independent, that is, if and only if $\{D, V \backslash D\} = \{A, B\}$ happens to be the bipartition of T. □

Cycles. The OS chromatic number of cycles equals their ordinary chromatic number.

Observation 3.9. *For every $n \geq 3$ it holds that*

$$\chi_{\text{os}}(C_n) = \chi(C_n) = \begin{cases} 2 & \text{if } n \text{ is even;} \\ 3 & \text{if } n \text{ is odd.} \end{cases} \tag{9}$$

Proof. The set $D = V(C_n)$ is an odd-dominating set of C_n. Hence C_n admits an OS coloring with color set $\{1, 3, \ldots, 2\chi(C_n) - 1\}$. Consequently, $\chi_{\text{os}}(C_n) \leq \chi(C_n)$. The reversed inequality is true for any graph. □

Excepting K_1, each of the graphs we already considered admitted an OS coloring with color set $\{1, 2, \ldots, 2\Delta\}$. This nice feature does not hold for cycles in general.

Observation 3.10. *C_n has proper odd-dominating sets (i.e., distinct from $V(C_n)$) if and only if $3 \mid n$. In the affirmative the number of such proper Ds is 3, and each is of size $\frac{n}{3}$.*

Proof. Let us look into the existence of an odd-dominating set $D \subsetneq V(C_n)$. Start by noting that the presence of two adjacent vertices (of C_n) within an odd-dominating set D implies that $D = V(C_n)$. Consequently, given $D \subsetneq V(C_n)$ is an odd-dominating set if and only if from every triplet of consecutive vertices along C_n precisely one belongs in D.

Therefore, C_n has a proper odd-dominating set D if and only if $3 \mid n$. In the affirmative, the number of such Ds is exactly three, and each has $|D| = \frac{n}{3}$. □

In regard to the existence of an OS coloring with color set $\{1,2,3,4\}$, Observation 3.10 implies the following.

(a) If $3 \mid n$ then C_n admits an OS coloring with color set $\{1,2,4\}$.

(b) If $3 \nmid n$ and $2 \mid n$ then C_n admits an OS coloring with color set $\{1,3\}$.

(c) If $3 \nmid n$ and $2 \nmid n$ then C_n does not admit an OS coloring with color set $\{1,2,3,4\}$.

Indeed, (a) is straightforward. As for (b) and (c), the assumption $3 \nmid n$ tells that $V(C_n)$ is the only odd-dominating set of C_n. Hence, an OS coloring of C_n is precisely a proper coloring that uses only odd integers as colors. The coloring is optimal if it uses $\chi(C_n)$ (odd) colors.

In view of the above and Observation 3.7 we deduce the following.

Proposition 3.11. *Let G be a connected graph of maximum degree $\Delta \leq 2$. Then G admits an OS coloring with color set $\{1,2,\dots,2\Delta\}$ if and only if G is neither K_1 nor C_n where $n \equiv 1$ or $5 \pmod 6$.*

Prisms over graphs. Given a graph G, a *prism* over G is the graph $G \square K_2$ consisting of two disjoint copies of G and a perfect matching between the corresponding vertices. We are confined to $G \in \{K_n, C_n\}$. Our findings shall demonstrate the sharpness of (7). First we consider the prism over $G = K_n$.

Proposition 3.12. *For every $n \geq 1$ it holds that*

$$\chi_{\text{os}}(K_n \square K_2) = \begin{cases} n & \text{if } n \text{ is even;} \\ 2n & \text{if } n \text{ is odd.} \end{cases} \tag{10}$$

Proof. Note that $K_n \square K_2$ is n-regular and has $\chi(K_n \square K_2) = n$. Let A and B be the vertex sets of the two copies of K_n comprising $K_n \square K_2$, and for every $v \in V(K_n)$ let the vertices $v_A \in A$ and $v_B \in B$ form an edge $v_A v_B$ in $K_n \square K_2$. We determine all possible odd-dominating sets of $K_n \square K_2$ by considering separately the two cases regarding parity of n.

Case 1: *n is even.* Then all vertex degrees are even, as $K_n \square K_2$ is n-regular. Consequently, $V = A \cup B$ is an odd-dominating set, implying that $K_n \square K_2$ admits an OS coloring with color set $\{1, 3, 5, \dots, 2n-1\}$. Hence $\chi_{\text{os}}(K_n \square K_2) = \chi(K_n \square K_2) = n$. In fact V is the only odd-dominating set. To demonstrate this, let D be an arbitrary odd-dominating set of $K_n \square K_2$. First we show that $D \cap A, D \cap B$ are even-sized. For argument's sake, suppose $D \cap B$ is odd-sized. So for any $v_B \in B$ the intersection $N[v_B] \cap (D \cap B)$ is odd-sized, implying that $v_A \in A \backslash D$. Consequently, $D \subseteq B$. But then for any vertex $w_B \in B \backslash D$ (here we use that $n = |B|$ is even), the vertex w_A is not dominated by D, a contradiction.

Now, since for any $v_A \in A$ the set $N[v_A] \cap (D \cap A)$ is even-sized, it must be that $v_B \in D \cap B$. Hence $B \subseteq D$. Analogously, $A \subseteq D$ and we conclude that $D = V$.

Case 2: *n is odd.* There are two obvious odd-dominating sets of $K_n \square K_2$, namely A and B. In fact these are the only ones, and we show this by considering an arbitrary odd-dominating set D. If $D \subseteq A$ then $D = A$, for otherwise there is a vertex in B which is not dominated by D. Similarly, if $A \subseteq D$ then $D = A$, for otherwise there is a vertex in $A = D \cap A$ having odd number of neighbors in D, a contradiction. So, supposing $D \neq A$ and $D \neq B$, we have that $D \cap A, D \cap B$ are non-empty proper subsets of A and B, respectively. The same argument from the previous case tells us that $D \cap A, D \cap B$ are even-sized. However, the latter implies that for every $v_B \in B$ we have $v_A \in D \cap A$. In other words, $A \subseteq D$, a contradiction.

Since A and B are the only odd-dominating sets of $K_n \square K_2$, we deduce that $\chi_{\text{os}}(K_n \square K_2) = 2n$. Note that $K_n \square K_2$ admits an OS coloring with color set $\{1, 2, \dots, 2n\}$ and it must be surjective. □

Note in passing that (10) demonstrates that for every $\chi \equiv 1 \pmod 2$ and for every $\Delta \equiv 1 \pmod 2$ there is a graph G such that $\chi_{\text{os}}(G) = |G| = 2\chi(G) = 2\Delta(G)$ showing the tightness of (7) and of Corollary 3.17 (which appears later).

Next we consider the prism over $G = C_n$.

Proposition 3.13. *For every $n \geq 3$ it holds that*

$$\chi_{\text{os}}(C_n \square K_2) = \begin{cases} 2 & \text{if } n \text{ is even;} \\ 6 & \text{if } n \text{ is odd.} \end{cases} \tag{11}$$

Proof. Note that $\chi(C_n \square K_2) = 2$ or 3 depending on whether n is even or odd. Let us enumerate the vertices in one copy of $V(C_n)$ with $u_1, u_2 \dots, u_n$, and the vertices in the other copy of $V(C_n)$ with $w_1, w_2, \dots, w_n$ so that $u_i w_i$ is an edge in $C_n \square K_2$ for every i. We consider separately the two possible parities of n.

Case 1: *n is even.* Then $D = \{u_1, u_3, \dots, u_{n-1}\} \cup \{w_2, w_4, \dots, w_n\}$ is an odd-dominating set. Indeed, for every vertex $v \in V(C_n \square K_2)$ it holds that $|N[v] \cap D|$

is either 1 or 3. Since for this particular choice, both D and $V(C_n \square K_2)\backslash D$ are independent sets, we conclude that $C_n \square K_2$ admits an OS coloring with color set $\{1, 2\}$. Hence, $\chi_{\text{os}}(C_n \square K_2) = \chi(C_n \square K_2) = 2$ in this case. (In fact, this follows from a more general observation that $\chi_{\text{os}}(G) = 2$ whenever G is bipartite and Eulerian; indeed, each color class in a proper 2-coloring of G is an odd-dominating set.)

Case 2: *n is odd.* Consider the characteristic function χ_D of an arbitrary odd-dominating set D of $C_n \square K_2$, that is, the function on $V(C_n \square K_2)$ defined by:

$$\chi_D(v) = \begin{cases} 1 & \text{if } v \in D\,; \\ 0 & \text{if } v \notin D\,. \end{cases}$$

In what follows we consider all indices i taken $(\bmod\, n)$ up to n. In view of the intersections $N[u_i] \cap D$ and $N[w_i] \cap D$, we have that

$$\chi_D(u_{i-1}) + \chi_D(w_{i-1}) \equiv \chi_D(u_{i+1}) + \chi_D(w_{i+1}) \pmod 2\,. \tag{12}$$

Namely, since both $N[u_i] \cap D$ and $N[w_i] \cap D$ are odd-sized, the sum

$$(\chi_D(u_{i-1}) + \chi_D(u_i) + \chi_D(w_i) + \chi_D(u_{i+1})) + (\chi_D(w_{i-1}) + \chi_D(u_i) + \chi_D(w_i) + \chi_D(w_{i+1}))$$

is even. Taking into account that n is odd, (12) implies that all $\chi_D(u_i) + \chi_D(w_i)$ are of the same parity. Indeed, the sequence $1, 3, 5, \dots n, n+2, n+4, \dots, 2n-1$ taken $(\bmod\, n)$ up to n reads $1, 3, 5, \dots, n, 2, 4, \dots, n-1$.

It cannot be that $\chi_D(u_i) + \chi_D(w_i)$ are all even. Otherwise, in view of $N[u_i] \cap D$, it would hold that $\chi_D(u_{i-1}) \neq \chi_D(u_{i+1})$ for every i. However, this yields a clear contradiction: the sequence $1, 3, 5, \dots n, n+2, n+4, \dots, 2n-1, 2n+1$ taken $(\bmod\, n)$ up to n reads $1, 3, 5, \dots, n, 2, 4, \dots, n-1, 1$ and it is of even length, implying that $\chi_D(u_1) \neq \chi_D(u_1)$.

So for every i we have $\chi_D(u_i) + \chi_D(w_i) = 1$. Consequently, $\chi_D(u_{i-1}) = \chi_D(u_{i+1})$. Invoking once again the fact that n is odd, we deduce $\chi_D(u_1) = \chi_D(u_2) = \dots = \chi_D(u_n)$. In view of the equality $\chi_D(u_i) + \chi_D(w_i) = 1$, it follows that either $D = \{u_1, u_2, \dots, u_n\}$ or $D = \{w_1, w_2, \dots, w_n\}$.

It is readily seen that both $\{u_1, u_2, \dots, u_n\}$ and $\{w_1, w_2, \dots, w_n\}$ are odd-dominating sets of $C_n \square K_2$ (regardless of the parity of n). What we established above is that in the case when n is odd, these two are the only odd-dominating sets. Since they are complementary to each other (in regard to $V(C_n \square K_2)$) and each induces an odd cycle (namely C_n), from (6) we conclude that $\chi_{\text{os}}(C_n \square K_2) = 2\chi(C_n \square K_2) = 6$. Every OS coloring of $C_n \square K_2$ with color set $\{1, 2, \dots, 6\}$ is surjective. □

We end our discussion concerning prisms over graphs by noting that (11) shows that Corollaries 3.4 and 3.17 are tight even for 3-regular 3-chromatic planar graphs.

Corona. The *corona* $C(G)$ of a graph G is the graph obtained from G by attaching a pendant edge to each vertex of G. Clearly, $\chi(C(G)) = \chi(G)$ if and only if $\chi(G) \geq 2$, and otherwise $\chi(C(G)) = \chi(G) + 1$.

Proposition 3.14. *For the corona of every graph G it holds that*

$$\chi(C(G)) \leq \chi_{\text{os}}(C(G)) \leq \chi(C(G)) + 1\,. \tag{13}$$

Moreover, both inequalities are attainable.

Proof. Since the added leaves form an independent odd-dominating set of $C(G)$, from the basic equality (6) it follows that $\chi(C(G)) \leq \chi_{\text{os}}(C(G)) \leq 1 + \chi(C(G))$.

Let us show that either of the inequalities in (13) is sharp. The left one is realized by any 2-chromatic graph G with all degrees even, as then $\chi_{\text{os}}(C(G)) = \chi(C(G)) = 2$. We proceed to prove that the right inequality is attained for any complete graph: namely, for every $n \geq 1$ it holds that $\chi_{\text{os}}(C(K_n)) = n + 1$.

Denote $A = V(K_n), B = V(C(K_n))\backslash A$ and for any $v \in A$ let $\bar{v} \in B$ be the leaf adjacent to v. Consider an odd-dominating set D of $C(K_n)$. Since $N[\bar{v}] = \{v, \bar{v}\}$, the set D contains exactly one member from each pair $\{v, \bar{v}\}$. From this it follows that at least one of the sets $A\backslash D$ $A \cap D$ is empty. Indeed, for any $v \in A\backslash D$ we have $N[v] \cap D = \{\bar{v}\} \cup (A \cap D)$, implying that $|A \cap D| \equiv 0 \pmod 2$. On the other hand, for any $v \in A \cap D$ we have $N[v] \cap D = A \cap D$, implying that $|A \cap D| \equiv 1 \pmod 2$.

Now, as each intersection $D \cap \{v, \bar{v}\}$ is a singleton, $A\backslash D = \varnothing$ if and only if $D = A$. For the same reason, $A \cap D = \varnothing$ if and only if $D = B$. It is readily seen that A is an odd-dominating set of $C(K_n)$ if and only if $n \equiv 1 \pmod 2$. On the other hand, B is always an odd-dominating set of $C(K_n)$ (regardless of the parity of n). Since $D \in \{A, B\}$, the basic equality (6) yields $\chi_{\text{os}}(C(K_n)) = \chi([A]) + \chi([B]) = n + 1$. $\square$

3.3 Upper bound in terms of maximum degree

Most of the graphs considered in the previous subsection were found to admit an OS coloring with color set $\{1, 2, \ldots, 2\Delta\}$. In fact, it easily follows from Propositions 2.2 and (6) that $\Delta \geq 3$ is sufficient for such colorability.

Proposition 3.15. *Let G be a graph of maximum degree $\Delta \geq 3$. Then G admits an OS coloring with color set $\{1, 2, \ldots, 2\Delta\}$.*

Proof. Consider a minimum counter-example G. In view of the conclusion in the proof of Observation 3.5, G is not a complete graph. Let us show that G is disconnected. Supposing the opposite, Proposition 2.2 applies to every subgraph of G. It follows that the chromatic number of any subgraph of G is at most Δ. In particular, for any odd-dominating set D of G, we have $\chi(G[D]) \leq \Delta$ and $\chi(G - D) \leq \Delta$. By combining a proper coloring of $G[D]$ with color set $\{1, 3, 5, \ldots, 2\Delta - 1\}$ and a proper coloring of $G - D$ with color set $\{2, 4, 6, \ldots, 2\Delta\}$, we construct an OS coloring of G with color set $\{1, 2, \ldots, 2\Delta\}$. The obtained contradiction confirms that G is disconnected.

Since G is a counter-example, there exists a component H of G which does not admit an OS coloring with color set $\{1, 2, \ldots, 2\Delta\}$. The minimality choice of G implies that the maximum degree $\Delta(H)$ of H is less than Δ. Consequently, the chromatic number of any subgraph of H is at most $\Delta(H) + 1 \leq \Delta$. However, then the same construction used above produces an OS coloring of H with color set $\{1, 2, \ldots, 2\Delta\}$, a contradiction. □

The requirement $\Delta \geq 3$ cannot be omitted, in view of Proposition 3.11. Hence, we conclude the following.

Theorem 3.16. *Let G be a connected non-trivial graph of maximum degree Δ. Then G admits an OS coloring with color set $\{1, 2, \ldots, 2\Delta\}$ if and only if G is not a cycle C_n with $n \equiv 1$ or $5 \pmod 6$.*

Note that the following straightforward consequence of Theorem 3.16 and Observation 3.9 is sharp for every odd value of Δ, by (10).

Corollary 3.17. *Let G be a non-empty graph of maximum degree Δ. Then $\chi_{\text{os}}(G) \leq 2\Delta$.*

We end this subsection by sharing some of our initial thoughts regarding the structure of (possible?) graphs G having sufficiently large maximum degree $\Delta \equiv 0 (\text{mod } 2)$ and $\chi_{\text{os}}(G) = 2\Delta$. Let D be a minimum (in terms of size) odd-dominating set of such G. Due to the minimality choice of D, no component of $G[D]$ is a copy of $K_{\Delta+1}$. Consequently, since $\chi(G[D]) + \chi(G - D) = 2\Delta$, it must be that $\chi(G[D]) = \chi(G - D) = \Delta$. From the equality $\chi(G[D]) = \Delta$ it follows that $\Delta(G[D]) = \Delta$. Indeed, for otherwise $\Delta(G[D]) \leq \Delta - 2$ (as Δ and $G[D]$ are both even) implying that $\chi(G[D]) \leq \Delta - 1$. Borodin and Kostochka [3] conjectured that if $\Delta(G) \geq 9$ and $\omega(G) \leq \Delta(G) - 1$ then $\chi(G) \leq \Delta - 1$. Beutelspacherf and Hering [1] independently conjectured that this statement holds for sufficiently large Δ. Reed [24] confirmed the latter by showing that $\Delta \geq 10^{14}$ suffices. So assuming that our even Δ is sufficiently large, it must be that $\omega(G[D]) = \Delta$. In other words, $K_\Delta \subseteq G[D]$. Any vertex $v \in V(K_\Delta)$ of such a complete subgraph of $G[D]$ has a neighbor $\overline{v}$ in $D \backslash V(K_\Delta)$ (again, since Δ and $G[D]$ are both even). Moreover, no two such $\overline{v'}$ and $\overline{v''}$ coincide, otherwise $D \backslash \{v', v''\}$ would be a smaller odd-dominating set of G. We conclude that a corona $C(K_\Delta)$ appears as a subgraph of $G[D]$. Finally, note that $\Delta((G - D)) \leq \Delta - 1$ (because every vertex in $V \backslash D$ is dominated by some vertex in D). Thus, Brooks' theorem implies $\omega(G - D) = \Delta$. In other words, a copy of K_Δ is present within $G - D$ as well.

3.4 Sufficient conditions for $\chi_{\text{os}} = \chi$

We present two such conditions, and for each we derive equality of the OS chromatic number and the ordinary chromatic number by using the basic equality (6) and the fact that the former graph parameter is never less than the latter. The first sufficient condition regards the parity of all vertex degrees.

Observation 3.18. *Let G be a graph with all degrees even. Then $\chi_{\text{os}}(G) = \chi(G)$.*

Proof. Clearly, for every vertex $v \in V(G)$ the closed neighborhood $N[v]$ is odd-sized. Consequently, $V(G)$ is an odd-dominating set of G. So, in view of (6), we have that $\chi_{\text{os}}(G) \le \chi(G)$. The reversed equality holds in general. □

The second sufficient condition refers to the presence of a vertex adjacent to every other vertex (i.e., a universal dominating vertex).

Observation 3.19. *Let G be a graph with a vertex v of degree $\deg(v) = |V(G)| - 1$. Then $\chi_{\text{os}}(G) = \chi(G)$.*

Proof. Clearly $\{v\}$ is an odd-dominating set of G. Noting that $\chi(G - v) = \chi(G) - 1$, it follows from (6) that $\chi_{\text{os}}(G) \le \chi(G)$. □

3.5 Complexity

It is well-known that the problem of determining the chromatic number of a graph G is $\mathcal{NP}$-complete as soon as $\chi(G) \ge 3$. Here we show $\mathcal{NP}$-hardness of the problem of determining the OS chromatic number.

Observation 3.20. *Determining if $\chi_{\text{os}}(G) = 3$ is $\mathcal{NP}$-complete for 4-regular planar graphs.*

Proof. By Observation 3.18, for any planar graph G with all degrees even we have $\chi_{\text{os}}(G) = \chi(G)$. Since for 4-regular planar graphs deciding 3-colorability is $\mathcal{NP}$-complete [16] it follows that for 4-regular planar graphs the problem of deciding odd-sum 3-colorability is also $\mathcal{NP}$-complete. □

Contrarily, similar to ordinary 2-colorability, OS 2-colorability can be efficiently decided. This is implied by the following characterization of the graphs G satisfying $\chi_{\text{os}}(G) = 2$.

Observation 3.21. *Let G be a non-trivial connected graph. Then $\chi_{\text{os}}(G) \le 2$ if and only if G is bipartite and there is a partition $D \cup B = V$ where D is an odd-dominating set of G such that either $D = V$ (and B is empty) or both D and B are the independent sets forming the bipartition of G.*

Proof. Assume $\chi_{\text{os}}(G) \le 2$. As $\chi_{\text{os}}(G) \ge \chi(G)$ we have that G is bipartite. By (6), there is an odd-dominating set D of G such that $\chi(G[D]) + \chi(G - D) \le 2$. Setting $B = V \backslash D$ we have the desired partition $D \cup B = V$. Indeed, if B is empty then $D = V$ (implying that G has all degrees even). Contrarily, if B is non-empty then both D and B are independent sets (and from the connectedness of G it follows that $\{D, B\}$ is precisely the bipartition of G).

In the other direction, assume that G is bipartite and that there is a partition $D \cup B = V$, where D is an odd-dominating set of G with either $D = V$ (and B empty) or D, B being the two independent sets forming the bipartition of G. In both cases, the basic equality (6) implies that $\chi_{\text{os}}(G) \le \chi(G[D]) + \chi(G[B]) \le 2$. □

4 Extremal problems

In view of Corollaries 3.4 and 3.17, for every non-empty graph G it holds that

$$\chi_{\text{os}}(G) \leq \min\{|G|, 2\Delta(G), 2\chi(G)\}\,. \tag{14}$$

Here we discuss possible realizations of certain aspects of (14). In particular, first we characterize the equality $\chi_{\text{os}}(G) = |G| = 2\Delta(G)$. Afterwards, we point out to two realizations of the equality $\chi_{\text{os}}(G) = |G| = 2\chi(G)$. Finally, we briefly comment on $\chi_{\text{os}}(G) = |G|$.

The equality $\chi_{\text{os}}(\mathbf{G}) = |\mathbf{G}| = \mathbf{2\Delta(G)}$. By Proposition 3.12, every graph $G = K_n \,\square\, K_2$ with odd n satisfies $\chi_{\text{os}}(G) = |G| = 2\Delta(G)$. We show that these particular prism graphs are the only realization of the considered equality.

Proposition 4.1. *A graph G satisfies $\chi_{\text{os}}(G) = |G| = 2\Delta(G)$ if and only if $G = K_n \,\square\, K_2$ and n is odd.*

Proof. Begin by observing that every graph G for which $\chi_{\text{os}}(G) = |G|$ must be connected. Indeed, for otherwise in any odd-sum $\chi_{\text{os}}(G)$-coloring of G there is an odd color with multiple occurrences (namely, in at least two components), making impossible the equality $\chi_{\text{os}}(G) = |G|$. Let us abbreviate $\Delta(G)$ to Δ and $\omega(G)$ to ω. So, assuming $\chi_{\text{os}}(G) = |G| = 2\Delta$, the graph G is surely connected. Note that $\Delta \geq 1$, and equality implies $G = K_2 = K_1 \,\square\, K_2$. If $\Delta = 2$ then $|G| = 4$, hence $G = P_4$ or $G = C_4$. However, neither works as $\chi_{\text{os}}(P_4) = 3$ and $\chi_{\text{os}}(C_4) = 2$. Consequently, if $\Delta \neq 1$ then $\Delta \geq 3$. By Observation 3.5, $G \neq K_{\Delta+1}$. Thus $\omega \leq \Delta$.

Consider an arbitrary odd-dominating set D of G. In view of (6), the equality $\chi_{\text{os}}(G) = |G|$ implies that $\chi(G[D]) = |D|$ and $\chi(G-D) = |V\backslash D|$. In other words, both $G[D]$ and $G-D$ are complete graphs. Let us show that each is of order Δ. First note that $\Delta(G-D) \leq \Delta - 1$ (because every vertex from $V\backslash D$ is dominated by some vertex of D). Consequently, $|V\backslash D| \leq \Delta$, which in turn gives $|D| \geq \Delta$. The strict inequality $|D| > \Delta$ would imply $G = K_{\Delta+1}$. Hence $|D| = |V\backslash D| = \Delta$, i.e., $G[D]$ and $G - D$ are copies of K_Δ. Since D is an odd-dominating set of G, the graph $G[D]$ is even. So Δ must be odd. Now, as $G[D]$ and $G - D$ are $(\Delta - 1)$-regular, the bipartite subgraph $H = G[D, V\backslash D]$ of G has the following two properties: (i) for every $x \in D$ the degree $\deg_H(x) \leq 1$, and (ii) for every $y \in V\backslash D$ the degree $\deg_H(y) = 1$. From $|D| = |V\backslash D|$ it follows that $E(H)$ is a perfect matching of G. In other words, $G = K_n \,\square\, K_2$ with $n = \Delta$ odd. □

The equality $\chi_{\text{os}}(\mathbf{G}) = |\mathbf{G}| = \mathbf{2}\chi(\mathbf{G})$. If G is a graph such that $\chi_{\text{os}}(G) = |G| = 2\chi(G)$, then G is connected (because of $\chi_{\text{os}}(G) = |G|$) and non-complete (because of $|G| = 2\chi(G)$). Once again, in view of the basic equality (6), for any odd-dominating set D of G it holds that $G[D]$ and $G-D$ are complete graphs of order $\frac{|G|}{2}$. Moreover, since $G[D]$ is even, $\frac{|G|}{2}$ must be odd, that is, $|G| \equiv 2 \pmod 4$. The challenging part of the task of characterizing the equality $\chi_{\rm os}(G) = |G| = 2\chi(G)$ is to determine all possible realizations of the bipartite subgraph $G[D, V\backslash D]$ of G. Unfortunately, we weren't able to entirely resolve this issue. One realization is immediate by Proposition 3.12: it comes with $E(H)$ being a perfect matching of G, that is, when $G = K_n \square K_2$ and $n = \frac{|G|}{2}$ is odd. But unlike the previous extremal problem, this time those particular prism graphs are not the only possible realization. Indeed, our next result demonstrates another realization: H is obtained from $K_{n,n}$ by removing the edges of a Hamilton cycle.

Proposition 4.2. *Let $G = \overline{C_{4k+2}}$, that is, G be the graph obtained from K_{4k+2} by deleting the edge set of a Hamilton cycle. Then*

$$\chi_{\rm os}(G) = |G| = 2\chi(G)\,.$$

Proof. Letting $n = 2k+1$, the graph G decomposes into two copies of K_n, with corresponding vertex sets A and B, and a bipartite graph $H[A, B] = K_{n,n} - E(C_{2n})$ between them. We refer to the edges of the removed Hamilton cycle as the 'missing edges'. Thus every vertex is incident with two missing edges, and the endpoints of every second missing edge on a traversing of C_{2n} form a pair of non-adjacent vertices in G (one from A and the other one from B). Hence, by assigning a distinctive color from the set $\{1, 2, \ldots, n\}$ to the members of each such pair we obtain a proper n-coloring of G, which shows that $\chi(G) = n$. Clearly, each of the sets A, B is an odd-dominating set of G, and $\chi(G[A]) + \chi(G[B]) = 2n = |G| = 2\chi(G)$. In view of the basic equality (6), in order to show $\chi_{\rm os}(G) = |G| = 2\chi(G)$ it suffices to prove that A, B are the only odd-dominating sets of G. So consider an arbitrary odd-dominating set D of G. Let $[D, V\backslash D]_{\rm mis}$ be the spanning subgraph of C_{2n} on the missing edges having one endpoint in D and the other in $V\backslash D$. Similarly, let $[D]_{\rm mis}$ and $[V\backslash D]_{\rm mis}$ be the induced subgraphs of C_{2n} with vertex set D and $V\backslash D$, respectively.

Suppose first that $|D| \equiv 0 \pmod 2$. Then $[D]_{\rm mis}$ is 1-regular. Indeed, as D is an odd-dominating set, every vertex $v \in D$ is adjacent in G to an even number of vertices from $D\backslash\{v\}$. Consequently, v misses an odd number of vertices from $D\backslash\{v\}$. And since v misses in total two vertices in $V\backslash\{v\}$, it follows that v is incident with exactly one missing edge whose other endpoint is also in D, which proves the 1-regularity of $[D]_{\rm mis}$. Moreover, we also deduce from this that every vertex of D is incident with exactly one missing edge whose other endpoints lies in $V\backslash D$. On the other hand, by applying the same reasoning to our supposition that D is even-sized, we have that every vertex of $V\backslash D$ is incident with an odd number (and thus exactly one) missing edge towards D. So $[D, V\backslash D]_{\rm mis}$ is 1-regular as well. Now consider the sets $A \cap D$, $B \cap D$ and $A\backslash D$. The established 1-regularity of $[D]_{\rm mis}$ implies that $|A \cap D| = |B \cap D|$. Analogously, the 1-regularity of $[D, V\backslash D]_{\rm mis}$ yields the equality $|A\backslash D| = |B \cap D|$. Therefore, $|A \cap D| = |A\backslash D|$, which in turn gives that $|A| = 2|A \cap D|$ is even. However, this clearly contradicts with $n = |A|$ being odd.

We conclude that $|D| \equiv 1 \pmod 2$. Then every vertex of D is incident with an even number (0 or 2) of missing edges whose other endpoint is in D. And similarly, every vertex of $V\backslash D$ is incident with an even number (0 or 2) of missing edges towards D. Consequently, each of the graphs $[D]_{\text{mis}}$, $[D, V\backslash D]_{\text{mis}}$ and $[V\backslash D]_{\text{mis}}$ is even (i.e., has all degrees even). But $[D]_{\text{mis}}$ and $[V\backslash D]_{\text{mis}}$ are also proper subgraphs of C_{2n} (since $D, V\backslash D \subsetneq V$). Therefore, they are both empty, implying that every missing edge has one endpoint in D and the other endpoint in $V\backslash D$. However, this means that $D, V\backslash D$ is a bipartition of the connected bipartite graph C_{2n}, proving that $\{D, V\backslash D\} = \{A, B\}$. □

The equality $\chi_{\text{os}}(\mathbf{G}) = |\mathbf{G}|$. Recall that every realization G must be connected. And of course every complete graph works since $\chi_{\text{os}}(K_n) = n = |K_n| = \chi(K_n)$. The problem of characterizing all graphs that satisfy the considered equality is interesting because, in a sense, those graphs are the cliques in regard to odd-sum colorings. Unfortunately, we were not able to go significantly beyond the straightforward observation that every odd-dominating set D of G as well as its complement $V\backslash D$ must induce (ordinary) cliques. Therefore, we confine here to only briefly discussing our initial thoughts which perhaps disclose the difficulties of this particular characterization problem.

Observation 4.3. *There is no graph G with $\chi_{\text{os}}(G) = |G|$ and $\chi(G) = |G| - 1$.*

Proof. Note that G is connected and non-complete. So $\omega \le |G| - 1$. It cannot be that $\Delta = |G| - 1$, otherwise G has a universal dominating vertex and Observation 3.19 applies giving $\chi_{\text{os}}(G) = \chi(G) < |G|$. So $\Delta \le |G| - 2$. If $\Delta \le 2$ then G is either a path or cycle, but Observations 3.7 and 3.9 discard this possibility. Hence $\Delta \ge 3$ and Brooks theorem applies, giving $|G| - 1 = \chi(G) \le \max\{\Delta, \omega\} \le |G| - 1$. Hence $\omega = |G| - 1 > \Delta$, contradicting the connectedness. □

One naturally wonders next whether there are graphs G satisfying $\chi_{\text{os}}(G) = |G|$ and $\chi(G) = |G| - 2$. Any possible realization must be a connected non-complete graph. Moreover, as $\chi(G) = |G| - 2$, it is implied that $\omega \le |G| - 2$. The absence of a universal dominating vertex guarantees that $\Delta \le |G| - 2$ as well. Consequently, $\max\{\Delta, \omega\} \le |G| - 2$. Since it is easily checked that $\Delta \ge 3$, Brooks theorem gives that $\max\{\Delta, \omega\} = |G| - 2$. Combined with the connectedness of G, this implies that $\Delta = |G| - 2$. So $\chi(G) = \Delta$, and we leave at this here since clarifying the structure of such graphs seems to be a very difficult point (see e.g. [20]).

5 Planar graphs

Let $\mathcal{P}$ be the class of all planar graphs, and let $\mathcal{P}_{\nexists\triangle}$, $\mathcal{O}$ and $\mathcal{B}$ be the subclasses of triangle-free planar graphs, of outerplanar graphs, and of bipartite planar graphs, respectively. Moreover, for an invariant χ^* and a graph class $\mathcal{C}$ define

$$\chi^*(\mathcal{C}) = \max\{\chi^*(G) \mid G \in \mathcal{C}\} .$$

It is well-known that $\chi(\mathcal{P}) = 4$ whereas $\chi(\mathcal{P}_{\nexists\triangle}) = \chi(\mathcal{O}) = 3$ and $\chi(\mathcal{B}) = 2$. In view of the basic inequality (6), for any graph class $\mathcal{C}$ that is closed under taking subgraphs it holds that

$$\chi_{\rm os}(\mathcal{C}) \leq 2\chi(\mathcal{C})\,.$$

Moreover, every graph $G \in \mathcal{C}$ admits an OS coloring with color set $\{1, 2, \ldots, 2\chi(\mathcal{C})\}$. In particular, every planar graph admits an OS coloring with color set $\{1, 2, 3, 4, 5, 6, 7, 8\}$, every triangle-free planar graph or outerplanar graph admits an OS coloring with color set $\{1, 2, 3, 4, 5, 6\}$, and every bipartite planar graph admits an OS coloring with color set $\{1, 2, 3, 4\}$. Consequently, $\chi_{\rm os}(\mathcal{P}) \leq 8$, $\chi(\mathcal{P}_{\nexists\triangle}), \chi(\mathcal{O}) \leq 6$ and $\chi(\mathcal{B}) \leq 4$. In this section we show that each of the last four inequalities is in fact an equality.

Theorem 5.1. *$\chi_{\rm os}(\mathcal{P}) = 8$.*

Proof. Consider the planar graph G depicted in Figure 1. It is readily seen that the set $D = \{v_1, v_3, v_4, v_5, v_6, v_9, v_{11}\}$ is an odd-dominating set of G. Moreover, both of the induced subgraphs $G[D]$ and $G - D$ contain a copy of K_4 as a subgraph: indeed, $\{v_1, v_3, v_4, v_6\} \subseteq D$ and $\{v_2, v_7, v_8, v_{10}\} \subseteq V \backslash D$. Hence $\chi(G[D]) = \chi(G - D) = 4$.

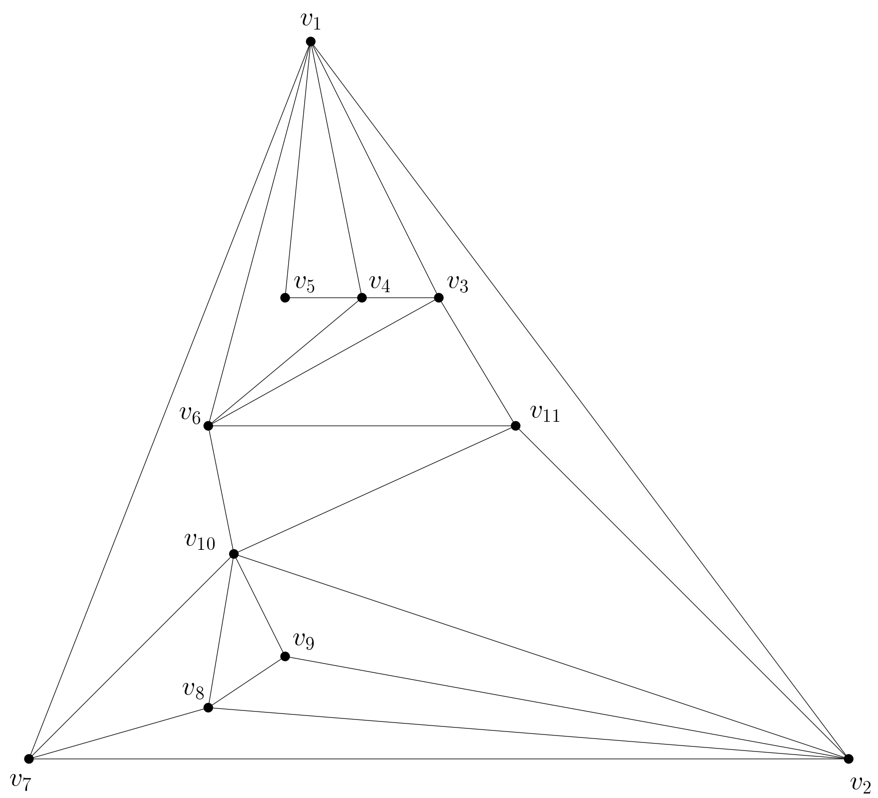

Figure 1: An example of a planar graph that is not OS 7-colorable.

A simple calculation shows that $\det(A + I) = 1$ over GF(2), meaning that $A + I$ is non-singular. (Here A denotes the adjacency matrix of G and I is the identity square matrix of order $|V(G)|$). Consequently, the above mentioned D is the only odd-dominating set of G. In view of (6), we conclude that $\chi_{\rm os}(G) = 4 + 4 = 8$.

We have found dozens of planar graphs with $\chi_{\rm os} = 8$ and maximum degree 6, 7 or 8. So the considered graph G is by no means unique in realizing the equality

$\chi_{os}(\mathcal{P}) = 8$. However it is of minimum possible order, as there are no such planar graphs on fewer than 11 vertices. □

Theorem 5.2. $\chi(\mathcal{P}_{\nexists\triangle}) = \chi(\mathcal{O}) = 6$.

Proof. The equality $\chi(\mathcal{P}_{\nexists\triangle}) = 6$ can be deduced from (11), as for every odd integer $n \geq 3$ the prism $C_n \square K_2$ is an example of a member of $\mathcal{P}_{\nexists\triangle}$ that requires six colors for an OS coloring.

Next we point out an infinite family of graphs that realize $\chi(\mathcal{O}) = 6$. Namely, for every positive integer n divisible by 6, we exhibit a connected maximal outerplanar graph G of order $6n + 1$ such that $\chi_{os}(G) = 6$. Start with a path $P_n : v_1v_2 \dots v_n$ and a copy of K_1, form the join $P_n \vee K_1$, and then attach a vertex w of degree 2 to the vertices v_{n-2} and v_{n-1}. The obtained graph is the promised G (see Figure 2).

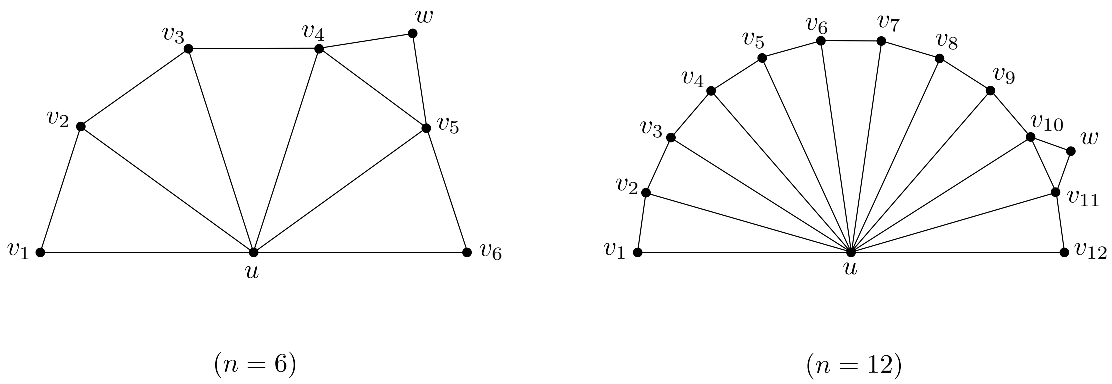

Figure 2: Graph G for $n = 6$ (left) and for $n = 12$ (right).

In what follows we show that G has only one odd-dominating set, namely $\{v_1, v_{n-2}, v_{n-1}, w\}$. Let us denote by u the vertex of degree n (coming from the copy of K_1 joined to P_n). Consider an arbitrary odd-dominating set of G.

Claim 1. $u \notin D$. Arguing by contradiction, suppose $u \in D$. Consequently, $v_1 \in D$ as well. Indeed, for otherwise it must be that $v_2 \notin D, v_3 \notin D, \dots, v_{n-2} \notin D$. However, then $\{v_{n-1}, w\} \cap D = N[w] \cap D$ is a singleton, implying $|N[v_{n-2}] \cap D| = 2$, a contradiction.

Now, since $uv_1 \in D$, it is forced that: $v_2 \in D$, $v_3 \notin D$, $v_4, v_5 \in D$, …, $v_{n-6} \notin D$, $v_{n-5}, v_{n-4} \in D$, $v_{n-3} \notin D$ and $v_{n-2} \in D$. There are two possibilities regarding w, each of which yields a contradiction. Namely, if $w \in D$ then $v_{n-1} \in D$ as well (because $|N[w] \cap D|$ is odd), and from this $v_n \in D$ (because $|N[v_n] \cap D|$ is odd); however, by then $|N[v_0 \cap D]| = \frac{2}{3}n + 2$ is even. Contrarily, if $w \notin D$ then $v_{n-1} \notin D$ as well, and from this $v_n \in D$; but then $|N[v_n] \cap D| = 2$.⋄

Claim 2. $v_1 \in D$. Again, for argument's sake, suppose the opposite, that is, let $v_1 \notin D$. Then $v_2 \in D$, $v_3, v_4 \notin D$, $v_5 \in D$, …, $v_{n-3}, v_{n-2} \notin D$. But now it is forced that $\{v_{n-1}, v_n\} \cap D$ is a singleton. Indeed, if $w \in D$ then $v_{n-1} \notin D$, implying $v_n \in D$.

Contrarily, if $w \notin D$ then $v_{n-1} \in D$, and thus $v_n \notin D$. Either way, it follows that $|N[v_0] \cap D| = \frac{n}{3}$ is even, a contradiction.◇

So far we have established that $u \notin D$ and $v_1 \in D$. Consequently $v_2, v_3 \notin D$, $v_4 \in D$, $v_5, v_6 \notin D$, …, $v_{n-4}, v_{n-3} \notin D$ and $v_{n-2} \in D$. If $w \notin D$ then $v_{n-1} \notin D$ as well, implying that $v_n \in D$. However this would make $|N[v_{n-1} \cap D]| = 2$, a contradiction. We deduce that $w \in D$, which in turn yields $v_{n-1} \in D$ and $v_n \notin D$. In conclusion, we have that $D = \{v_1, v_{n-2}, v_{n-1}, w\}$.

Since both $G[D]$ and $G - D$ are outerplanar graphs containing a triangle, we have that $\chi(G[D]) = \chi(G - D) = 3$. In view of the uniqueness of D, the basic equality (6) proves that $\chi_{\text{os}}(G) = 6$. □

All graphs considered in the proof of Theorem 5.2 are 2-connected. Note that among graphs of connectivity 1 there are more obvious examples that realize the equalities in Theorem 5.2. In fact, for any given odd value $\Delta \geq 3$ there exists an outerplanar graph G of arbitrary large girth and maximum degree Δ such that $\chi_{\text{os}}(G) = 6$. Indeed, simply take two copies of C_n with n odd, connect them by an edge and to all other vertices add a leaf. It is easily seen that there are precisely two odd-dominating sets of G: each consists of the vertex set of one copy of C_n and the leaves attached to the other copy of C_n.

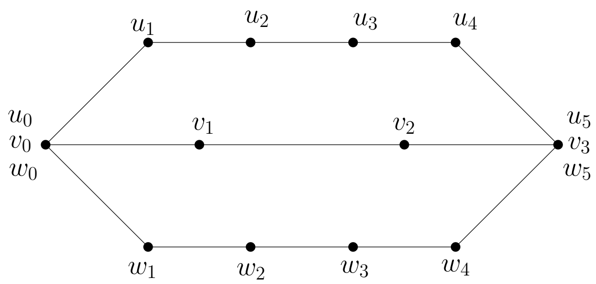

Figure 3: The theta graph $\Theta_{3,5,5}$.

In the proof of the next result we exhibit a family of bipartite planar graphs of arbitrary large girth that require 4 colors for any OS coloring. Recall that a *theta graph* $\Theta_{k,l,m}$ is a graph obtained by joining two vertices by three internally-disjoint paths of length k, l and m. We use $\{v_0v_1\cdots v_k, u_0u_1\cdots u_l, w_0w_1\cdots w_m\}$ with $v_0 = u_0 = w_0$ and $v_k = u_l = w_m$ to denote a $\Theta_{k,l,m}$ (see Figure 3).

Theorem 5.3. $\chi_{\text{os}}(\mathcal{B}) = 4$.

Proof. Let $G = \Theta_{k,l,m}$ with $k \equiv 3 \pmod 6$ and $l, m \equiv 5 \pmod 6$. Note that G is planar and of girth $= \min\{k+l, l+m, k+m\}$. Since G is also bipartite, $\chi_{\text{os}}(G) \leq 4$ by (7). We show that $\chi_{\text{os}}(G) = 4$. For argument's sake suppose the opposite, that is, let there be an odd-dominating set D of G such that $\chi(G[D]) + \chi(G - D) \leq 3$.

Claim 1. $\chi(G - D) = 2$. We may assume that $u_2 \in D$, otherwise either u_1u_2 or u_2u_3 is an edge in $G - D$. If $u_1 \notin D$ then also $u_3 \notin D$ and $u_4 \notin D$, hence u_3u_4 is an edge in

$G - D$. So let $u_1 \in D$. Consequently, $u_i \in D$ for all $i = 0, 1, \ldots, k$. Then $v_1, v_{k-1} \notin D$ or $w_1, w_{m-1} \notin D$, otherwise $D = V(G)$ and thus $|N[u_0] \cap D| = 4$, a contradiction. Since both $l, m \equiv 5 \pmod 6$, by symmetry assume that $v_1, v_{k-1} \notin D$. From $v_0 \in D$ and $v_1 \notin D$ it follows that $v_2 \notin D$. So $v_1 v_2 \in E(G - D)$.$\diamond$

Since $\chi(G[D]) + \chi(G - D) \leq 3$, it follows from Claim 1 that D is an independent set.

Claim 2. $v_0, v_k \in D$. Suppose first that $\{v_0, v_k\} \cap D$ is a singleton. By symmetry let it be that $v_0 \in D$ and $v_k \notin D$. So $v_1 \notin D$ (because D is independent). It follows that $v_2 \notin D$, $v_3 \in D$, $v_4, v_5 \notin D$, $\ldots$, $v_{k-3} \in D$ and $v_{k-2}, v_{k-1} \notin D$. However then $N[v_{k-1}] \cap D = \varnothing$, a contradiction. Suppose now that $\{v_0, v_k\} \cap D$ is empty. It follows that $\{v_1, v_{k-1}\} \cap D$ is a singleton. Indeed, if $v_1 \in D$ then $v_2, v_3 \notin D$, $v_4 \in D$, $\ldots$, $v_{k-2} \in D$ and $v_{k-1} \notin D$; contrarily, if $v_1 \notin D$ then $v_2 \in D$, $v_3, v_4 \notin D$, $\ldots$, $v_{k-3}, v_{k-2} \notin D$ and $v_{k-1} \in D$. By symmetry, assume $v_1 \notin D$. Note that $u_1 \in$, for otherwise $u_2 \in D$, $u_3, u_4 \notin D$, $u_5 \in D$, $\ldots$, $u_{l-3} \in D$ and $u_{l-2}, u_{l-1} \notin D$; but then $N[u_{l-1}] \cap D = \varnothing$, a contradiction. Now looking at $N[v_0] \cap D$, we conclude that $w_1 \notin D$. However, this yields a similar contradiction as it implies that $w_2 \in D$, $w_3, w_4 \notin D$, $w_5 \in D$, $\ldots$, $w_{m-3} \in D$ and $w_{m-2}, w_{m-1} \notin D$, giving $N[w_{m-1}] \cap D = \varnothing$.$\diamond$

As D is independent, if follows from Claim 2 that $v_1, v_{k-1}, u_1, u_{l-1}, w_1, w_{m-1} \notin D$. In particular, since $u_0 \in D$ and $u_1 \notin D$, we conclude that $u_2 \notin D$, $u_3 \in D$, $u_4, u_5 \notin D$, $\ldots$, $u_{l-2} \in D$ and $u_{l-1} \notin D$. But then $|N[u_{l-1}] \cap D| = 2$, a contradiction. □

The graphs considered in the proof of Theorem 5.3 are of maximum degree $\Delta = 3$. Note that for any odd value of $\Delta \geq 3$ the generalized theta graph $\Theta_{k,l_1,l_2,\ldots,l_{\Delta-1}}$ with $k \equiv 3 \pmod 6$ and $l_1, l_2, \ldots, l_{\Delta-1} \equiv 5 \pmod 6$ realizes the equality $\chi_{\mathrm{os}}(\mathcal{B}) = 4$ amongst graphs of maximum degree Δ.

6 Further work

We noted after Proposition 3.12 that for every $\chi \equiv 1 \pmod 2$ there is a graph G such that $\chi_{\mathrm{os}}(G) = 2\chi(G)$. For $\chi = 2$ there are analogous examples: one is $G = C_4 \,\square\, K_2$ and another is $G = P_4 \,\dot\cup\, C_4$. As for $\chi = 4$ such an example is the graph used in the proof of Theorem 5.1.

Problem 6.1. *For each $k \geq 3$ find graphs G such that $\chi_{\mathrm{os}}(G) = 2\chi(G) = 4k$.*

The analogous issue regarding realizations of the equality $\chi_{\mathrm{os}} = 2\Delta$ amongst graphs with maximum degree Δ is settled for every odd value of Δ (e.g. $K_n \,\square\, K_2$ with n odd) and also for $\Delta = 2$ (e.g. $P_4 \,\dot\cup\, C_4$), but open for any even $\Delta \geq 4$.

Problem 6.2. *For each $k \geq 2$ find graphs G such that $\chi_{\mathrm{os}}(G) = 2\Delta(G) = 4k$.*

In Section 4 we discussed a possible characterization of the equality $\chi_{\mathrm{os}}(G) = 2\chi(G) = |G|$, and pointed out to at least two realizations: namely, $G = K_n \,\square\, K_2$ for $n \equiv 1 \pmod 2$ and $G = \overline{C_n}$ for $n \equiv 2 \pmod 4$.

Problem 6.3. *Find other realizations of the equality* $\chi_{\text{os}}(G) = 2\chi(G) = |G|$.

As to the third possible upper bound $\chi_{\text{os}}(G) \leq |G|$ and in view of Observation 4.3 we suggest the following.

Problem 6.4. *Find more realizations of the equality* $\chi_{\text{os}}(G) = |G|$.

We concluded the proof of Theorem 5.1 by mentioning that we have found dozens of planar graphs with $\chi_{\text{os}} = 8$ and maximum degree 6, 7 or 8. A peculiar feature of all examples that we have found is that each has a unique odd-dominating set. Perhaps this is no coincidence.

Problem 6.5. *Find a planar graph G with $\chi_{\text{os}}(G) = 8$ that has at least two odd-dominating sets.*

For 4-chromatic planar graphs we propose the following.

Conjecture 6.6. *Every planar graph G maximum degree $\Delta \leq 5$ has $\chi_{\text{os}}(G) \leq 7$.*

In view of the remark given after Theorem 5.2, one naturally wonders the following.

Question 6.7. *Given an even value $\Delta \geq 4$, is there a sufficiently large value g such that all connected planar (or connected outerplanar) graphs of maximum degree Δ and girth $\geq g$ are odd-sum 5-colorable?*

Similarly, in the light of the last remark in the previous section, we ask the following.

Question 6.8. *Given an even value $\Delta \geq 4$, is there a sufficiently large value g such that all connected bipartite planar graphs of maximum degree Δ and girth $\geq g$ are odd-sum 3-colorable?*

For a surface Σ, let us define the *odd-sum chromatic number* of Σ,

$$\chi_{\text{os}}(\Sigma) = \max_{G \hookrightarrow \Sigma} \chi_{\text{os}}(G),$$

as the maximum of $\chi_{\text{os}}(G)$ over all graphs G embedded into Σ.

The basic equality (6) implies that $\chi_{\text{os}}(\Sigma) \leq 2\chi(\Sigma)$, and Theorem 5.1 gives that $\chi_{\text{os}}(S_0) = 2\chi(S_0) = 8$, where S_0 is the sphere. Our result motivates the study of this invariant for graphs on other surfaces. It would be interesting to have a similar characterization to the Heawood number for other surfaces of higher genus.

Problem 6.9. *Determine $\chi_{\text{os}}(\Sigma_g)$, where g is the Euler genus.*

It is our belief that for some positive constant C, it turns out that $H(\Sigma_g)+C$ colors always suffice, where $H(\Sigma_g) = \left\lfloor \frac{7+\sqrt{1+28g}}{2} \right\rfloor$ is the Heawood number of the surface Σ_g.

Recall that for any graph G, $\bar{G}$ denotes the complement of G, that is, the graph defined on the vertex set of G so that an edge belongs to $\overline{G}$ if and only if it does not belong to G. Nordhaus and Gaddum [21] studied the chromatic number in a graph and in its complement together. They proved sharp lower and upper bounds on the sum and on the product of $\chi(G)$ and $\chi(\overline{G})$ in terms of the order n of G. For example, they showed that $\chi(G) + \chi(\overline{G}) \leq n + 1$. Since then, any bound on the sum and/or the product of an invariant in a graph G and the same invariant in the complement $\overline{G}$ of G is called a Nordhaus-Gaddum type inequality or relation.

Question 6.10. *Is there a constant C such that for every graph G of order n it holds that*

$$\chi_{\mathrm{os}}(G) + \chi_{\mathrm{os}}(\overline{G}) \leq n + C\,?$$

We hope that the provided generous list of questions, conjectures and open problems will spur further research in this direction. Recently, several questions posed above were answered by D. W. Cranston. We refer the interested reader to [14].

Acknowledgments

This work is partially supported by ARRS P1-0383 and J1-3002. We thank the anonymous referees for their many valuable suggestions, which improved the presentation of this paper.

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(Received 21 Sep 2022; revised 11 Jan 2023)