AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **85(3)** (2023), Pages 350–359

On improved upper bounds on the transversal number of hypergraphs

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Abstract

A subset T of vertices in a hypergraph H is a transversal if T has a nonempty intersection with every edge of H. The transversal number of H is the minimum size of a transversal in H. A subset S of vertices in a graph G with no isolated vertex is a total dominating set if every vertex of G is adjacent to a vertex of S. The minimum cardinality of a total dominating set in G is the total domination number of G. In this note, we improve previous probabilistic upper bounds given for the transversal number of a hypergraph and the total domination number of a graph given by the first two authors in [*Discrete Math. Algorithms Appl.* 11 (1) (2019), 1950004, 6 pp.].

ISSN: 2202-3518

1 Introduction

A subset T of vertices in a hypergraph H is a *transversal* (also called *vertex cover* or *hitting set* in many papers) if T has a nonempty intersection with every edge of H. The *transversal number* $\tau(H)$ is the minimum size of a transversal in H. The notion of transversal is fundamental in hypergraph theory and has been studied extensively. Two of the five chapters in the classical 1989 monograph on hypergraph theory by Berge [5] are on transversals. For a more recent book on transversals in hypergraphs we refer the reader to [18]. We refer to [3, 6, 7, 15, 16, 21–24] for recent results and further references. In this paper, we obtain a new (improved) probabilistic upper bound for the transversal number of a hypergraph. As a consequence of this result, we obtain a new (improved) probabilistic upper bound for the total domination number of a graph.

A *hypergraph* $H = (V, E)$ is a finite set $V = V(H)$ of elements, called vertices, together with a finite multiset $E = E(H)$ of subsets of $V(H)$, called hyperedges or simply edges. A k-edge in H is an edge of size k. The hypergraph H is k-*uniform* if every edge of H is a k-edge. We assume throughout this paper that $|e| \geq 2$ holds for every edge $e \in E$. The degree of a vertex v in H, denoted by $d_H(v)$, is the number of edges of H which contain v. The minimum and maximum degrees among the vertices of H is denoted by $\delta_v(H)$ and $\Delta_v(H)$, respectively. We also denote the minimum and maximum size among the edges of H by $\delta_e(H)$ and $\Delta_e(H)$, respectively. If X is a subset of vertices of a hypergraph H, then the *sub-hypergraph* of H induced by X, denoted by $H[X]$, is the hypergraph on the vertex set X with edge set $\{e \in E(H) \colon e \subseteq X\}$.

A graph without an isolated vertex is called an *isolate-free graph*. A *total dominating set* in an isolate-free graph G is a set S of vertices of G such that every vertex is adjacent to a vertex in S. The *total domination number* of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a total dominating set of G. Total domination in graphs is now well studied in graph theory. The literature on this subject has been surveyed and detailed in the 2013 book on this topic that can be found in [19]. A survey of total domination in graphs can be found in [13]. For a state of the art on domination in graphs we refer the reader to the recent books [10, 11, 12].

For a graph G, the *open neighborhood hypergraph*, abbreviated ONH, of G is the hypergraph H_G with vertex set $V(H_G) = V(G)$ and with edge set $E(H_G) = \{N_G(x) \mid x \in V(G)\}$ consisting of the open neighborhoods of vertices in G. As first observed in [23] (see also [19]), the transversal number of the ONH of a graph is precisely the total domination number of the graph.

Observation 1.1 (Thomassé, Yeo [23]). *If G is a graph with no isolated vertex and H_G is the* ONH *of G, then $\gamma_t(G) = \tau(H_G)$.*

A subset of vertices in a hypergraph H is an *independent set* if it contains no edge of H. Equivalently, a set of vertices S is an independent set in H if and only if $V(H) \setminus S$ is a transversal in H. The *independence number* $\alpha(H)$ of H is the maximum cardinality of an independent set in H.

We note that $n(H) = \tau(H)+\alpha(H)$. Independence in hypergraphs is very well studied (see, for example, [1, 4, 8, 9, 17, 20]).

For the probabilistic methods notation and terminology we refer the reader to [2].

2 Known results

Alon [3] established the following important upper bound on the transversal number of a uniform hypergraph.

Theorem 2.1 ([3]). *If H is a k-uniform hypergraph with n vertices and m edges, where $k > 1$, then for any positive real α,*

$$\tau(H) \leq \left(\frac{\alpha \ln(k)}{k}\right) n + \frac{m}{k^{\alpha}}.$$

Eustis [8] proved the following lower bound on the independence number of a uniform hypergraph.

Theorem 2.2 ([8]). *If H is a k-uniform hypergraph with average degree $d \geq 1$, then*

$$\alpha(H) \geq \left(1 - \frac{1}{k}\right) \frac{n}{d^{\frac{1}{k-1}}}.$$

Henning and Yeo [19] gave a simple heuristic that finds a total dominating set in a graph. As a consequence of this heuristic, they established the following upper bound on the total domination number of a graph in terms of its minimum degree.

Theorem 2.3 ([19]). *If G is a graph with $\delta = \delta(G) \geq 2$, then*

$$\gamma_t(G) \leq \left(\frac{1 + \ln \delta}{\delta}\right) n.$$

Henning and Jafari Rad [14] generalized Theorem 2.2.

Theorem 2.4 ([14]). *If H is a hypergraph on n vertices with m edges and with maximum vertex degree $\Delta_v = \Delta_v(H)$ such that every edge of H has size at least $\delta_e = \delta_e(H) > 2$, then*

$$\alpha(H) \geq \left(1 - \frac{1}{\delta_e}\right) \left(\frac{1}{\Delta_v}\right)^{\frac{1}{\delta_e - 1}} n.$$

The authors in [14] also presented a slight improvement of Theorem 2.1.

Theorem 2.5 ([14]). *If H is a hypergraph with n vertices, m edges and maximum vertex degree $\Delta_v = \Delta_v(H)$ such that every edge of H has size at least $\delta_e = \delta_e(H) > 1$, then for any positive real α,*

$$\tau(H) \leq n\left(\frac{\alpha \ln \delta_e}{\delta_e}\right) + \frac{m}{\delta_e^{\alpha}} - n\left(1 - \frac{1}{\delta_e(H)}\right)\left(\frac{1}{\Delta_v(H)}\right)^{\frac{1}{\delta_e(H)-1}} \left(\frac{\alpha \ln \delta_e}{\delta_e}\right)^{1+\Delta_v(\Delta_e - 1)}.$$

As consequences of Theorem 2.5 they obtained the following.

Corollary 2.6 ([14]). *If H is a k-uniform hypergraph with n vertices and m edges, where $k > 1$, with maximum vertex degree Δ, then for any positive real α,*

$$\tau(H) \leq n\left(\frac{\alpha \ln k}{k}\right) + \frac{m}{k^\alpha} - n\left(1 - \frac{1}{k}\right)\left(\frac{1}{\Delta}\right)^{\frac{1}{k-1}}\left(\frac{\alpha \ln k}{k}\right)^{1+\Delta(k-1)}.$$

Corollary 2.7 ([14]). *If G is a graph with minimum degree $\delta \geq 2$ and maximum degree Δ, then*

$$\gamma_t(G) \leq n\left(\frac{1 + \ln \delta}{\delta}\right) - n\left(1 - \frac{1}{\delta}\right)\left(\frac{1}{\Delta}\right)^{\frac{1}{\delta-1}}\left(\frac{\ln \delta}{\delta}\right)^{1+\Delta(\delta-1)}.$$

Our main aim in this paper is to present an improvement of Theorem 2.5, which will enable us to improve the upper bounds given in both Corollaries 2.6 and 2.7.

3 Main result

We will prove the following new probabilistic upper bound for the transversal number in a hypergraph.

Theorem 3.1. *If H is a hypergraph with n vertices, m edges and maximum vertex degree $\Delta_v = \Delta_v(H)$ such that every edge of H has size at least $\delta_e = \delta_e(H) > 1$, then for any integer $k \geq 0$ and reals $0 < p, q < 1$,*

$$\tau(H) \;\leq\; np + m(1-p)^{\delta_e} - np^{1+\Delta_v(\Delta_e-1)}\left(q - \frac{\Delta_v}{\delta_e}q^{\delta_e}\right)\sum_{i=0}^{k}(1-q)^{i(\Delta_v(\Delta_e-1)+1)}.$$

3.1 Preliminary result

In order to prove our main result, namely Theorem 3.1, we first present the following lower bound for the independence number of a hypergraph. We remark that our proof of Theorem 3.2 below initially follows the proof of Theorem 2.4 before a deeper analysis enables us to improve the bound given in this theorem. In what follows, we assume that no two edges in a hypergraph are equal since duplicated edges play no role in determining the independence number (and transversal number) of a hypergraph.

Theorem 3.2. *Let H be a hypergraph on n vertices with m distinct edges and with maximum vertex degree $\Delta_v = \Delta_v(H)$ and minimum vertex degree $\delta_v = \delta_v(H) \geq 1$ such that every edge of H has size at least $\delta_e = \delta_e(H) \geq 2$ and at most $\Delta_e = \Delta_e(H)$. For each real $0 < p < 1$ and each integer $k \geq 0$,*

$$\alpha(H) \geq n\left(p - \frac{\Delta_v}{\delta_e}p^{\delta_e}\right)\sum_{i=0}^{k}(1-p)^{i(\Delta_v(\Delta_e-1)+1)}.$$

Proof. The proof is by induction on k. For the base step of the induction assume that $k = 0$. We follow initially the proof of Theorem 2.4 given in [14]. Create a subset $X \subseteq V(H)$ by choosing each vertex $v \in V(H)$ independently with probability p. Let $H[X]$ be the sub-hypergraph induced by X, and $m_{H[X]}$ be the number of edges in $H[X]$. We compute the expectation of the random variable $|X| - m_{H[X]}$. Clearly,

$$E(|X|) = np \quad \text{and} \quad E(m_{H[X]}) \le mp^{\delta_e}.$$

The number m of edges in H is bounded above by the following inequality.

$$m \le \frac{n\Delta_v}{\delta_e}.$$

Thus,

$$E(|X| - m_{H[X]}) = E(|X|) - E(m_{H[X]}) \ge np - mp^{\delta_e} \ge np - n\left(\frac{\Delta_v}{\delta_e}\right)p^{\delta_e}.$$

We remove one vertex from each edge of $H[X]$ to obtain an independent subset I_X of vertices in X, implying that

$$\alpha(H) \ge |I_X| \ge np - n\left(\frac{\Delta_v}{\delta_e}\right)p^{\delta_e} = n\left(p - \frac{\Delta_v}{\delta_e}p^{\delta_e}\right)\sum_{i=0}^{k}(1-p)^{i(\Delta_v(\Delta_e-1)+1)}.$$

This establishes the base case when $k = 0$. Let $k \ge 1$ and assume the result holds for all integers k' where $0 \le k' < k$. As in the base case, we create a subset $X \subseteq V(H)$ by choosing each vertex $v \in V(H)$ independently with probability p, and let $H[X]$ be the sub-hypergraph induced by X. As it was seen in the base step, there is an independent set I_X of vertices in X satisfying

$$|I_X| \ge np - n\left(\frac{\Delta_v}{\delta_e}\right)p^{\delta_e}.$$

Let $Y = V(H) \setminus N_H[X]$, that is,

$$Y = \{y \in V(H) \setminus X : \text{ no edge of } H \text{ containing } y \text{ intersects } X\}.$$

Let $H[Y]$ be the sub-hypergraph induced by Y. By the inductive hypothesis,

$$\alpha(H[Y]) \ge |Y|\left(p - \frac{\Delta_v}{\delta_e}p^{\delta_e}\right)\sum_{i=0}^{k-1}(1-p)^{i(\Delta_v(\Delta_e-1)+1)}.$$

Thus

$$E(\alpha(H[Y])) \ge \sum_{i=0}^{k-1}(1-p)^{i(\Delta_v(\Delta_e-1)+1)}\left(p - \frac{\Delta_v}{\delta_e}p^{\delta_e}\right)E(|Y|).$$

We next compute a lower bound on $E(|Y|)$. Let v be an arbitrary vertex of H. The vertex v has degree at most Δ_v, and therefore there are at most Δ_v edges incident with v in H. Each such edge incident with v contains at most $\Delta_e - 1$ vertices different from v. Hence, $|N_H[v]| \le \Delta_v(\Delta_e - 1) + 1$. We note that $v \in Y$ if and only if $N_H[v]$ is disjoint from X. Hence, the probability that v belongs to Y is $(1-p)^{|N_H[v]|} \ge (1-p)^{\Delta_v(\Delta_e-1)+1}$. Since there are n vertices in H, we therefore infer that

$$E(|Y|) \ge n(1-p)^{\Delta_v(\Delta_e-1)+1}.$$

Hence,

$$E(\alpha(H[Y])) \ge \sum_{i=0}^{k-1}(1-p)^{i(\Delta_v(\Delta_e-1)+1)}\left(p - \frac{\Delta_v}{\delta_e}p^{\delta_e}\right)n(1-p)^{\Delta_v(\Delta_e-1)+1}.$$

Thus there is an independent set I_Y in Y such that

$$|I_Y| \ge \sum_{i=0}^{k-1}(1-p)^{i(\Delta_v(\Delta_e-1)+1)}\left(p - \frac{\Delta_v}{\delta_e}p^{\delta_e}\right)n(1-p)^{\Delta_v(\Delta_e-1)+1}.$$

The set $I_X \cup I_Y$ is an independent set in H, implying that

$$\begin{aligned}
\alpha(H) &\ge |I_X| + |I_Y| \\
&\ge np - n\left(\frac{\Delta_v}{\delta_e}\right)p^{\delta_e} + \sum_{i=0}^{k-1}(1-p)^{i(\Delta_v(\Delta_e-1)+1)}\left(p - \frac{\Delta_v}{\delta_e}p^{\delta_e}\right)n(1-p)^{\Delta_v(\Delta_e-1)+1} \\
&= n\left(p - \frac{\Delta_v}{\delta_e}p^{\delta_e}\right) + n\left(p - \frac{\Delta_v}{\delta_e}p^{\delta_e}\right)\sum_{i=0}^{k-1}(1-p)^{(i+1)(\Delta_v(\Delta_e-1)+1)} \\
&= n\left(p - \frac{\Delta_v}{\delta_e}p^{\delta_e}\right)\left(1 + \sum_{i=1}^{k}(1-p)^{(i+1)(\Delta_v(\Delta_e-1)+1)}\right) \\
&= n\left(p - \frac{\Delta_v}{\delta_e}p^{\delta_e}\right)\sum_{i=0}^{k}(1-p)^{i(\Delta_v(\Delta_e-1)+1)}.
\end{aligned}$$

This completes the proof of Theorem 3.2. □

3.2 Proof of main result

In this section, we present a proof of our main result, namely Theorem 3.1. Recall its statement.

Theorem 3.1. *If H is a hypergraph with n vertices, m edges and maximum vertex degree $\Delta_v = \Delta_v(H)$ such that every edge of H has size at least $\delta_e = \delta_e(H) > 1$, then for any integer $k \ge 0$ and reals $0 < p, q < 1$,*

$$\tau(H) \le np + m(1-p)^{\delta_e} - np^{1+\Delta_v(\Delta_e-1)}\left(q - \frac{\Delta_v}{\delta_e}q^{\delta_e}\right)\sum_{i=0}^{k}(1-q)^{i(\Delta_v(\Delta_e-1)+1)}.$$

Proof. Our proof initially follows the proof of Theorem 2.5 given in [14]. Create a subset $X \subseteq V(H)$ by choosing each vertex $v \in V(H)$ independently with probability p. Let $F = F_X \subseteq E(H)$ be the set of all edges $e \in E(H)$ that do not intersect X. For every edge $e \in E(H)$ of size k, we have

$$\Pr(e \in F_X) = (1-p)^k \leq (1-p)^{\delta_e}.$$

Let X_F be a set obtained by picking, arbitrarily, a vertex from each edge in F_X. Thus, $|X_F| \leq |F_X|$. Let X^* be the set of all vertices v of X such that $V(e) \subseteq X$ for every edge e that contains the vertex v. Let $H^* = H[X^*]$ be the sub-hypergraph induced by X^*, that is, the hypergraph with vertex set $V(H^*) = X^*$ and $E(H^*) = \{e \in E(H) \colon V(e) \subseteq X^*\}$.

Recall that $\delta_v = \delta_v(H)$, $\Delta_v = \Delta_v(H)$, $\delta_e = \delta_e(H)$, and $\Delta_e = \Delta_e(H)$. For notational simplicity, let $\delta_v^* = \delta_v(H^*)$, $\Delta_v^* = \Delta_v(H^*)$, $\delta_e^* = \delta_e(H^*)$, and $\Delta_e^* = \Delta_e(H^*)$. By construction of the hypergraph H^*, we note that $\delta_e^* \geq \delta_e$, $\Delta_e^* \leq \Delta_e$, and $\Delta_v^* \leq \Delta_v$. Let I^* be a maximum independent set of H^*, and so $|I^*| = \alpha(H^*)$. Applying Theorem 3.2 to the hypergraph H^* we have

$$\begin{aligned} |I^*| &\geq |X^*|\left(q - \frac{\Delta_v^*}{\delta_e^*}q^{\delta_e^*}\right)\sum_{i=0}^{k}(1-q)^{i(\Delta_v^*(\Delta_e^*-1)+1)} \\ &\geq |X^*|\left(q - \frac{\Delta_v}{\delta_e}q^{\delta_e}\right)\sum_{i=0}^{k}(1-q)^{i(\Delta_v(\Delta_e-1)+1)}. \end{aligned}$$

Now, $T = (X \setminus I^*) \cup X_F$ is a transversal for H. We calculate the expectation of $|T|$ as follows. We note that

$$E(|X|) = np \quad \text{and} \quad E(|X_F|) \leq E(|F_X|) \leq m(1-p)^{\delta_e},$$

as $e \in F_X$ if and only if $V(e) \cap X = \emptyset$ for arbitrary edge $e \in E(H)$. Furthermore, since $v \in X^*$ if and only if $N_H[v] \subseteq X$ and $|N_H[v]| \leq 1 + \Delta_v(\Delta_e - 1)$ for any vertex $v \in V(H)$, it follows that

$$\Pr(v \in X^*) \geq p^{1+\Delta_v(\Delta_e-1)}.$$

Thus by linearity of expectation,

$$\begin{aligned} E(|I^*|) &\geq E\left(|X^*|\left(q - \frac{\Delta_v}{\delta_e}q^{\delta_e}\right)\sum_{i=0}^{k}(1-q)^{i(\Delta_v(\Delta_e-1)+1)}\right) \\ &= \left(\left(q - \frac{\Delta_v}{\delta_e}q^{\delta_e}\right)\sum_{i=0}^{k}(1-q)^{i(\Delta_v(\Delta_e-1)+1)}\right)E(|X^*|) \\ &\geq \left(\left(q - \frac{\Delta_v}{\delta_e}q^{\delta_e}\right)\sum_{i=0}^{k}(1-q)^{i(\Delta_v(\Delta_e-1)+1)}\right)np^{1+\Delta_v(\Delta_e-1)}. \end{aligned}$$

We deduce that

$$\tau(H) \le E(|T|) \le np + m(1-p)^{\delta_e} - np^{1+\Delta_v(\Delta_e-1)}\left(q - \frac{\Delta_v}{\delta_e} q^{\delta_e}\right)\sum_{i=0}^{k}(1-q)^{i(\Delta_v(\Delta_e-1)+1)},$$

as desired, completing the proof of Theorem 3.1. □

4 Consequences of main result

In the statement of Theorem 3.1, we let

$$p = \frac{\alpha \ln \delta_e}{\delta_e} \quad \text{and} \quad q = \left(\frac{1}{\Delta_v}\right)^{\frac{1}{\delta_e - 1}},$$

noting that $\left(1 - \frac{\alpha \ln \delta_e}{\delta_e}\right)^{\delta_e} \le \frac{1}{\delta_e^{\alpha}}$, to obtain the following result which is an improvement of Theorem 2.5.

Theorem 4.1. *Let H be a hypergraph with n vertices, m edges and maximum vertex degree $\Delta_v = \Delta_v(H)$ such that every edge of H has size at least $\delta_e = \delta_e(H) > 1$ and let α be a positive real. For any integer $k \ge 0$,*

$$\begin{aligned}\tau(H) \ \le\ & n\left(\frac{\alpha \ln \delta_e}{\delta_e}\right) + \frac{m}{\delta_e^{\alpha}} \\ & - n\left(\frac{\alpha \ln \delta_e}{\delta_e}\right)^{1+\Delta_v(\Delta_e-1)}\left(\left(\frac{1}{\Delta_v}\right)^{\frac{1}{\delta_e-1}}\left(1 - \frac{1}{\delta_e}\right)\right)\sum_{i=0}^{k}\left(1 - \left(\frac{1}{\Delta_v}\right)^{\frac{1}{\delta_e-1}}\right)^{i(\Delta_v(\Delta_e-1)+1)}.\end{aligned}$$

As a consequence of Theorem 4.1, we obtain the following improvement of Corollary 2.6.

Corollary 4.2. *If H is a k-uniform hypergraph with n vertices and m edges, where $k > 1$, with maximum vertex degree Δ and minimum vertex degree δ, then for any positive real α and any integer $\ell \ge 0$,*

$$\begin{aligned}\tau(H) \ \le\ & n\left(\frac{\alpha \ln k}{k}\right) + \frac{m}{k^{\alpha}} \\ & - n\left(\frac{\alpha \ln k}{k}\right)^{1+\Delta(k-1)}\left(\left(\frac{1}{\Delta}\right)^{\frac{1}{k-1}}\left(1 - \frac{1}{k}\right)\right)\sum_{i=0}^{\ell}\left(1 - \left(\frac{1}{\Delta}\right)^{\frac{1}{k-1}}\right)^{i(\Delta(k-1)+1)}.\end{aligned}$$

From Observation 1.1 and Corollary 4.2 (letting $\alpha = 1$) we obtain the following improvement of Corollary 2.7.

Corollary 4.3. *If G is a graph with minimum degree $\delta \ge 2$ and maximum degree Δ, then*

$$\gamma_t(G) \le n\left(\frac{1+\ln \delta}{\delta}\right) - n\left(1 - \frac{1}{\delta}\right)\left(\frac{1}{\Delta}\right)^{\frac{1}{\delta-1}}\left(\frac{\ln \delta}{\delta}\right)^{1+\Delta(\delta-1)}\sum_{i=0}^{\ell}\left(1 - \left(\frac{1}{\Delta}\right)^{\frac{1}{\delta-1}}\right)^{i(\Delta(\delta-1)+1)}.$$

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(Received 16 Sep 2022; revised 21 Feb 2023)