# How many non-isospectral integral circulant graphs are there?

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### Abstract

The answer to the question in the title is contained in the following conjecture by So [*Discrete Math.* 306 (2006), 153–158]:

There are exactly $2^{\tau(n)-1}$ non-isospectral integral circulant graphs of order $n$, where $\tau(n)$ is the number of divisors of $n$.

In this paper we review some background about this conjecture, which is still open. Moreover, we affirm this conjecture for some special cases of $n$, namely, $n = p^k, pq^k, p^2q$ with primes $2 \leq p < q$ and integer $k \geq 1$; and $n = pqr$ with primes $2 \leq p < q < r$. Our approach is basically a case-by-case study, but a common technique used in the proofs of these different cases is the notion of a super sequence: a positive sequence in which each term is greater than the partial sum of all previous terms. An immediate consequence of this conjecture is a result of Klin and Kovacs [*Electron. J. Combin.* 19 (2012), #P35], which asserts that there are exactly $2^{\tau(n)-1}$ non-isomorphic integral circulant graphs of order $n$.

## 1 Background

All graphs considered in this paper are simple, i.e., undirected graphs without self loops and multi-edges. A simple graph is called *circulant* if it has a circulant adjacency matrix [3]. By definition, circulant matrices are determined by their first rows, and so each circulant graph is uniquely determined by its so-called *symbol S*.

Symbols for circulant graphs of order $n$ can be derived by the first row of the respective $n \times n$ circulant adjacency matrices and can be characterized as those subsets $S \subseteq \{1, 2, \ldots, n-1\}$ with the property that $S = n-S$, where $n-S = \{n-k : k \in S\}$.

Let $CG_n(S)$ denote a circulant graph of order $n$ with symbol $S$. The spectrum of $CG_n(S)$ [6] is the multiset

$$Sp(CG_n(S)) = \{\lambda_0(S), \lambda_1(S), \ldots, \lambda_{n-1}(S)\}$$

which can be computed as follows: for $0 \le t \le n-1$,

$$\lambda_t(S) = \sum_{j \in S} (\omega^t)^j$$

where $\omega = e^{\frac{2\pi i}{n}}$ is the $n$-th root of unity. Since $S = n - S$, each eigenvalue $\lambda_t(S)$ is a real number, and

$$\lambda_t(S) = \sum_{j \in S} (\omega^t)^j = \sum_{j \in S} (\omega^t)^{n-j} = \sum_{j \in S} (\omega^{-t})^j = \sum_{j \in S} (\omega^{n-t})^j = \lambda_{n-t}(S)$$

for $1 \le t \le n-1$. Note that $CG_n(S)$ is a regular graph of regularity

$$|S| = \sum_{j \in S} 1 = \sum_{j \in S} (\omega^0)^j = \lambda_0(S),$$

which is the largest eigenvalue because, for any $t$,

$$\lambda_t(S) \le |\lambda_t(S)| \le \sum_{j \in S} |(\omega^t)^j| = \sum_{j \in S} 1 = |S| = \lambda_0(S).$$

By Perron-Frobenius Theory, the multiplicity of $\lambda_0(S)$ is 1 if and only if $CG_n(S)$ is connected. Indeed, we can say more about the connectivity of a circulant graph [1].

**Lemma 1.1** *Let $S$ be a symbol with $\gcd(S, n) = m$. Then $CG_n(S)$ is a union of $m$ copies of the connected circulant graph $CG_{\frac{n}{m}}(\frac{1}{m}S)$, where $\frac{1}{m}S = \{\frac{k}{m} : k \in S\} \subseteq \{1, 2, \ldots, \frac{n}{m} - 1\}$. Moreover, $Sp(CG_n(S))$ consists of $m$ copies of $Sp(CG_{\frac{n}{m}}(\frac{1}{m}S))$, and so $\lambda_0(S)$ has a multiplicity $m$ in $Sp(CG_n(S))$.*

**Corollary 1.2** *If $Sp(CG_n(S)) = Sp(CG_n(T))$ has the largest eigenvalue $\lambda_0(S) = \lambda_0(T)$ of a mulitplicity $m$ then $Sp(CG_{\frac{n}{m}}(\frac{1}{m}S)) = Sp(CG_{\frac{n}{m}}(\frac{1}{m}T))$.*

**Lemma 1.3** *If $CG_{2h}(S)$ is a connected graph then the eigenvalue $\lambda_h(S)$ must have an odd multiplicity.*

*Proof:* For $1 \le t \le h-1$,
$$\lambda_t(S) = \lambda_{2h-t}(S).$$

Therefore all eigenvalues have an even multiplicity except $\lambda_0(S)$ (which has multiplicity 1 due to connectedness) and $\lambda_h(S)$ (which must have an odd multiplicity due to even order $2h$).                                                                                   $\square$

*Integral circulant graphs* (ICG) are circulant graphs with integer eigenvalues only. Each integral circulant graph comes from a special type of symbol called *integral symbol*. In [6], it is proved that there are a total of $\tau(n) - 1$ basic integral symbols $\{G_n(d) : d|n, d < n\}$, where $G_n(d) = \{k : \gcd(n, k) = d\} \subseteq \{1, 2, \ldots, n-1\}$. Note that the collection of basic integral symbols is a partition of the set $\{1, 2, \ldots, n-1\}$. The eigenvalues of $CG_n(G_n(d))$ can be computed effectively using the Euler function $\phi(\cdot)$ and the Mobius function $\mu(\cdot)$ as follows: for $0 \leq t \leq n-1$,

$$\lambda_t(G_n(d)) = \frac{\phi(n/d)}{\phi(\frac{n/d}{\gcd(t,n/d)})} \mu\left(\frac{n/d}{\gcd(t, n/d)}\right) \in \mathbf{Z}.$$

In particular, $\lambda_0(G_n(d)) = |G_n(d)| = |G_{\frac{n}{d}}(1)| = \phi(\frac{n}{d})$. These basic integral symbols are used to generate all $2^{\tau(n)-1}$ integral symbols $\cup_{d \in D} G_n(d)$ where $D \subseteq \{d : d|n, d < n\}$ is a subset of proper divisors of $n$. From now on, we denote by $ICG_n(D)$ the integral circulant graph of order $n$ with integral symbol $\cup_{d \in D} G_n(d)$, i.e., $ICG_n(D) = CG_n(\cup_{d \in D} G_n(d))$, and its eigenvalues are given by, $0 \leq t \leq n-1$,

$$\lambda_t(D) = \lambda_t(\cup_{d \in D} G_n(d)) = \sum_{d \in D} \frac{\phi(n/d)}{\phi(\frac{n/d}{\gcd(t,n/d)})} \mu\left(\frac{n/d}{\gcd(t, n/d)}\right) \in \mathbf{Z}.$$

Complement graphs are useful tools; in particular, we use them heavily to simplify the proof of Theorem 5.4. First, we notice that the complement graph of $CG_n(S)$ is also circulant and indeed $\overline{CG_n(S)} = CG_n(\overline{S})$ where $\overline{S}$ denotes the complement of $S$ with respect to the set $\{1, 2, \ldots, n-1\}$. The complement graph of $ICG_n(D)$ is also an integral circulant graph and indeed $\overline{ICG_n(D)} = ICG_n(D^*)$ where $D^*$ is the complement of $D$ with respect to the set $\{d : d|n, d < n\}$. Moreover, we have the following.

**Lemma 1.4** (i) *If* $Sp(CG_n(S)) = Sp(CG_n(T))$ *then* $Sp(CG_n(\overline{S})) = Sp(CG_n(\overline{T}))$.
  (ii) *If* $Sp(ICG_n(D_S)) = Sp(ICG_n(D_T))$ *then* $Sp(ICG_n(D_S^*)) = Sp(CG_n(D_T^*))$.

*Proof:*  (i) Since $CG_n(S)$ and $CG_n(T)$ are regular graphs, $Sp(\overline{CG_n(S)})$ and $Sp(\overline{CG_n(T)})$ are determined by $Sp(CG_n(S))$ and $Sp(CG_n(T))$ respectively. Hence $Sp(CG_n(S)) = Sp(CG_n(T))$ implies that

$$Sp(\overline{CG_n(S)}) = Sp(\overline{CG_n(T)})$$

and so

$$Sp(CG_n(\overline{S})) = Sp(CG_n(\overline{T})).$$

(ii) Since $ICG_n(D_S)$ and $ICG_n(D_T)$ are regular graphs, $Sp(\overline{ICG_n(D_S)})$ and $Sp(\overline{ICG_n(D_T)})$ are determined by $Sp(ICG_n(D_S))$ and $Sp(ICG_n(D_T))$ respectively. Hence $Sp(ICG_n(D_S)) = Sp(ICG_n(D_T))$ implies that

$$Sp(\overline{ICG_n(D_S)}) = Sp(\overline{ICG_n(D_T)})$$

and so

$$Sp(ICG_n(D_S^*)) = Sp(ICG_n(D_T^*)).$$

$\square$

We adopt the convention that $ICG_n(\emptyset)$ is the empty graph with all eigenvalues equal to 0. In theory, there should be $2^{\tau(n)-1}$ integral circulant graphs, one from each integral symbol. However, there is a catch: we need to make sure that it is impossible that two different integral symbols produce isomorphic integral circulant graphs. Note that it is possible to have two different symbols producing isomorphic circulant graphs; fortunately they are *not* integral symbols. Indeed, Klin and Kovacs [4] proved that different integral symbols always produce non-isomorphic integral circulant graphs by using the techniques from Schur ring and group theory. Hence we know that there are exactly $2^{\tau(n)-1}$ non-isomorphic integral circulant graphs of order $n$. However, this result provides mere evidence but no affirmation for the conjecture stated in the abstract because it is possible for non-isomorphic graphs to be isospectral. Note that if we can confirm the conjecture by other means then we potentially have an alternate proof for the result of Klin and Kovacs.

As far as we know, the conjecture is still open. In the rest of the paper, we affirm the conjecture for the following special cases of $n$:

- $n = p^k$ with prime $p$ and integer $k \geq 1$;
- $n = pq^k$ with primes $p < q$ and integer $k \geq 1$;
- $n = p^2q$ with primes $p < q$;
- $n = pqr$ with primes $p < q < r$.

To confirm the conjecture, we need to show that two different integral symbols produce integral circulant graphs with different spectra (as multisets), or contrapositively, two integral symbols producing the same spectrum must be identical. That is,

$$Sp(ICG_n(D_S)) = Sp(ICG_n(D_T)) \text{ implies } D_S = D_T$$

where $D_S$ and $D_T$ are subsets of $\{d : d|n, d < n\}$ because the corresponding integral symbols are equal: $\bigcup_{d \in D_S} G_n(d) = \bigcup_{d \in D_T} G_n(d)$. A common technique employed in our proofs is the notion of super sequences.

**Definition 1.5** A finite sequence $(x_0, x_1, x_2, \ldots, x_J)$ of nonnegative numbers is called a *super sequence* if $x_t > \sum_{j=0}^{t-1} x_j$ for any $1 \leq t \leq J$.

**Lemma 1.6** *Let $p \geq 2$. Then the sequence $(1, p, p^2, \ldots, p^J)$ is a super sequence, and so is $(p - 1, p^2 - p, p^3 - p^2, \ldots, p^{J+1} - p^J)$.*

The following lemma shows an interesting property that all partial sums of a super sequence are distinct.

**Lemma 1.7** *Let $(x_0, x_1, x_2, \ldots, x_J)$ be a super sequence. If $a_j$, $b_j \in \{0, 1\}$ for all $0 \leq j \leq J$ such that $\sum_{j=0}^{J} a_j x_j = \sum_{j=0}^{J} b_j x_j$, then $a_j = b_j$ for all $0 \leq j \leq J$. In other words, if $x^T = [x_0, x_1, x_2, \ldots, x_J]$ is a vector such that $x^T a = x^T b$ for some $(0, 1)$-vectors $a, b \in \{0, 1\}^{J+1}$, then $a = b$.*

*Proof:* Assume that there exists $j$ such that $a_j \neq b_j$. Then the set $R = \{0 \leq j \leq J : a_j \neq b_j\}$ is non-empty and let $m$ be the largest element of $R$. Suppose without loss of generality that $a_m = 1$ and $b_m = 0$. Then we have the contradiction:

$$
\begin{aligned}
\sum_{j=0}^{J} a_j x_j &= \sum_{j=0}^{m} a_j x_j + \sum_{j=m+1}^{J} a_j x_j \\
&= \sum_{j=0}^{m} a_j x_j + \sum_{j=m+1}^{J} b_j x_j \text{ (since $m$ is the largest element of $R$)} \\
&\geq x_m + \sum_{j=m+1}^{J} b_j x_j \text{ (since $a_m = 1$ and $x_j \geq 0$ )} \\
&> \sum_{j=0}^{m-1} x_j + \sum_{j=m+1}^{J} b_j x_j \text{ (due to the hypothesis on $x_j$)} \\
&\geq \sum_{j=0}^{m} b_j x_j + \sum_{j=m+1}^{J} b_j x_j \text{ (since $b_m = 0, b_j \leq 1$)} \\
&= \sum_{j=0}^{J} b_j x_j.
\end{aligned}
$$

$\square$

**Lemma 1.8** *Let $d_1 = 1, d_2, \ldots, d_J < n$ be all the proper divisors of $n$ such that*

$$
\left( \phi(\frac{n}{d_J}), \phi(\frac{n}{d_{J-1}}), \ldots, \phi(\frac{n}{d_1}) \right)
$$

*is a super sequence. If $D_S$ and $D_T$ are subsets of $\{d_1, \ldots, d_J\}$ such that*

$$
Sp(ICG_n(D_S)) = Sp(ICG_n(D_T))
$$

*then $D_S = D_T$.*

*Proof:* Since $Sp(ICG_n(D_S)) = Sp(ICG_n(D_T))$, we have $\lambda_0(D_S) = \lambda_0(D_T)$. Hence

$$\sum_{d \in D_S} \phi(\frac{n}{d}) = \sum_{d \in D_S} |G_n(d)| = \sum_{d \in D_T} |G_n(d)| = \sum_{d \in D_T} \phi(\frac{n}{d}).$$

Because of the super sequence hypothesis and Lemma 1.7, we conclude that $D_S = S_T$.
$\square$

The following lemma from [5] is needed in Sections 4 and 5.

**Lemma 1.9** *Let $p$ be an odd prime which is a proper divisor of $n$. If*

$$Sp(ICG_n(D_S)) = Sp(ICG_n(D_T))$$

*then $\frac{n}{p} \notin (D_S - D_T)$, where $D_S - D_T = \{k : k \in D_S \text{ and } k \notin D_T\}$.*

## 2  The case $n = p^k$ with prime $p$ and integer $k \geq 1$

In [6], So not only proposed the conjecture for all $n$ but also reported (without proof) that the conjecture is true for a prime power $n$. In this section, we give the proof of such assertion.

**Theorem 2.1** *Let $D_S$ and $D_T$ be two subsets of proper divisors of $p^k$ with prime $p$ and integer $k \geq 1$. If $Sp(ICG_{p^k}(D_S)) = Sp(ICG_{p^k}(D_T))$ then $D_S = D_T$.*

*Proof:* The proper divisors of $n = p^k$ are $1, p, p^2, \ldots, p^{k-1}$, and so

$$\left(\phi(\frac{n}{p^{k-1}}), \phi(\frac{n}{p^{k-2}}), \ldots, \phi(\frac{n}{1})\right) = \left(\phi(p), \phi(p^2), \ldots, \phi(p^k)\right)$$
$$= (p - 1, p^2 - p, \ldots, p^k - p^{k-1})$$

is a super sequence by Lemma 1.6. Hence $D_S = D_T$ by Lemma 1.8.     $\square$

## 3  The case $n = pq^k$ with primes $p < q$ and integer $k \geq 1$

In [6], So also reported (without proof) that the conjecture is true for $n$ being a product of two distinct primes. In this section, we extend this result to $n$ of the form $pq^k$ where $p < q$ are distinct primes and $k \geq 1$ is an integer. We consider the cases $p > 2$ and $p = 2$ in Theorems 3.2 and 3.3 respectively. Both theorems were originally published in the Master's thesis of Chris Cusanza [2]. However, we present different proofs using super sequences.

**Lemma 3.1** *For primes $p, q$ such that $2 < p < q$ and integer $k \geq 1$, define*

$$x_j = \begin{cases} p - 1 & \text{if } j = 0, \\ (q-1)q^{\frac{j+1}{2}-1} & \text{if } j = 1, 3, \ldots, 2k - 1, \\ (p-1)(q-1)q^{\frac{j}{2}-1} & \text{if } j = 2, 4, \ldots, 2k. \end{cases}$$

*Then the sequence $(x_0, x_1, x_2, \ldots, x_{2k}) =$*

$$\left(p - 1, q - 1, (p-1)(q-1), \ldots, (p-1)(q-1)q^{k-1}\right)$$

*is a super sequence.*

*Proof:* We prove that $x_t > \sum_{j=0}^{t-1} x_j$ for $t = 1, \ldots, 2k$ by induction on $t$.

For $t = 1$, $x_1 = q - 1 > p - 1 = x_0 = \sum_{j=0}^{1-1} x_j$ since $p < q$.

Consider $t \geq 2$. If $t$ is even, then

$$\begin{aligned} \sum_{j=0}^{t} x_j &= x_t + x_{t-1} + \sum_{j=0}^{t-2} x_j \\ &< x_t + x_{t-1} + x_{t-1} \quad \text{(by induction hypothesis)} \\ &= (p-1)(q-1)q^{\frac{t}{2}-1} + 2(q-1)q^{\frac{t}{2}-1} \\ &= (p+1)(q-1)q^{\frac{t}{2}-1} \\ &\leq (q-1)q^{\frac{t+2}{2}-1} \quad \text{(since } p < q) \\ &= x_{t+1}. \end{aligned}$$

Similarly, if $t$ is odd, then

$$\begin{aligned} \sum_{j=0}^{t} x_j &= x_t + \sum_{j=0}^{t-1} x_j \\ &< x_t + x_t \quad \text{(by induction hypothesis)} \\ &= 2(q-1)q^{\frac{t+1}{2}-1} \\ &\leq (p-1)(q-1)q^{\frac{t+1}{2}-1} \quad \text{(since } 2 < p) \\ &= x_{t+1}. \end{aligned}$$

$\square$

**Theorem 3.2** *Let $D_S$ and $D_T$ be two subsets of proper divisors of $pq^k$ with primes $p, q$ such that $2 < p < q$ and integer $k \geq 1$. If $Sp(ICG_{pq^k}(D_S)) = Sp(ICG_{pq^k}(D_T))$ then $D_S = D_T$.*

*Proof:* The order $pq^k$ has $2k + 1$ proper divisors $\{1, q, \ldots, q^k, p, pq, \ldots, pq^{k-1}\}$. Let us label the proper divisors as follows:

$$d_j = \begin{cases} q^k & \text{if } j = 0, \\ pq^{k-\frac{j+1}{2}} & \text{if } j = 1, 3, \ldots, 2k - 1, \\ q^{k-\frac{j}{2}} & \text{if } j = 2, 4, \ldots, 2k. \end{cases}$$

Consequently, for $0 \leq j \leq 2k$, we have $\phi(\frac{pq^k}{d_j}) = x_j$, which is a member of the super sequence defined in Lemma 3.1. Hence $D_S = D_T$ by Lemma 1.8. $\qquad\square$

**Theorem 3.3** *Let $D_S$ and $D_T$ be two subsets of proper divisors of order $2q^k$ with a prime $q > 2$ and an integer $k \geq 1$. If $Sp(ICG_{2q^k}(D_S)) = Sp(ICG_{2q^k}(D_T))$ then $D_S = D_T$.*

*Proof:* Let $m$ be the multiplicity of $\lambda_0(D_S) = \lambda_0(D_T)$. Then $m|2q^k$ and so we have the following five cases.

**Case 1:**　$m = 1$. That is, (i) $\lambda_0(D_S) = \lambda_0(D_T)$ has multiplicity 1. By Lemma 1.3, both $\lambda_{q^k}(D_S)$ and $\lambda_{q^k}(D_T)$ have odd multiplicity. From the hypothesis

$$Sp(ICG_{2q^k}(D_S)) = Sp(ICG_{2q^k}(D_T)),$$

we have (ii) $\lambda_{q^k}(D_S) = \lambda_{q^k}(D_T)$.

Now the order $2q^k$ has $2k+1$ proper divisors $\{1, q, \ldots, q^k, 2, 2q, \ldots, 2q^{k-1}\}$. Let us label the proper divisors as follows:

$$d_j = \begin{cases} q^k & \text{if } j = 0, \\ q^{k-j} & \text{if } j = 1, 2, \ldots, k, \\ 2q^{2k-j} & \text{if } j = k+1, k+2, \ldots, 2k. \end{cases}$$

Since both $D_S, D_T$ are subsets of $\{1, q, \ldots, q^k, 2, 2q, \ldots, 2q^{k-1}\}$, for $0 \leq j \leq 2k$, define $a_j = 1$ if $d_j \in D_S$, $a_j = 0$ otherwise. Similarly, define $b_j$ using $D_T$. Note that $D_S = D_T$ if and only if $a_j = b_j$ for all $0 \leq j \leq 2k$.

Note that $\lambda_0(D_S) = \sum_{j=0}^{2k} a_j |G_{pq^k}(d_j)| = \sum_{j=0}^{2k} a_j \phi(\frac{pq^k}{d_j}) = a_0 + Qx_a + Qy_a$ where $Q = [q-1 \ (q-1)q \ \cdots (q-1)q^{k-1}]$, $x_a = [a_1, \ldots, a_k]^T$ and $y_a = [a_{k+1}, \ldots, a_{2k}]^T$. Similarly, $\lambda_0(D_T) = b_0 + Qx_b + Qy_b$ where $x_b = [b_1, \ldots, b_k]^T$ and $y_b = [b_{k+1}, \ldots, b_{2k}]^T$. Hence, from (i),

$$a_0 + Qx_a + Qy_a = b_0 + Qx_b + Qy_b.$$

Note that

$$\begin{aligned} \lambda_{q^k}(D_S) &= \sum_{d \in D_S} \sum_{i \in G_{2q^k}(d)} (\omega^{q^k})^i \\ &= \sum_{j=0}^{2k} a_j \sum_{i \in G_{2q^k}(d_j)} (-1)^i \\ &= -a_0 - Qx_a + Qy_a. \end{aligned}$$

Similarly, $\lambda_{q^k}(D_T) = -b_0 - Qx_b + Qy_b$. Hence, from (ii),

$$-a_0 - Qx_a + Qy_a = -b_0 - Qx_b + Qy_b.$$

Consequently, by adding the two equations and then simplifying, we have $Qy_a = Qy_b$. By Lemma 1.7 (because the sequence of entries from $Q$ is a super sequence

by Lemma 1.6), $y_a = y_b$. It follows that $a_0 + Qx_a = b_0 + Qx_b$. By Lemma 1.7 (because the sequence of entries from $Q$ with an additional 1 in the beginning is a super sequence by Lemma 1.6), $a_0 = b_0$ and $x_a = x_b$. Finally $[a_0 \ x_a \ y_a] = [b_0 \ x_b \ y_b]$ and so $D_S = D_T$.

**Case 2:** $m = 2$. By Corollary 1.2, $Sp(ICG_{q^k}(\frac{1}{2}D_S)) = Sp(ICG_{q^k}(\frac{1}{2}D_T))$. Now, by Theorem 2.1, $\frac{1}{2}D_S = \frac{1}{2}D_T$, and so $D_S = D_T$.

**Case 3:** $m = q^r$ with $k > r \geq 1$. By Corollary 1.2, $Sp(ICG_{2q^{k-r}}(\frac{1}{q^r}D_S)) = Sp(ICG_{2q^{k-r}}(\frac{1}{q^r}D_T))$. Moreover the multiplicity of $\lambda_0(\frac{1}{q^r}D_S) = \lambda_0(\frac{1}{q^r}D_T)$ is 1. By Case 1, $\frac{1}{q^r}D_S = \frac{1}{q^r}D_T$, and so $D_S = D_T$.

**Case 4:** $m = q^k$. By Corollary 1.2, $Sp(ICG_2(\frac{1}{q^k}D_S)) = Sp(ICG_2(\frac{1}{q^k}D_T))$. Now, by Theorem 2.1, $\frac{1}{q^k}D_S = \frac{1}{q^k}D_T$, and so $D_S = D_T$.

**Case 5:** $m = 2q^r$ with $k > r \geq 1$. By Corollary 1.2, $Sp(ICG_{q^{k-r}}(\frac{1}{2q^r}D_S)) = Sp(ICG_{q^{k-r}}(\frac{1}{2q^r}D_T))$. Now, by Theorem 2.1, $\frac{1}{2q^r}D_S = \frac{1}{2q^r}D_T$, and so $D_S = D_T$. $\quad\square$

## 4 The case $n = p^2q$ with primes $p < q$

The results in this section also appeared in [2] and [5], but both had different proofs than ours. We consider three cases depending on the values of $p$ and $q$: (i) $p = 2$ and $q = 3$, (ii) $p = 2$ and $q > 3$, (iii) $2 < p < q$.

**Theorem 4.1** *Let $D_S$ and $D_T$ be two subsets of proper divisors of $2^2 \cdot 3 = 12$. If $Sp(ICG_{12}(D_S)) = Sp(ICG_{12}(D_T))$ then $D_S = D_T$.*

*Proof:* The order 12 has five proper divisors $\{1, 2, 3, 4, 6\}$. Hence there are five basic integral symbols: $G_{12}(1), G_{12}(2), G_{12}(3), G_{12}(4), G_{12}(6)$, and so $2^5 = 32$ integral symbols. Using the formulas in Section 1, we can compute the spectra of all 32 integral circulant graphs explicitly and see that they are all different multisets. $\quad\square$

**Theorem 4.2** *Let $D_S$ and $D_T$ be two subsets of proper divisors of $2^2q = 4q$ with odd prime $q > 3$. If $Sp(ICG_{4q}(D_S)) = Sp(ICG_{4q}(D_T))$ then $D_S = D_T$.*

*Proof:* The order $4q$ has five proper divisors $\{1, 2, 4, q, 2q\}$ with an odd prime $q > 3$. Hence there are five basic integral symbols:

$$G_{4q}(1), \ G_{4q}(2), \ G_{4q}(4), \ G_{4q}(q), \ G_{4q}(2q).$$

By Lemma 1.9, $D_S - D_T \subseteq \{1, 2, q, 2q\}$ and so $|\bigcup_{d \in D_S - D_T} G_{4q}(d)| = a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4$ for some $a_i \in \{0, 1\}$ and $x_1 = |G_{4q}(2q)| = 1$, $x_2 = |G_{4q}(q)| = 2$, $x_3 = |G_{4q}(2)| = q - 1$, $x_4 = |G_{4q}(1)| = 2(q - 1)$. Similarly, $|\bigcup_{d \in D_T - D_S} G_{4q}(d)| = b_1x_1 + b_2x_2 + b_3x_3 + b_4x_4$ for some $b_i \in \{0, 1\}$. Since $Sp(ICG_{4q}(D_S)) = Sp(ICG_{4q}(D_T))$, we have

$$\left| \bigcup_{d \in D_S} G_{4q}(d) \right| = \lambda_0(D_S) = \lambda_0(D_T) = \left| \bigcup_{d \in D_T} G_{4q}(d) \right|,$$

and so

$$\sum_i a_i x_i = \Big| \bigcup_{d \in D_S - D_T} G_{4q}(d) \Big| = \Big| \bigcup_{d \in D_T - D_S} G_{4q}(d) \Big| = \sum_i b_i x_i.$$

Note that $x_1 < x_2 < x_3 < x_4$ is a super sequence because $q \geq 5$. By Lemma 1.7, $a_i = b_i$ for all $i$, i.e., $D_S - D_T = D_T - D_S$. Since $(D_S - D_T) \cap (D_T - D_S) = \emptyset$, we have $D_S - D_T = D_T - D_S = \emptyset$. Consequently, $D_S = D_T$. □

**Theorem 4.3** *Let $D_S$ and $D_T$ be two subsets of proper divisors of $p^2 q$ with primes $p, q$ such that $2 < p < q$. If $Sp(ICG_{p^2 q}(D_S)) = Sp(ICG_{p^2 q}(D_T))$ then $D_S = D_T$.*

*Proof:* The order $p^2 q$ has five proper divisors $\{1, p, p^2, q, pq\}$ with primes $p, q$ such that $2 < p < q$. Hence there are five basic integral symbols:

$$G_{p^2 q}(1), \ G_{p^2 q}(p), \ G_{p^2 q}(p^2), \ G_{p^2 q}(q), \ G_{p^2 q}(pq).$$

By Lemma 1.9, $D_S - D_T \subseteq \{1, p, q\}$ and so $\Big| \bigcup_{d \in D_S - D_T} G_{p^2 q}(d) \Big| = a_1 x_1 + a_2 x_2 + a_3 x_3$ for some $a_i \in \{0, 1\}$ and $x_1 = |G_{p^2 q}(q)| = p(p-1)$, $x_2 = |G_{p^2 q}(p)| = (p-1)(q-1)$, $x_3 = |G_{p^2 q}(1)| = p(p-1)(q-1)$. Similarly, $\Big| \bigcup_{d \in D_T - D_S} G_{p^2 q}(d) \Big| = b_1 x_1 + b_2 x_2 + b_3 x_3$ for some $b_i \in \{0, 1\}$. Since $Sp(ICG_{p^2 q}(D_S)) = Sp(ICG_{p^2 q}(D_T))$, we have

$$\Big| \bigcup_{d \in D_S} G_{p^2 q}(d) \Big| = \lambda_0(D_S) = \lambda_0(D_T) = \Big| \bigcup_{d \in D_T} G_{p^2 q}(d) \Big|,$$

and so

$$\sum_i a_i x_i = \Big| \bigcup_{d \in D_S - D_T} G_{p^2 q}(d) \Big| = \Big| \bigcup_{d \in D_T - D_S} G_{p^2 q}(d) \Big| = \sum_i b_i x_i.$$

Note that $x_1 < x_2 < x_3$ is a super sequence because $2 < p < q$. By Lemma 1.7, $a_i = b_i$ for all $i$, i.e., $D_S - D_T = D_T - D_S$. Since $(D_S - D_T) \cap (D_T - D_S) = \emptyset$, we have $D_S - D_T = D_T - D_S = \emptyset$. Consequently, $D_S = D_T$. □

## 5   The case $n = pqr$ with primes $p < q < r$

The results in this section are new. We give the complete proof when $n = pqr$ with primes $p, q, r$ such that $2 < p < q < r$. Since the proof for $n = pqr$ with primes $2 = p < q < r$ are similar except for some minor differences, we omit its inclusion.

The set of proper divisors of $n = pqr$ is $\{1, p, q, r, pq, pr, qr\}$. Hence there are seven basic integral symbols:

$$G_n(1), \ G_n(p), \ G_n(q), \ G(r), \ G_n(pq), \ G_n(pr), \ G_n(qr).$$

Let $x_1 = |G_n(r)| = (p-1)(q-1), x_2 = |G_n(q)| = (p-1)(r-1), x_3 = |G_n(p)| = (q-1)(r-1), x_4 = |G_n(1)| = (p-1)(q-1)(r-1)$. Then $x_1 < x_2 < x_3 < x_4$ because $2 < p < q < r$.

We prove our result in two cases according to the values of $p, q, r$:

(i) $(q - p)(r - p) > (p - 1)^2$, i.e., $x_1 + x_2 < x_3$,
(ii) $(q - p)(r - p) \leq (p - 1)^2$, i.e., $x_1 + x_2 \geq x_3$.

**Theorem 5.1** *Let $D_S$ and $D_T$ be two subsets of proper divisors of $pqr$ with primes $p, q, r$ such that $2 < p < q < r$ and $(q - p)(r - p) > (p - 1)^2$. If $Sp(ICG_{pqr}(D_S)) = Sp(ICG_{pqr}(D_T))$ then $D_S = D_T$.*

*Proof:* Since $Sp(ICG_{pqr}(D_S)) = Sp(ICG_{pqr}(D_T))$, by Lemma 1.9, $D_S - D_T \subseteq \{1, p, q, r\}$ and so $|\bigcup_{d \in D_S - D_T} G_{pqr}(d)| = a_1 x_1 + a_2 x_2 + a_3 x_3 + a_4 x_4$ for some $a_i \in \{0, 1\}$. Similarly, $|\bigcup_{d \in D_T - D_S} G_{pqr}(d)| = b_1 x_1 + b_2 x_2 + b_3 x_3 + b_4 x_4$ for some $b_i \in \{0, 1\}$. Moreover, we have

$$\Big| \bigcup_{d \in D_S} G_{pqr}(d) \Big| = \lambda_0(D_S) = \lambda_0(D_T) = i \Big| \bigcup_{d \in D_T} G_{pqr}(d) \Big|,$$

and so

$$\sum_i a_i x_i = \Big| \bigcup_{d \in D_S - D_T} G_{pqr}(d) \Big| = \Big| \bigcup_{d \in D_T - D_S} G_{pqr}(d) \Big| = \sum_i b_i x_i.$$

Note that $x_1 < x_2 < x_3 < x_4$ is a super sequence because $2 < p < q < r$ are primes such that $(q - p)(r - p) > (p - 1)^2$, i.e., $x_1 + x_2 < x_3$. By Lemma 1.7, $a_i = b_i$ for all $i$, i.e., $D_S - D_T = D_T - D_S$. Since $(D_S - D_T) \cap (D_T - D_S) = \emptyset$, we have $D_S - D_T = D_T - D_S = \emptyset$. Consequently, $D_S = D_T$. $\qquad\square$

**Lemma 5.2** *Let $D_S$ and $D_T$ be two subsets of proper divisors of $pqr$ with primes $p, q, r$ such that $2 < p < q < r$ and $(q - p)(r - p) \leq (p - 1)^2$. If $Sp(ICG_{pqr}(D_S)) = Sp(ICG_{pqr}(D_T))$ and $D_S \neq D_T$ with $|D_S| \leq |D_T|$ then $(q - p)(r - p) = (p - 1)^2$, and*

$$D_S - D_T = \{p\}, \quad D_T - D_S = \{q, r\}.$$

*Proof:* Since $Sp(ICG_{pqr}(D_S)) = Sp(ICG_{pqr}(D_T))$, by Lemma 1.9, $D_S - D_T, D_T - D_S \subseteq \{1, p, q, r\}$. Moreover, let $S = \bigcup_{d \in D_S} G_{pqr}(d)$ and $T = \bigcup_{d \in D_T} G_{pqr}(d)$, then

$$|S| = \Big| \bigcup_{d \in D_S} G_{pqr}(d) \Big| = \lambda_0(D_S) = \lambda_0(D_T) = \Big| \bigcup_{d \in D_T} G_{pqr}(d) \Big| = |T|,$$

and so

$$|S - T| = \Big| \bigcup_{d \in D_S - D_T} G_{pqr}(d) \Big| = \Big| \bigcup_{d \in D_T - D_S} G_{pqr}(d) \Big| = |T - S|.$$

Since $D_S \neq D_T$, both $S - T$ and $T - S$ are not empty. Using $|D_S| \leq |D_T|$, we have the following cases to consider.

(i) $|S - T| = x_i$ and $|T - S| = x_j$ where $i, j$ are distinct.

Hence $x_i = |S - T| = |T - S| = x_j$, which is impossible because $x_1 < x_2 < x_3 < x_4$.

(ii) $|S - T| = x_i$ and $|T - S| = x_j + x_k$ where $i, j, k$ are distinct.

    (a) $x_4 = |S - T| = |T - S| = x_1 + x_2$
        Hence $(p - 1)(q - 1)(r - 1) = (p - 1)(q - 1) + (p - 1)(r - 1)$, and so $(q - 1)(r - 1) = q - 1 + r - 1$, i.e., $(q - 2)(r - 2) = 1$, which is impossible because $2 < p < q < r$.

    (b) $x_4 = |S - T| = |T - S| = x_1 + x_3$
        Hence $(p - 1)(q - 1)(r - 1) = (p - 1)(q - 1) + (q - 1)(r - 1)$, and so $(p - 1)(r - 1) = p - 1 + r - 1$, i.e., $(p - 2)(r - 2) = 1$, which is impossible because $2 < p < q < r$.

    (c) $x_4 = |S - T| = |T - S| = x_2 + x_3$
        Hence $(p - 1)(q - 1)(r - 1) = (p - 1)(r - 1) + (q - 1)(r - 1)$, and so $(p - 1)(q - 1) = p - 1 + q - 1$, i.e., $(p - 2)(q - 2) = 1$, which is impossible because $2 < p < q < r$.

    (d) $x_3 = |S - T| = |T - S| = x_1 + x_2$
        This is the only remaining possible subcase, and $(q - p)(r - p) = (p - 1)^2$. Hence $S - T = G_{pqr}(p)$ and $T - S = G_{pqr}(q) \cup G_{pqr}(r)$, i.e., $D_S - D_T = \{p\}$ and $D_T - D_S = \{q, r\}$.

(iii) $|S - T| = x_i$ and $|T - S| = x_j + x_k + x_l$ where $i, j, k, l$ are distinct.

    Because $|S - T| = |T - S|$ and $x_1 < x_2 < x_3 < x_4$, we must have $x_4 = |S - T| = |T - S| = x_1 + x_2 + x_3$. Since $(q - p)(r - p) \leq (p - 1)^2$, i.e., $x_1 + x_2 \geq x_3$,

$$(p - 1)(q - 1)(r - 1) = x_4 = x_1 + x_2 + x_3 \leq 2x_1 + 2x_2 = 2(p - 1)(q + r - 2)$$

    and so $(q - 1)(r - 1) \leq 2(q + r - 2)$, i.e., $4 \geq (q - 3)(r - 3)$, which is impossible because $2 < p < q < r$.

(iv) $|S - T| = x_i + x_j$ and $|T - S| = x_k + x_l$ where $i, j, k, l$ are distinct.

    Because $|S - T| = |T - S|$ and $x_1 < x_2 < x_3 < x_4$, we must have $x_1 + x_4 = |S - T| = |T - S| = x_2 + x_3$. Hence $(r - 1)(p + q - 2) = (p - 1)(r - 1) + (q - 1)(r - 1) = x_2 + x_3 = x_1 + x_4 > x_4 = (p - 1)(q - 1)(r - 1) > (r - 1)(p + q - 2)$, which is impossible. $\square$

**Lemma 5.3** *We list the spectra of several integral circulant graphs of order $pqr$.*

- *For $ICG_{pqr}(\{p\})$,*

$$\lambda_i(\{p\}) = \begin{cases} (q - 1)(r - 1) & \textit{if } qr|i, \\ -(q - 1) & \textit{if } q|i \textit{ and } qr \nmid i, \\ -(r - 1) & \textit{if } r|i \textit{ and } qr \nmid i, \\ 1 & \textit{otherwise}. \end{cases}$$

- *For $ICG_{pqr}(\{q,r\})$,*

$$
\lambda_i(\{q,r\}) =
\begin{cases}
(p-1)(q+r-2) & \text{if } pqr|i, \quad i.e. \ i=0, \\
(p-1)q & \text{if } pq|i \text{ and } i>0, \\
(p-1)r & \text{if } pr|i \text{ and } i>0, \\
-(q-1)(r-1) & \text{if } qr|i \text{ and } i>0, \\
-2(p-1) & \text{if only } p|i, \\
-(q-1)+1=2-q & \text{if only } q|i, \\
-(r-1)+1=2-r & \text{if only } r|i, \\
2, & \text{otherwise.}
\end{cases}
$$

- *For $ICG_{pqr}(\{1\})$,*

$$
\lambda_i(\{1\}) =
\begin{cases}
(p-1)(q-1)(r-1) & \text{if } pqr|i, \ i.e. \ i=0, \\
-(p-1)(q-1) & \text{if } pq|i \text{ and } i>0, \\
-(p-1)(r-1) & \text{if } pr|i \text{ and } i>0, \\
-(q-1)(r-1) & \text{if } qr|i \text{ and } i>0, \\
p-1 & \text{if only } p|i, \\
q-1 & \text{if only } q|i, \\
r-1 & \text{if only } r|i, \\
-1 & \text{otherwise.}
\end{cases}
$$

- *For $ICG_{pqr}(\{pq\})$,*

$$
\lambda_i(\{pq\}) =
\begin{cases}
r-1 & \text{if } r|i, \\
-1 & \text{otherwise.}
\end{cases}
$$

- *For $ICG_{pqr}(\{pr\})$,*

$$
\lambda_i(\{pr\}) =
\begin{cases}
q-1 & \text{if } q|i, \\
-1 & \text{otherwise.}
\end{cases}
$$

- *For $ICG_{pqr}(\{qr\})$,*

$$
\lambda_i(\{qr\}) =
\begin{cases}
p-1 & \text{if } p|i, \\
-1 & \text{otherwise.}
\end{cases}
$$

**Theorem 5.4** *Let $D_S$ and $D_T$ be two subsets of proper divisors of $pqr$ with primes $p,q,r$ such that $2 < p < q < r$ and $(q-p)(r-p) \leq (p-1)^2$. If $Sp(ICG_{pqr}(D_S)) = Sp(ICG_{pqr}(D_T))$ then $D_S = D_T$.*

*Proof:* Without loss of generality, let $|D_S| \leq |D_T|$. Assume the contrary that $D_S \neq D_T$. Then, by Lemma 5.2, $(q-p)(r-p) = (p-1)^2$, and $D_S - D_T = \{p\}$, $D_T - D_S = \{q,r\}$. Hence, $p \geq 5$ and

$$D_S = \{p\} \cup D, \qquad D_T = \{q,r\} \cup D$$

where $D \subseteq \{1, pq, pr, qr\}$. We have five cases to consider according to the cardinality of $D$.

**Case 1:**   $|D| = 0$.

Then $D_S = \{p\}$ and $D_T = \{q, r\}$. From Lemma 5.3, $1 \in Sp(ICG_{pqr}(D_S))$, but $1 \notin Sp(ICG_{pqr}(D_T))$ because $2 < p < q < r$. Hence $Sp(ICG_{pqr}(D_S)) \neq Sp(ICG_{pqr}(D_T))$, a contradiction!

**Case 2:**   $|D| = 1$.

Then $D_S = \{p, y\}$ and $D_T = \{q, r, y\}$ for some $y \in \{1, pq, pr, qr\}$. From Lemma 5.3, we have $1 = 2 + (-1) = \lambda_1(\{q, r\}) + \lambda_1(\{y\}) = \lambda_1(D_T)$. However, again by Lemma 5.3, $1 \neq \lambda_i(\{p\}) + \lambda_i(\{y\})$ for any $i$ because $5 \leq p < q < r$. Hence $Sp(ICG_{pqr}(D_S)) \neq Sp(ICG_{pqr}(D_T))$, a contradiction!

**Case 3:** $|D| = 2$.

**Subcase 3.1:** $D = \{pq, pr\}$.

Then $D_S = \{p, pq, pr\}$ and $D_T = \{q, r, pq, pr\}$. By Lemma 5.3, $\lambda_p(D_T) = \lambda_p(\{q, r\}) + \lambda_p(\{pq\}) + \lambda_p(\{pr\}) = -2(p-1) + (-1) + (-1) = -2p$, but $\lambda_i(D_S) = (q-1)(r-1) + (q-1) + (r-1) > 0$ if $qr|i$, $-1$ otherwise. Hence $-2p \notin Sp(ICG_{pqr}(D_S))$, and so $Sp(ICG_{pqr}(D_S)) \neq Sp(ICG_{pqr}(D_T))$, a contradiction!

**Subcase 3.2:** $D = \{pq, qr\}$.

Then $D_S = \{p, pq, qr\}$ and $D_T = \{q, r, pq, qr\}$. By Lemma 5.3, $\lambda_1(D_T) = \lambda_1(\{q, r\}) + \lambda_1(\{pq\}) + \lambda_1(\{qr\}) = 2 + (-1) + (-1) = 0$, but

$$
\lambda_i(D_S) = \begin{cases}
q(r-1) + (p-1) & \text{if } pqr|i, \\
q(r-1) - 1 & \text{if } qr|i, \ p{\nmid}i, \\
p - 1 & \text{if } q{\nmid}i, \ p|i, \\
-1 & \text{if } q{\nmid}i, \ p{\nmid}i, \\
-q + (p-1) & \text{if } q|i, \ r{\nmid}i, \ p|i, \\
-q - 1 & \text{if } q|i, \ r{\nmid}i, \ p{\nmid}i.
\end{cases}
$$

Hence $0 \notin Sp(ICG_{pqr}(D_S))$, and so $Sp(ICG_{pqr}(D_S)) \neq Sp(ICG_{pqr}(D_T))$, a contradiction!

**Subcase 3.3:** $D = \{pr, qr\}$.

Then $D_S = \{p, pr, qr\}$ and $D_T = \{q, r, pr, qr\}$. By Lemma 5.3, $\lambda_1(D_T) = \lambda_1(\{q, r\}) + \lambda_1(\{pr\}) + \lambda_1(\{qr\}) = 2 + (-1) + (-1) = 0$, but

$$
\lambda_i(D_S) = \begin{cases}
(q-1)r + (p-1) & \text{if } pqr|i, \\
(q-1)r - 1 & \text{if } qr|i, \ p{\nmid}i, \\
p - 1 & \text{if } r{\nmid}i, \ p|i, \\
-1 & \text{if } r{\nmid}i, \ p{\nmid}i, \\
-r + (p-1) & \text{if } r|i, \ q{\nmid}i, \ p|i, \\
-r - 1 & \text{if } r|i, \ q{\nmid}i, \ p{\nmid}i.
\end{cases}
$$

Hence $0 \notin Sp(ICG_{pqr}(D_S))$, and so $Sp(ICG_{pqr}(D_S)) \neq Sp(ICG_{pqr}(D_T))$, a contradiction!

**Subcase 3.4:** $D = \{1, pq\}$.

Then $D_S = \{p, 1, pq\}$ and $D_T = \{q, r, 1, pq\}$. Since $Sp(ICG_{pqr}(D_S)) = Sp(ICG_{pqr}(D_T))$, by Lemma 1.4, $Sp(ICG_{pqr}(D_S^*)) = Sp(ICG_{pqr}(D_T^*))$, and so

$$Sp(ICG_{pqr}(\{q, r, pr, qr\})) = Sp(ICG_{pqr}(\{p, pr, qr\}))$$

which leads to a contracdiction by Subcase 3.3.

**Subcase 3.5:** $D = \{1, pr\}$.

Then $D_S = \{p, 1, pr\}$ and $D_T = \{q, r, 1, pr\}$. Since $Sp(ICG_{pqr}(D_S)) = Sp(ICG_{pqr}(D_T))$, by Lemma 1.4, $Sp(ICG_{pqr}(D_S^*)) = Sp(ICG_{pqr}(D_T^*))$, and so

$$Sp(ICG_{pqr}(\{q, r, pq, qr\})) = Sp(ICG_{pqr}(\{p, pq, qr\}))$$

which leads to a contracdiction by Subcase 3.2.

**Subcase 3.6:** $D = \{1, qr\}$.

Then $D_S = \{p, 1, qr\}$ and $D_T = \{q, r, 1, qr\}$. Since $Sp(ICG_{pqr}(D_S)) = Sp(ICG_{pqr}(D_T))$, by Lemma 1.4, $Sp(ICG_{pqr}(D_S^*)) = Sp(ICG_{pqr}(D_T^*))$, and so

$$Sp(ICG_{pqr}(\{q, r, pr, pq\})) = Sp(ICG_{pqr}(\{p, pr, pq\}))$$

which leads to a contracdiction by Subcase 3.1.

**Case 4:** $|D| = 3$.

Then $D_S^* = (\{p\} \cup D)^* = \{q, r\} \cup E$, and similarly $D_T^* = \{p\} \cup E$ where $E = \{1, pq, pr, qr\} - D$. Since $Sp(ICG_{pqr}(D_S)) = Sp(ICG_{pqr}(D_T))$, by Lemma 1.4, $Sp(ICG_{pqr}(D_S^*)) = Sp(ICG_{pqr}(D_T^*))$ and so

$$Sp(ICG_{pqr}(\{q, r\} \cup E)) = Sp(ICG_{pqr}(\{p\} \cup E))$$

with $|E| = 1$. By Case 2, it leads to a contradiction.

**Case 5:** $|D| = 4$ i.e. $D = \{1, pq, pr, qr\}$.

Then $D_S^* = (\{p\} \cup D)^* = \{q, r\}$ , and similarly $D_T^* = \{p\}$.
Since $Sp(ICG_{pqr}(D_S)) = Sp(ICG_{pqr}(D_T))$, by Lemma 1.4, $Sp(ICG_{pqr}(D_S^*)) = Sp(ICG_{pqr}(D_T^*))$ and so

$$Sp(ICG_{pqr}(\{q, r\})) = Sp(ICG_{pqr}(\{p\})).$$

By Case 1, it leads to a contradiction. $\square$

## Acknowledgements

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