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# On loose 3-cycle decompositions of complete 4-uniform hypergraphs

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**Abstract**

The complete 4-uniform hypergraph of order $v$ has a set $V$ of size $v$ as its vertex set and the set of all 4-element subsets of $V$ as its edge set. A 4-uniform loose 3-cycle is a hypergraph of order 9 with vertex set $\{a, b, c, d, e, f, g, h, i\}$ and edge set $\big\{\{a, b, c, d\},\ \{d, e, f, g\},\ \{g, h, i, a\}\big\}$. We give necessary and sufficient conditions for the existence of a decomposition of the complete 4-uniform hypergraph of order $v$ into subgraphs isomorphic to a loose 3-cycle.

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# 1 Introduction

A *hypergraph* $H$ consists of a finite set $V$ of *vertices* and a finite collection (possibly multiset) $E = \{e_1, e_2, \ldots, e_m\}$ of nonempty subsets of $V$ called *hyperedges* or simply *edges*. If no edge in $E$ is repeated, then $H$ is *simple*. For a given hypergraph $H$, we use $V(H)$ and $E(H)$ to denote the vertex set and the edge set (or multiset) of $H$, respectively. We call $|V(H)|$ and $|E(H)|$ the *order* and *size* of $H$, respectively. If $H$ is not simple, the hypergraph with vertex set $V(H)$ and edge set the set of distinct edges in $E(H)$ is referred to as the simple hypergraph *underlying* $H$. If for each $e \in E(H)$ we have $|e| = t$, then $H$ is said to be *$t$-uniform*. Thus $t$-uniform hypergraphs are generalizations of the concept of a graph (where $t = 2$). The hypergraph with vertex set $V$ and with edge set the set of all $t$-element subsets of $V$ is called the *complete $t$-uniform hypergraph on* $V$ and is denoted by $K_V^{(t)}$. If $v = |V|$, then $K_v^{(t)}$ is called the *complete $t$-uniform hypergraph of order $v$* and is used to denote any hypergraph isomorphic to $K_V^{(t)}$. When $t = 2$, we may use $K_v$ in place of $K_v^{(2)}$. If $H'$ is a subgraph of $H$, then $H \setminus H'$ denotes the hypergraph obtained from $H$ by deleting the edges of $H'$. We may refer to $H \setminus H'$ as the hypergraph $H$ with a *hole* $H'$. The vertices in $H'$ may be referred to as the vertices in the hole.

A commonly studied problem in combinatorics concerns decompositions of graphs into edge-disjoint subgraphs. A *decomposition* of a graph $K$ is a set $\Delta = \{G_1, G_2, \ldots, G_s\}$ of subgraphs of $K$ such that $\{E(G_1), E(G_2), \ldots, E(G_s)\}$ is a partition of $E(K)$. If each element of $\Delta$ is isomorphic to a fixed graph $G$, then $\Delta$ is called a *$G$-decomposition* of $K$. A $G$-decomposition of $K_v$ is also known as a *$G$-design of order* $v$. A $K_k$-design of order $v$ is usually known as a 2-$(v, k, 1)$ *design* or as a *balanced incomplete block design of index* 1 or a $(v, k, 1)$-*BIBD*. The problem of determining all $v$ for which there exists a $G$-design of order $v$ is of special interest (see [1] for a survey).

The notion of decompositions of graphs naturally extends to hypergraphs. A *decomposition* of a hypergraph $K$ is a set $\Delta = \{H_1, H_2, \ldots, H_s\}$ of subgraphs of $K$ such that $\{E(H_1), E(H_2), \ldots, E(H_s)\}$ is a partition of $E(K)$. Any element of $\Delta$ isomorphic to a fixed hypergraph $H$ is called an *$H$-block*. If all elements of $\Delta$ are $H$-blocks, then $\Delta$ is called an *$H$-decomposition* of $K$, and we may also say $H$ *decomposes* $K$. An $H$-decomposition of $K_v^{(t)}$ is called an *$H$-design of order* $v$. The problem of determining all $v$ for which there exists an $H$-design of order $v$ is called the *spectrum problem* for $H$-designs.

A $K_k^{(t)}$-design of order $v$ is a generalization of 2-$(v, k, 1)$ designs and is known as a $t$-$(v, k, 1)$ *design* or simply as a *$t$-design*. A summary of results on $t$-designs appears in [20]. A $t$-$(v, k, 1)$ design is also known as a *Steiner system* and is denoted by $S(t, v, k)$ (see [13] for a summary of results on Steiner systems). Keevash [19] has recently shown that for all $t$ and $k$ the obvious necessary conditions for the existence of an $S(t, k, v)$-design are sufficient for sufficiently large values of $v$. Similar results were obtained by Glock, Kühn, Lo, and Osthus [14, 15] and extended to include the corresponding asymptotic results for $H$-designs of order $v$ for all uniform hypergraphs $H$. These results for $t$-uniform hypergraphs mirror the celebrated results of Wilson [25] for graphs. Although these asymptotic results assure the existence of $H$-designs for sufficiently large values of $v$ for any uniform hypergraph $H$, the spectrum problem has been settled for very few hypergraphs of uniformity larger than 2.

In the study of graph decompositions, a fair amount of the focus has been on $G$-decompositions of $K_v$ where $G$ is a graph with a relatively small number of edges (see [1] and [7] for known results). Some authors have investigated the corresponding problem for 3-uniform hypergraphs. For example, in [5], the spectrum problem is settled for all 3-uniform hypergraphs on 4 or fewer vertices. More recently, the spectrum problem was settled in [6] for all 3-uniform hypergraphs with at most 6 vertices and at most 3 edges. In [6], they also settle the spectrum problem for the 3-uniform hypergraph of order 6 whose edges form the lines of the Pasch configuration. Authors have also considered $H$-designs where $H$ is a 3-uniform hypergraph whose edge set is defined by the faces of a regular polyhedron. Let $T$, $O$, and $I$ denote the tetrahedron, octahedron, and icosahedron hypergraphs, respectively. The hypergraph $T$ is the same as $K_4^{(3)}$, and its spectrum was settled in 1960 by Hanani [16]. In another paper [17], Hanani settled the spectrum problem for $O$-designs and gave necessary conditions for the existence of $I$-designs.

Using the approach in [6], the spectrum problem has recently been settled for several individual 3-uniform hypergraphs $H$. These include when $H$ is a loose $m$-cycle for $3 \leq m \leq 5$ (see [8], [10], [12]) and when $H$ is a tight 6-cycle [2] or a tight 9-cycle [9].

There are also several articles on decompositions of complete $t$-uniform hypergraphs (see [4] and [23]) and of $t$-uniform $t$-partite hypergraphs (see [21] and [24]) into variations on the concept of a Hamilton cycle. There are also several results on decompositions of 3-uniform hypergraphs into structures known as Berge cycles with a given number of edges (see for example [18] and [22]). We note however that the Berge cycles in these decompositions are not required to be isomorphic.

Perhaps the best known result on decompositions of complete $t$-uniform hypergraphs is a result by Baranyai [3] on the existence of 1-factorizations of $K_{mt}^{(t)}$ for all positive integers $m$. A more general result of Baranyai [3] subsumes the 1-factorization result and represents the only known spectrum problem type result for all $t$-uniform hypergraphs.

**Theorem 1.1.** *Let $H$ be a $t$-uniform matching of size $m$. There exists an $H$-decomposition of $K_n^{(t)}$ if and only if $m \mid \binom{n}{t}$ and $n \geq mt$.*

In this work, we settle the spectrum problem for $H$-designs where $H$ is the 4-uniform hypergraph known as a loose 3-cycle. A 4-uniform *loose $m$-cycle*, denoted $LC_m^{(4)}$, is a hypergraph of order $3m$ and size $m$ with vertex set $\{v_1, v_2, \ldots, v_{3m}\}$ and edge set $\big\{\{v_{3i-2}, v_{3i-1}, v_{3i}, v_{3i+1}\} : 1 \leq i \leq m-1\big\} \cup \{v_{3m-2}, v_{3m-1}, v_{3m}, v_1\}$. In general, for $t > 2$ and $m \geq 3$, a *$t$-uniform loose $m$-cycle*, denoted $LC_m^{(t)}$, can be viewed as the $t$-uniform hypergraph obtained by appending $t-2$ degree 1 (loose) vertices to each edge in a 2-uniform $m$-cycle. An illustration of $LC_3^{(4)}$ is shown in Figure 1.

**Additional Notation and Terminology**

Let $\mathbb{Z}_n$ denote the group of integers modulo $n$. If $a$ and $b$ are integers, we define $[a, b]$ to be $\{r \in \mathbb{Z} : a \leq r \leq b\}$.

For any edge-disjoint $t$-uniform hypergraphs $H_1$ and $H_2$, we use $H_1 \cup H_2$ to indicate the hypergraph with vertex set $V(H_1) \cup V(H_2)$ and edge set $E(H_1) \cup E(H_2)$. Similarly, if $H$ is a hypergraph and $r$ is a nonnegative integer, then an edge-disjoint union of $r$ copies of $H$ is denoted with $rH$.

We next define some notation for certain types of multipartite-like 4-uniform hypergraphs. Let $A, B, C, D$ be pairwise disjoint sets. The hypergraph with vertex set $A \cup B \cup C \cup D$ and edge set consisting of all 4-element sets having exactly one vertex in each of $A, B, C, D$ is denoted by $K^{(4)}_{A,B,C,D}$. The hypergraph with vertex set $A \cup B$ and edge set consisting of all 4-element sets having at least one vertex in each of $A$ and $B$ is denoted by $L^{(4)}_{A,B}$. Furthermore, if $t_1$ and $t_2$ are positive integers with $t_1 + t_2 = 4$, we use $L^{(t_1,t_2)}_{A,B}$ to denote the subgraph of $L^{(4)}_{A,B}$ where each edge consists of $t_1$ elements from $A$ and $t_2$ elements from $B$. The hypergraph with vertex set $A \cup B \cup C$ and edge set consisting of all 4-element sets having at least one vertex in each of $A, B, C$ is denoted by $L^{(4)}_{A,B,C}$. Moreover, if $t_1$, $t_2$, and $t_3$ are positive integers with $t_1 + t_2 + t_3 = 4$, we use $L^{(t_1,t_2,t_3)}_{A,B,C}$ to denote the subgraph of $L^{(4)}_{A,B,C}$ where each edge consists of $t_1$ elements from $A$, $t_2$ elements from $B$, and $t_3$ from $C$. If $|A| = a$, $|B| = b$, $|C| = c$, and $|D| = d$, we may use $K^{(4)}_{a,b,c,d}$ to denote any hypergraph that is isomorphic to $K^{(4)}_{A,B,C,D}$, $L^{(4)}_{a,b}$ to denote any hypergraph that is isomorphic to $L^{(4)}_{A,B}$, and $L^{(4)}_{a,b,c}$ to denote any hypergraph that is isomorphic to $L^{(4)}_{A,B,C}$. We use $K^{(4)}_{a,\overline{b},\overline{c},\overline{d}} \cup L^{(4)}_{\overline{b},\overline{c},\overline{d}}$ to denote any hypergraph isomorphic to $K^{(4)}_{A,B,C,D} \cup L^{(4)}_{B,C,D}$. Similarly, we use $L^{(4)}_{a,\overline{b},\overline{c}} \cup L^{(4)}_{\overline{b},\overline{c}}$ to denote any hypergraph isomorphic to $L^{(4)}_{A,B,C} \cup L^{(4)}_{B,C}$.

It is simple to observe that if $A$, $B$, $C$, $D$, and $D'$ are pairwise-disjoint, then $K^{(4)}_{A,B,C,D\cup D'} = K^{(4)}_{A,B,C,D} \cup K^{(4)}_{A,B,C,D'}$. Thus we have the following lemma.

**Lemma 1.2.** *If $a$, $b$, $c$, $d$, $w$, $x$, $y$, and $z$ are positive integers, then there is a decomposition of $K^{(4)}_{wa,xb,yc,zd}$ into $wxyz$ copies of $K^{(4)}_{a,b,c,d}$.*

Similarly, we observe that if $A$, $B$, $C$, and $C'$ are pairwise-disjoint, then $L^{(4)}_{A,B,C\cup C'} = L^{(4)}_{A,B,C} \cup L^{(4)}_{A,B,C'} \cup K^{(4)}_{A,B,C,C'}$. Thus we have the following lemma.

**Lemma 1.3.** *If $a$, $b$, $c$, $x$, $y$, and $z$ are positive integers, then there is a decomposition of $L^{(4)}_{xa,yb,zc}$ into $xyz$ copies of $L^{(4)}_{a,b,c}$, $\binom{x}{2}yz$ copies of $K^{(4)}_{a,a,b,c}$, $x\binom{y}{2}z$ copies of $K^{(4)}_{a,b,b,c}$, and $xy\binom{z}{2}$ copies of $K^{(4)}_{a,b,c,c}$.*

Now, consider a subgraph $H$ of a hypergraph $K$ with $A \subseteq V(K)$. The *restriction* of $H$ to $A$ is the hypergraph with vertex set $A \cap V(H)$ and edge multiset $\{e \cap A : e \in E(H)\}$. We note that if $H$ is a subgraph of $L^{(t_1,t_2,\ldots,t_m)}_{A_1,A_2,\ldots,A_m}$, then the restriction of $H$ to $A_i$, for $i \in [1, m]$, is $t_i$-uniform.

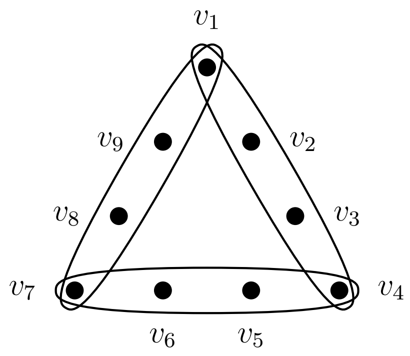


Figure 1: The 4-uniform loose 3-cycle, $LC_3^{(4)}$, denoted by $[v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9]$.

Finally, let $\mathcal{A}$ and $\mathcal{B}$ be sets of $t_1$-element sets and $t_2$-element sets, respectively, such that $A \cap B = \varnothing$ for all $A \in \mathcal{A}$ and all $B \in \mathcal{B}$, and let $\mathcal{A}^* = \bigcup_{A\in\mathcal{A}} A$ and $\mathcal{B}^* = \bigcup_{B\in\mathcal{B}} B$. We use $U_{\mathcal{A},\mathcal{B}}$ to denote the $(t_1+t_2)$-uniform hypergraph with vertex set $\mathcal{A}^* \cup \mathcal{B}^*$ and edge set $\{A \cup B : A \in \mathcal{A}, B \in \mathcal{B}\}$. Thus, for example, if $C$ and $D$ are disjoint sets, then $U_{E\left(K_C^{(t_1)}\right), E\left(K_D^{(t_2)}\right)}$ is isomorphic to $L_{C,D}^{(t_1,t_2)}$.

## 2 Some Small Examples

As illustrated in Figure 1, we will use $[v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9]$ to denote any hypergraph isomorphic to the $LC_3^{(4)}$ with vertex set $\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9\}$ and edge set $\{\{v_1, v_2, v_3, v_4\}, \{v_4, v_5, v_6, v_7\}, \{v_7, v_8, v_9, v_1\}\}$.

Next, we give some examples of $LC_3^{(4)}$-decompositions that are used in proving our main result. For the most part, these decompositions, as well as the ones found in the Appendix, are either cyclic or $r$-pyramidal as defined in [11]. They were found either by hand or by computer searches.

**Example 2.1.** Let $V\left(K_9^{(4)}\right) = \mathbb{Z}_7 \cup \{\infty_1, \infty_2\}$ and let

$$B = \big\{[\infty_2, 0, 1, \infty_1, 5, 2, 4, 3, 6], [\infty_2, 0, 2, \infty_1, 5, 4, 1, 3, 6], [3, \infty_1, \infty_2, 0, 1, 2, 4, 5, 6], [4, \infty_2, 6, 0, 1, 2, 5, 3, \infty_1], [6, \infty_1, 2, 0, 1, 3, 4, 5, \infty_2], [2, \infty_1, 4, 0, 1, 3, 5, 6, \infty_2]\big\}.$$

Then an $LC_3^{(4)}$-decomposition of $K_9^{(4)}$ consists of the $LC_3^{(4)}$-blocks in $B$ under the action of the map $\infty_i \mapsto \infty_i$ and $j \mapsto j+1 \pmod 7$.

Decompositions of $K_{10}^{(4)}$, $K_{11}^{(4)}$, $K_{12}^{(4)}$, and $K_{15}^{(4)}$ are accomplished in a similar fashion to that seen in the above example. For the sake of brevity, these larger $LC_3^{(4)}$-designs (Examples 2.6–2.9) are found in the Appendix.

**Example 2.2.** Let $V\left(L_{3,3,3}^{(4)}\right) = \mathbb{Z}_9$ with vertex partition $\{\{0,3,6\}, \{1,4,7\}, \{2,5,8\}\}$ and let

$$B = \big\{[0, 1, 2, 3, 4, 6, 8, 5, 7], [0, 1, 2, 5, 3, 7, 4, 6, 8], [0, 1, 5, 6, 3, 8, 7, 2, 4]\big\}.$$

Then an $LC_3^{(4)}$-decomposition of $L_{3,3,3}^{(4)}$ consists of the $LC_3^{(4)}$-blocks in $B$ under the action of the map $j \mapsto j+1 \pmod 9$.

**Example 2.3.** Let $V\left(K_{1,\overline{3},\overline{3},\overline{3}}^{(4)} \cup L_{\overline{3},\overline{3},\overline{3}}^{(4)}\right) = \mathbb{Z}_9 \cup \{\infty\}$ with vertex partition $\big\{\{\infty\}$, $\{0,3,6\}$, $\{1,4,7\}$, $\{2,5,8\}\big\}$ and let

$$B = \big\{[0,1,2,3,4,5,7,8,\infty],\ [0,1,2,5,3,6,7,4,8],\\ [0,1,4,5,3,8,7,2,6],\ [0,1,5,\infty,6,8,4,2,7]\big\}.$$

Then an $LC_3^{(4)}$-decomposition of $K_{1,\overline{3},\overline{3},\overline{3}}^{(4)} \cup L_{\overline{3},\overline{3},\overline{3}}^{(4)}$ consists of the $LC_3^{(4)}$-blocks in $B$ under the action of the map $\infty \mapsto \infty$ and $j \mapsto j+1 \pmod 9$.

**Example 2.4.** Let $V\left(K_{2,3,3,3}^{(4)}\right) = \mathbb{Z}_9 \cup \{\infty_1, \infty_2\}$ with vertex partition $\big\{\{\infty_1, \infty_2\}$, $\{0,3,6\}$, $\{1,4,7\}$, $\{2,5,8\}\big\}$ and let

$$B = \big\{[\infty_1,0,1,2,\infty_2,3,4,6,8],\ [\infty_2,2,6,1,\infty_1,0,5,7,3]\big\}.$$

Then an $LC_3^{(4)}$-decomposition of $K_{2,3,3,3}^{(4)}$ consists of the $LC_3^{(4)}$-blocks in $B$ under the action of the map $\infty_i \mapsto \infty_i$ and $j \mapsto j+1 \pmod 9$.

**Example 2.5.** Let $V\left(K_{3,3,3,3}^{(4)}\right) = \mathbb{Z}_9 \cup \{\infty_1, \infty_2, \infty_3\}$ with the vertex partition $\big\{\{\infty_1, \infty_2, \infty_3\}$, $\{0,3,6\}$, $\{1,4,7\}$, $\{2,5,8\}\big\}$ and let

$$B = \big\{[0,1,2,\infty_1,3,4,8,7,\infty_2], [0,1,2,\infty_3,3,7,5,4,\infty_2], [0,1,\infty_3,5,3,\infty_1,7,2,\infty_2]\big\}.$$

Then an $LC_3^{(4)}$-decomposition of $K_{3,3,3,3}^{(4)}$ consists of the $LC_3^{(4)}$-blocks in $B$ under the action of the map $\infty_i \mapsto \infty_i$ and $j \mapsto j+1 \pmod 9$.

## 3 Main Constructions

We begin with a lemma that allows us to derive some $LC_3^{(4)}$-decompositions from known 3- and 2-uniform decompositions.

**Lemma 3.1.** *Let $t_1$ and $t_2$ be positive integers and let $A$ and $B$ be disjoint sets. Let $H$ be a $(t_1+t_2)$-uniform subgraph of $L_{A,B}^{(t_1,t_2)}$ and let $H'$ and $H''$ denote the restrictions of $H$ to $A$ and to $B$, respectively, such that $H'$ is simple and such that the simple hypergraph underlying $H''$ is a subgraph of a matching $M$. If there exists an $H'$-decomposition of some $t_1$-uniform hypergraph $K$, then there exists an $H$-decomposition of $U_{E(K),E(M)}$.*

*Proof.* Let $n = |E(M)|$ and let $E(M) = \{W_i : i \in \mathbb{Z}_n\}$. For each $i \in \mathbb{Z}_n$, let $E_i = \{e \in E(H) : W_i \subset e\}$ and let $E'_i = \{e \cap A : e \in E_i\}$. Note that for any $W_i \notin E(H'')$ we have $E_i = E'_i = \varnothing$. Furthermore, the multiplicity of $W_i$ in $E(H'')$ is the cardinality of both $E_i$ and $E'_i$. Thus, each $E_i$ represents the (possibly empty) set of edges of $H$ that contain $W_i$ while the set of nonempty $E'_i$'s is a partition of $E(H')$.

Without loss of generality, we may assume there exists some $s \in \mathbb{Z}_n$ such that $W_0, W_1, \ldots, W_s$ have positive multiplicty in $E(H'')$ and, if $s \neq n-1$, that $W_{s+1}, \ldots, W_{n-1}$ are not in $E(H'')$. Hence, $E'_i$ is empty if only if $s < i \leq n-1$.

Let $\Delta' = \{H'_1, H'_2, \ldots\}$ be an $H'$-decomposition of $K$. For each $H'_j$ in $\Delta'$, we construct the $(t_1+t_2)$-uniform hypergraph $H_{j,0}$ by appending the elements of $W_i$, for $i \in [0,s]$, to each edge in the copy of $E'_i$ in $H'_j$. Hence, each copy of $E'_i$ in $H'_j$ becomes a copy of $E_i$, and $H_{j,0}$ is isomorphic to $H$. We repeat this process to construct $H_{j,k}$ for each $k \in [1, n-1]$ by appending the elements of $W_{i+k \pmod n}$, for $i \in [0,s]$, to each edge in the copy of $E'_i$. Thus we arrive at the set $\Delta = \{H_{1,0}, H_{1,1}, \ldots, H_{1,n-1}, H_{2,0}, H_{2,1}, \ldots, H_{2,n-1}, \ldots\}$, and $\Delta$ is an $H$-decomposition of $U_{E(K),E(M)}$. ☐

**Corollary 3.2.** *There exists an $LC_3^{(4)}$-decomposition of $L_{m,n}^{(3,1)}$ for $m \equiv 0$, 1, or 2 $\pmod 9$ and $n \geq 3$.*

*Proof.* Let $H$ denote the $LC_3^{(4)}$ illustrated in Figure 1. Let $A = \{v_1, v_3, v_4, v_6, v_7, v_9\}$ and $B = \{v_2, v_5, v_8\}$. Then $H$ is a subgraph of $L_{A,B}^{(3,1)}$. The restriction of $H$ to $A$ is a 3-uniform loose 3-cycle $LC_3^{(3)}$. Moreover, the restriction of $H$ to $B$ is a 1-uniform matching. It is shown in [6] that there exists a nontrivial $LC_3^{(3)}$-decomposition of $K_m^{(3)}$ if and only if $m \equiv 0$, 1, or 2 $\pmod 9$, $m \geq 9$. Hence, by Lemma 3.1, if $m \equiv 0$, 1, or 2 $\pmod 9$, $m \geq 9$, and if $n \geq 3$, then there exists an $LC_3^{(4)}$-decomposition of $U_{E\left(K_m^{(3)}\right), E\left(K_n^{(1)}\right)}$. But here, $U_{E\left(K_m^{(3)}\right), E\left(K_n^{(1)}\right)}$ is isomorphic to $L_{m,n}^{(3,1)}$. ☐

**Corollary 3.3.** *There exists an $LC_3^{(4)}$-decomposition of $L_{m,n}^{(2,2)}$ for $m \equiv 1$ or 3 $\pmod 6$ and $n \geq 6$.*

*Proof.* Let $H$ denote the $LC_3^{(4)}$ illustrated in Figure 1. Let $A = \{v_1, v_4, v_7\}$ and $B = \{v_2, v_3, v_5, v_6, v_8, v_9\}$. Then $H$ is a subgraph of $L_{A,B}^{(2,2)}$. The restriction of $H$ to $A$ is the 2-uniform complete graph $K_3$. Moreover, the restriction of $H$ to $B$ is a 2-uniform matching with 3 edges.

Let $n \geq 6$ be an integer and let $n' = \lfloor n/2 \rfloor \geq 3$. It is well-known that there is a nontrivial $K_3$-decomposition of $K_m$ (i.e., a Steiner triple system of order $m$) if and only if $m \equiv 1$ or 3 $\pmod 6$, $m \geq 3$. It is also simple to see that $K_n$ has a decomposition $\Delta$ into matchings with $n'$ edges.

For each matching $M \in \Delta$, we apply Lemma 3.1 with $K = K_m$ to get an $LC_3^{(4)}$-decomposition of $U_{E(K_m),E(M)}$. Since $\bigcup_{M \in \Delta} U_{E(K_m),E(M)} = U_{E(K_m),E(K_n)} = L_{m,n}^{(2,2)}$, the result follows. ☐

**Corollary 3.4.** *There exists an $LC_3^{(4)}$-decomposition of $L_{9,9}^{(4)}$.*

*Proof.* First, we note that $L_{9,9}^{(4)} = L_{9,9}^{(2,2)} \cup L_{9,9}^{(3,1)} \cup L_{9,9}^{(1,3)}$. By Corollary 3.2, $LC_3^{(4)}$ decomposes $L_{9,9}^{(3,1)}$, which is isomorphic to $L_{9,9}^{(1,3)}$. By Corollary 3.3, $LC_3^{(4)}$ decomposes $L_{9,9}^{(2,2)}$. Thus the result follows. ☐

For the next two lemmas, it is helpful to recall that we use $K^{(4)}_{a,\overline{b},\overline{c},\overline{d}} \cup L^{(4)}_{\overline{b},\overline{c},\overline{d}}$ and $L^{(4)}_{a,\overline{b},\overline{c}} \cup L^{(4)}_{\overline{b},\overline{c}}$ to denote hypergraphs isomorphic to $K^{(4)}_{A,B,C,D} \cup L^{(4)}_{B,C,D}$ and $L^{(4)}_{A,B,C} \cup L^{(4)}_{B,C}$, respectively, where $A$, $B$, $C$, and $D$ are pairwise disjoint sets of cardinality $a$, $b$, $c$, and $d$, respectively.

**Lemma 3.5.** *There exists an $LC_3^{(4)}$-decomposition of $K^{(4)}_{1,\overline{9},\overline{9},\overline{9}} \cup L^{(4)}_{\overline{9},\overline{9},\overline{9}}$.*

*Proof.* By Lemma 1.2, $K^{(4)}_{1,9,9,9}$ can be decomposed into 27 copies of $K^{(4)}_{1,3,3,3}$. Similarly, by Lemma 1.3, $L^{(4)}_{9,9,9}$ can be decomposed into 27 copies of $L^{(4)}_{3,3,3}$ and 81 copies of $K^{(4)}_{3,3,3,3}$. Thus, $K^{(4)}_{1,\overline{9},\overline{9},\overline{9}} \cup L^{(4)}_{\overline{9},\overline{9},\overline{9}}$ can be decomposed into 27 copies of $K^{(4)}_{1,\overline{3},\overline{3},\overline{3}} \cup L^{(4)}_{\overline{3},\overline{3},\overline{3}}$ and 81 copies of $K^{(4)}_{3,3,3,3}$. An $LC_3^{(4)}$-decomposition of $K^{(4)}_{1,\overline{3},\overline{3},\overline{3}} \cup L^{(4)}_{\overline{3},\overline{3},\overline{3}}$ is given in Example 2.3, and an $LC_3^{(4)}$-decomposition of $K^{(4)}_{3,3,3,3}$ is given in Example 2.5. The result now follows. □

We proceed by proving a lemma that is fundamental to our constructions.

**Lemma 3.6.** *Let $n \geq 1$, $x \geq 1$, and $r \geq 0$ be integers and let $v = nx + r$. There exists a decomposition of $K_v^{(4)}$ into the following:*

- *1 copy of $K^{(4)}_{n+r}$,*
- *$x - 1$ copies of $K^{(4)}_{n+r} \setminus K^{(4)}_r$ if $x \geq 2$ (these are isomorphic to $K^{(4)}_{n+r}$ if $r \leq 3$),*
- *$\binom{x}{2}$ copies of $L^{(4)}_{r,\overline{n},\overline{n}} \cup L^{(4)}_{\overline{n},\overline{n}}$ if $x \geq 2$ (here $L^{(4)}_{r,n,n}$ is empty if $r = 0$),*
- *$\binom{x}{3}$ copies of $K^{(4)}_{r,\overline{n},\overline{n},\overline{n}} \cup L^{(4)}_{\overline{n},\overline{n},\overline{n}}$ if $x \geq 3$ (here $K^{(4)}_{r,n,n,n}$ is empty if $r = 0$), and*
- *$\binom{x}{4}$ copies of $K^{(4)}_{n,n,n,n}$ if $x \geq 4$.*

*Proof.* If $x \in \{0, 1\}$, the decomposition is trivial. Thus we may assume that $x \geq 2$. Let $V_0, V_1, \ldots, V_x$ be pairwise disjoint sets of vertices with $|V_0| = r$ and $|V_1| = |V_2| = \cdots = |V_x| = n$ and let $V = V_0 \cup V_1 \cup \cdots \cup V_x$. Then, $K_V^{(4)}$ can be viewed as the (edge-disjoint) union

$$K^{(4)}_{V_1 \cup V_0} \cup \bigcup_{2 \leq i \leq x} \left( K^{(4)}_{V_i \cup V_0} \setminus K^{(4)}_{V_0} \right) \cup \bigcup_{1 \leq i < j \leq x} \left( L^{(4)}_{V_0, V_i, V_j} \cup L^{(4)}_{V_i, V_j} \right)$$
$$\cup \bigcup_{1 \leq i < j < k \leq x} \left( K^{(4)}_{V_0, V_i, V_j, V_k} \cup L^{(4)}_{V_i, V_j, V_k} \right) \cup \bigcup_{1 \leq i < j < k < \ell \leq x} \left( K^{(4)}_{V_i, V_j, V_k, V_\ell} \right)$$

Thus the result follows. □

We now have what we need to settle the spectrum problem for $LC_3^{(4)}$-designs.

**Theorem 3.7.** *There exists an $LC_3^{(4)}$-decomposition of $K_v^{(4)}$ if and only if $v \equiv 0$, 1, 2, 3, or 6 (mod 9) and $v \geq 9$.*

*Proof.* If $LC_3^{(4)}$ with its 3 edges decomposes $K_v^{(4)}$, then we must have $3 \mid \binom{v}{4}$. Therefore we have $v \equiv 0$, 1, 2, 3, or 6 (mod 9). Since $LC_3^{(4)}$ has order 9, we must further have $v \geq 9$. We now show these conditions are sufficient.

Let $v = 9x + r$, where $r \in \{0, 1, 2, 3, 6\}$ and $x \geq 1$. By Lemma 3.6, it suffices to give $LC_3^{(4)}$-decompositions of $K_{9+r}^{(4)}$, of $K_{9+r}^{(4)} \setminus K_r^{(4)}$, of $L_{r,\overline{9},\overline{9}}^{(4)} \cup L_{\overline{9},\overline{9}}^{(4)}$, of $K_{r,\overline{9},\overline{9},\overline{9}}^{(4)} \cup L_{\overline{9},\overline{9},\overline{9}}^{(4)}$, and of $K_{9,9,9,9}^{(4)}$.

We give $LC_3^{(4)}$-decompositions of $K_9^{(4)}$ in Example 2.1, of $K_{10}^{(4)}$ in Example 2.6, of $K_{11}^{(4)}$ in Example 2.7, of $K_{12}^{(4)}$ in Example 2.8, and of $K_{15}^{(4)}$ in Example 2.9. Thus we may assume that $x \geq 2$.

If $r \in \{0, 1, 2, 3\}$, then $K_{9+r}^{(4)} \setminus K_r^{(4)}$ is isomorphic to $K_{9+r}^{(4)}$. We give an $LC_3^{(4)}$-decomposition of $K_{15}^{(4)} \setminus K_6^{(4)}$ in Example 2.12.

The hypergraph $L_{r,\overline{9},\overline{9}}^{(4)} \cup L_{\overline{9},\overline{9}}^{(4)}$ is isomorphic to $L_{\overline{9},\overline{9}}^{(4)}$ when $r = 0$. We give an $LC_3^{(4)}$-decomposition of $L_{9,9}^{(4)}$ in Corollary 3.4. An $LC_3^{(4)}$-decomposition of $L_{1,\overline{9},\overline{9}}^{(4)} \cup L_{\overline{9},\overline{9}}^{(4)}$ is given in Example 2.10. An $LC_3^{(4)}$-decomposition of $L_{2,9,9}^{(4)}$ is given in Example 2.11. Thus $LC_3^{(4)}$ decomposes $L_{2,\overline{9},\overline{9}}^{(4)} \cup L_{\overline{9},\overline{9}}^{(4)}$. By Lemma 1.3, there is a decomposition of $L_{3,9,9}^{(4)}$ into 9 copies of $L_{3,3,3}^{(4)}$ and 18 copies of $K_{3,3,3,3}^{(4)}$. Similarly, there is a decomposition of $L_{6,9,9}^{(4)}$ into copies of $L_{3,3,3}^{(4)}$ and of $K_{3,3,3,3}^{(4)}$. We give $LC_3^{(4)}$-decompositions of $L_{3,3,3}^{(4)}$ and $K_{3,3,3,3}^{(4)}$ in Examples 2.2 and 2.5, respectively. Thus we have that $LC_3^{(4)}$ decomposes $L_{3,\overline{9},\overline{9}}^{(4)} \cup L_{\overline{9},\overline{9}}^{(4)}$ and $L_{6,\overline{9},\overline{9}}^{(4)} \cup L_{\overline{9},\overline{9}}^{(4)}$.

Finally, only $LC_3^{(4)}$-decompositions of $K_{r,\overline{9},\overline{9},\overline{9}}^{(4)} \cup L_{\overline{9},\overline{9},\overline{9}}^{(4)}$ and of $K_{9,9,9,9}^{(4)}$ remain. We prove an $LC_3^{(4)}$-decomposition of $K_{1,\overline{9},\overline{9},\overline{9}}^{(4)} \cup L_{\overline{9},\overline{9},\overline{9}}^{(4)}$ in Lemma 3.5. By Lemma 1.3, there is a decomposition of $L_{9,9,9}^{(4)}$ into 27 copies of $L_{3,3,3}^{(4)}$ and 81 copies of $K_{3,3,3,3}^{(4)}$. By Lemma 1.2, there is a decomposition of $K_{2,9,9,9}^{(4)}$ into 27 copies of $K_{2,3,3,3}^{(4)}$. Similarly, if $r \in \{3, 6, 9\}$, then there is a decomposition of $K_{r,9,9,9}^{(4)}$ into $27 \cdot r/3$ copies of $K_{3,3,3,3}^{(4)}$. Since $LC_3^{(4)}$ decomposes each of $L_{3,3,3}^{(4)}$ (Example 2.2), $K_{2,3,3,3}^{(4)}$ (Example 2.4), and $K_{3,3,3,3}^{(4)}$ (Example 2.5), we have that $LC_3^{(4)}$ decomposes each of $L_{9,9,9}^{(4)}$, $K_{2,9,9,9}^{(4)}$, $K_{3,9,9,9}^{(4)}$, $K_{6,9,9,9}^{(4)}$, and $K_{9,9,9,9}^{(4)}$. This completes the proof. □

# Appendix

We give several additional examples of $LC_3^{(4)}$-decompositions that are used in proving our main result. This is a continuation of the list of examples seen in Section 2.

**Example 2.6.** Let $V\left(K_{10}^{(4)}\right) = \mathbb{Z}_7 \cup \{\infty_1, \infty_2, \infty_3\}$ and let

$$\begin{aligned} B = \big\{&[\infty_1, \infty_3, 2, 3, 4, 5, 0, 1, \infty_2],\ [\infty_1, 6, 4, 5, 2, 1, 0, \infty_3, \infty_2], \\ &[\infty_2, \infty_3, 1, 3, 4, 6, 0, 2, \infty_1],\ [\infty_2, 1, 5, 4, 2, \infty_3, 0, 3, \infty_1], \\ &[\infty_3, \infty_2, 3, 6, 5, 1, 0, 2, \infty_1],\ [\infty_3, \infty_2, 5, 4, 2, 6, 0, 3, \infty_1], \end{aligned}$$

$$[0, \infty_1, 1, 3, \infty_3, 4, 5, 6, \infty_2],\ [0, \infty_1, 1, 4, \infty_3, 5, 2, 6, \infty_2],$$
$$[0, \infty_1, 1, 5, \infty_3, 2, 6, 4, \infty_2],\ [0, \infty_1, 2, 4, \infty_3, 6, 3, 5, \infty_2]\big\}.$$

Then an $LC_3^{(4)}$-decomposition of $K_{10}^{(4)}$ consists of the $LC_3^{(4)}$-blocks in $B$ under the action of the map $\infty_i \mapsto \infty_i$ and $j \mapsto j+1 \pmod 7$.

**Example 2.7.** Let $V\Big(K_{11}^{(4)}\Big) = \mathbb{Z}_{11}$ and let

$$\begin{aligned} B = \big\{&[4, 6, 7, 3, 2, 1, 0, 10, 5],\ [5, 3, 10, 0, 1, 2, 4, 9, 7],\ [9, 8, 6, 0, 1, 2, 5, 4, 3],\\ &[6, 4, 10, 0, 1, 2, 7, 3, 5],\ [5, 7, 10, 0, 1, 8, 2, 3, 9],\ [8, 7, 1, 0, 6, 4, 2, 3, 5],\\ &[3, 8, 1, 0, 4, 7, 2, 6, 9],\ [7, 3, 10, 0, 2, 4, 8, 1, 5],\ [4, 3, 8, 0, 2, 5, 7, 6, 10],\\ &[0, 2, 5, 8, 6, 3, 9, 7, 1]\big\}.\end{aligned}$$

Then an $LC_3^{(4)}$-decomposition of $K_{11}^{(4)}$ consists of the $LC_3^{(4)}$-blocks in $B$ under the action of the map $j \mapsto j+1 \pmod{11}$.

**Example 2.8.** Let $V\Big(K_{12}^{(4)}\Big) = \mathbb{Z}_{11} \cup \{\infty\}$ and let

$$\begin{aligned} B = \big\{&[\infty, 1, 2, 4, 5, 6, 7, 9, 10],\ [1, 2, 5, 4, 10, 8, \infty, 7, 3],\ [0, 1, 5, 2, 4, 8, 6, 9, 10],\\ &[0, 1, 3, 5, 2, 9, 6, 8, 10],\ [3, 0, 1, 8, 2, 5, 4, 9, 10],\ [1, 3, 0, 7, 9, 4, 2, 10, 6],\\ &[1, 3, 6, \infty, 5, 8, 10, 9, 7],\ [\infty, 1, 5, 2, 3, 7, 6, 9, 10],\ [1, 2, 8, \infty, 3, 9, 10, 7, 4],\\ &[0, 1, 6, 4, 3, 8, 10, 7, 5],\ [0, 2, 8, 4, \infty, 1, 9, 7, 3],\ [0, 1, 6, 3, \infty, 7, 5, 8, 10],\\ &[0, 4, 7, 1, 2, 9, 6, 5, \infty],\ [0, \infty, 1, 2, 3, 4, 8, 7, 9],\ [0, 1, 2, 9, 3, 8, 7, \infty, 4]\big\}.\end{aligned}$$

Then an $LC_3^{(4)}$-decomposition of $K_{12}^{(4)}$ consists of the $LC_3^{(4)}$-blocks in $B$ under the action of the map $\infty \mapsto \infty$ and $j \mapsto j+1 \pmod{11}$.

**Example 2.9.** Let $V\Big(K_{15}^{(4)}\Big) = \mathbb{Z}_{13} \cup \{\infty_1, \infty_2\}$ and let

$$\begin{aligned} B = \big\{&[0, 1, 2, \infty_2, 5, 6, \infty_1, 11, 12],\ [0, 1, 3, \infty_2, 5, 7, \infty_1, 10, 11],\ [0, 1, 4, \infty_2, 5, 8, \infty_1, 9, 10],\\ &[0, 1, 5, \infty_2, 3, 7, \infty_1, 8, 9],\ [6, 0, 1, \infty_2, 3, 8, \infty_1, 12, 7],\ [0, 2, \infty_2, 3, 4, 9, 10, \infty_1, 12],\\ &[0, 3, \infty_2, 4, 5, 10, 9, \infty_1, 12],\ [0, 4, \infty_2, 5, 6, 9, 8, \infty_1, 12],\ [0, 5, \infty_2, 6, 8, 9, 7, \infty_1, 12],\\ &[0, 1, 7, \infty_2, 3, 9, \infty_1, 6, 12],\ [0, 2, \infty_2, 4, 5, 7, 9, \infty_1, 11],\ [0, 2, \infty_2, 5, 6, 7, 10, 8, \infty_1],\\ &[0, 2, \infty_2, 6, 12, 8, 7, \infty_1, 9],\ [0, 5, \infty_2, 7, 8, 1, 6, \infty_1, 11],\ [0, \infty_2, 8, 3, \infty_1, 6, 11, 10, 5],\\ &[0, \infty_2, 7, 3, 12, 10, 9, 6, \infty_1],\ [0, 3, \infty_2, 6, 12, 9, 7, \infty_1, 10],\ [0, 3, \infty_2, 5, 9, 10, 8, \infty_1, 11],\\ &[0, 4, \infty_2, 6, 5, 1, 7, \infty_1, 11],\ [0, 4, \infty_2, 7, 9, 2, 6, \infty_1, 10],\ [1, 4, 0, 9, 8, 12, 5, 3, 11],\\ &[0, 1, 4, 11, 12, 3, 2, 5, 7],\ [0, 1, 5, 7, 8, 12, 2, 4, 11],\ [1, 0, 9, 5, 2, 6, 10, 4, 8],\\ &[1, 0, 11, 5, 8, 12, 2, 7, 9],\ [1, 11, 6, 4, 2, 0, 5, 7, 12],\ [2, 5, 0, 6, 9, 11, 12, 7, 4],\\ &[3, 0, 6, 7, 11, 2, 4, 10, 1],\ [6, 0, 1, 11, 2, 9, 3, 8, 12],\ [1, 0, 9, 6, 4, 10, 2, 7, 11],\\ &[0, 4, \infty_2, 8, 3, 12, 9, 5, \infty_1],\ [0, 1, 2, 4, 6, 7, 8, 10, 3],\ [0, 2, 5, 11, 9, 8, 6, 3, 4],\\ &[4, 0, 1, 6, 2, 10, 8, 5, 11],\ [0, \infty_2, 2, 7, 12, 5, 6, \infty_1, 8]\big\}.\end{aligned}$$

Then an $LC_3^{(4)}$-decomposition of $K_{15}^{(4)}$ consists of the $LC_3^{(4)}$-blocks in $B$ under the action of the map $\infty_i \mapsto \infty_i$ and $j \mapsto j+1 \pmod{13}$.

**Example 2.10.** Let $V\left(L^{(4)}_{1,\overline{9},\overline{9}} \cup L^{(4)}_{\overline{9},\overline{9}}\right) = \mathbb{Z}_{18} \cup \{\infty\}$ with vertex partition $\big\{\{\infty\}, \{0,2,4,6,8,10,12,14,16\}, \{1,3,5,7,9,11,13,15,17\}\big\}$ and let

$$\begin{aligned}
B = \big\{&[\infty_1,0,1,9,16,11,8,3,2],\ [\infty_1,0,5,9,8,1,4,15,10],\ [\infty_1,0,11,9,14,13,16,15,4],\\
&[\infty_1,0,7,9,4,5,2,3,14],\ [\infty_1,0,13,9,10,17,14,3,8],\ [\infty_1,0,17,13,3,10,12,15,16],\\
&[\infty_1,0,17,9,2,7,10,15,16],\ [\infty_1,0,5,7,3,4,12,15,10],\ [\infty_1,0,11,1,3,16,12,15,4],\\
&[\infty_1,0,7,17,15,2,6,3,14],\ [\infty_1,0,13,11,15,14,6,3,8],\ [\infty_1,0,1,5,15,8,6,3,2],\\
&[0,\infty_1,1,3,2,6,15,4,9],\ [0,\infty_1,5,15,10,12,3,2,9],\ [0,\infty_1,7,3,14,6,15,10,9],\\
&[0,\infty_1,11,15,4,12,3,8,9],\ [0,\infty_1,13,3,8,6,15,16,9],\ [0,\infty_1,17,15,16,12,3,14,9],\\
&[0,1,3,6,4,5,13,12,11],\ [0,5,15,12,2,7,11,6,1],\ [0,7,3,6,10,17,1,12,5],\\
&[0,11,15,12,8,1,17,6,13],\ [0,13,3,6,16,11,7,12,17],\ [0,17,15,12,14,13,5,6,7],\\
&[0,1,2,5,4,10,14,16,9],\ [0,5,10,7,2,14,16,8,9],\ [0,7,14,17,10,16,8,4,9],\\
&[0,11,4,1,8,2,10,14,9],\ [0,13,8,11,16,4,2,10,9],\ [0,17,16,13,14,8,4,2,9],\\
&[0,1,5,7,6,10,16,12,13],\ [0,5,7,17,12,14,8,6,11],\ [0,7,17,13,6,16,4,12,1],\\
&[0,11,1,5,12,2,14,6,17],\ [0,13,11,1,6,4,10,12,7],\ [0,17,13,11,12,8,2,6,5],\\
&[0,1,3,15,14,17,7,4,16],\ [0,5,15,3,16,13,17,2,8],\ [0,7,3,15,8,11,13,10,4],\\
&[0,11,15,3,10,7,5,8,14],\ [0,13,3,15,2,5,1,16,10],\ [0,17,15,3,4,1,11,14,2],\\
&[0,1,3,7,6,9,11,12,10],\ [0,5,15,17,12,9,1,6,14],\ [0,7,3,13,6,9,5,12,16],\\
&[0,11,15,5,12,9,13,6,2],\ [0,13,3,1,6,9,17,12,4],\ [0,17,15,11,12,9,7,6,8],\\
&[0,1,2,4,5,6,14,16,17],\ [0,5,10,2,7,12,16,8,13],\ [0,7,14,10,17,6,8,4,11],\\
&[0,6,2,1,5,8,12,16,17],\ [0,12,10,5,7,4,6,8,13],\ [0,6,14,7,17,2,12,4,11],\\
&[0,\infty_1,1,2,3,7,16,11,13],\ [0,\infty_1,5,10,15,17,8,1,11],\ [0,\infty_1,7,14,3,13,4,5,1],\\
&[12,\infty_1,9,0,2,5,15,1,4],\ [3,\infty_1,6,0,1,4,15,16,2],\ [15,\infty_1,9,0,3,6,12,13,16],\\
&[0,1,7,13,10,2,5,11,12],\ [0,1,6,12,13,15,10,11,17]\big\},\\
B' = \big\{&[0,1,9,10,16,3,5,6,17],\ [1,2,10,11,17,4,6,7,0],\ [2,3,11,12,0,5,7,8,1],\\
&[3,4,12,13,1,6,8,9,2],\ [4,5,13,14,2,7,9,10,3],\ [5,6,14,15,3,8,10,11,4],\\
&[6,7,15,16,4,9,11,12,5],\ [7,8,16,17,5,10,12,13,6],\ [8,9,17,0,6,11,13,14,7],\\
&[9,3,0,12,7,1,14,15,8],\ [10,4,1,13,8,2,15,16,9],\ [11,5,2,14,9,3,16,17,10],\\
&[12,6,3,15,10,4,17,0,11],\ [13,7,4,16,11,5,0,1,12],\ [14,8,5,17,12,6,1,2,13],\\
&[15,9,6,0,13,7,2,3,14], [16,10,7,1,14,8,3,4,15],\ [17,11,8,2,15,9,4,5,16],\\
&[0,2,11,9,15,12,6,13,1],\ [1,3,12,10,16,13,7,14,2],\ [2,4,13,11,17,14,8,15,3],\\
&[3,5,14,12,0,15,9,16,4],\ [4,6,15,13,1,16,10,17,5],\ [5,7,16,14,2,17,11,0,6],\\
&[6,8,17,15,3,0,12,1,7],\ [7,9,0,16,4,1,13,2,8],\ [8,10,1,17,5,2,14,3,9],\\
&[9,13,4,0,6,3,15,4,10],\ [10,14,5,1,7,4,16,5,11],\ [11,15,6,2,8,5,17,6,12],\\
&[12,16,7,3,9,6,0,7,13],\ [13,17,8,4,10,7,1,8,14],\ [14,0,9,5,11,8,2,9,15],\\
&[15,1,10,6,12,9,3,10,16],\ [16,2,11,7,13,10,4,11,17],\ [17,3,12,8,14,11,5,12,0]\big\}.
\end{aligned}$$

Then an $LC_3^{(4)}$-decomposition of $L^{(4)}_{1,\overline{9},\overline{9}} \cup L^{(4)}_{\overline{9},\overline{9}}$ consists of the $LC_3^{(4)}$-blocks in $B$ under the action of the map $\infty \mapsto \infty$ and $j \mapsto j+1 \pmod{18}$ along with the $LC_3^{(4)}$-blocks in $B'$.

**Example 2.11.** Let $V\left(L^{(4)}_{2,9,9}\right) = \mathbb{Z}_{18} \cup \{\infty_1, \infty_2\}$ with vertex partition $\big\{\{\infty_1, \infty_2\}$, $\{0, 2, 4, 6, 8, 10, 12, 14, 16\}$, $\{1, 3, 5, 7, 9, 11, 13, 15, 17\}\big\}$ and let

$$
\begin{aligned}
B = \big\{&[\infty_1, 0, 1, 3, \infty_2, 7, 10, 6, 5],\ [\infty_1, 0, 5, 15, \infty_2, 17, 14, 12, 7],\\
&[\infty_1, 0, 7, 3, \infty_2, 13, 16, 6, 17],\ [\infty_1, 0, 11, 15, \infty_2, 5, 2, 12, 1],\\
&[\infty_1, 0, 13, 3, \infty_2, 1, 4, 6, 11],\ [\infty_1, 0, 17, 15, \infty_2, 11, 8, 12, 13],\\
&[\infty_2, 0, 5, 1, \infty_1, 13, 8, 3, 2],\ [\infty_2, 0, 7, 5, \infty_1, 11, 4, 15, 10],\\
&[\infty_2, 0, 17, 7, \infty_1, 1, 2, 3, 14],\ [\infty_2, 0, 1, 11, \infty_1, 17, 16, 15, 4],\\
&[\infty_2, 0, 11, 13, \infty_1, 7, 14, 3, 8],\ [\infty_2, 0, 13, 17, \infty_1, 5, 10, 15, 16],\\
&[3, 11, 2, \infty_2, 1, 0, \infty_1, 12, 4],\ [15, 1, 10, \infty_2, 5, 0, \infty_1, 6, 2],\\
&[3, 5, 14, \infty_2, 7, 0, \infty_1, 12, 10],\ [12, 17, 4, \infty_2, 0, 3, \infty_1, 6, 9],\\
&[\infty_1, 0, 3, 9, 2, \infty_2, 13, 4, 1],\ [\infty_1, 0, 10, 5, \infty_2, 12, 3, 16, 1],\\
&[\infty_2, 0, 1, 4, 15, \infty_1, 8, 5, 3],\ [\infty_1, 0, 1, 4, \infty_2, 13, 8, 5, 3],\\
&[\infty_1, 0, 1, 15, 6, \infty_2, 9, 7, 16],\ [\infty_2, 0, 3, 10, \infty_1, 7, 17, 4, 2],\\
&[\infty_2, 6, 3, 0, 9, \infty_1, 13, 4, 1],\ [\infty_2, 0, 1, 15, \infty_1, 6, 14, 5, 4]\big\},\\
B' = \big\{&[3, 4, 5, \infty_1, 0, 9, \infty_2, 1, 2],\ [4, 5, 6, \infty_1, 1, 10, \infty_2, 2, 3],\\
&[5, 6, 7, \infty_1, 2, 11, \infty_2, 3, 4],\ [6, 7, 8, \infty_1, 3, 12, \infty_2, 4, 5],\\
&[7, 8, 9, \infty_1, 4, 13, \infty_2, 5, 6],\ [8, 9, 10, \infty_1, 5, 14, \infty_2, 6, 7],\\
&[9, 10, 11, \infty_1, 6, 15, \infty_2, 7, 8],\ [10, 11, 12, \infty_1, 7, 16, \infty_2, 8, 9],\\
&[11, 12, 13, \infty_1, 8, 17, \infty_2, 9, 10],\ [\infty_1, 1, 4, 12, 11, \infty_2, 10, 13, 3],\\
&[\infty_1, 2, 5, 13, 12, \infty_2, 11, 14, 4],\ [\infty_1, 3, 6, 14, 13, \infty_2, 12, 15, 5],\\
&[\infty_1, 4, 7, 15, 14, \infty_2, 13, 16, 6],\ [\infty_1, 5, 8, 16, 15, \infty_2, 14, 17, 7],\\
&[\infty_1, 6, 9, 17, 16, \infty_2, 15, 0, 8],\ [\infty_1, 7, 10, 0, 17, \infty_2, 16, 1, 9],\\
&[\infty_1, 8, 11, 1, 0, \infty_2, 17, 2, 10],\ [\infty_1, 9, 12, 2, 1, \infty_2, 0, 3, 11],\\
&[\infty_2, 3, 6, 14, 13, \infty_1, 12, 15, 5],\ [\infty_2, 4, 7, 15, 14, \infty_1, 13, 16, 6],\\
&[\infty_2, 5, 8, 16, 15, \infty_1, 14, 17, 7],\ [\infty_2, 6, 9, 17, 16, \infty_1, 15, 0, 8],\\
&[\infty_2, 7, 10, 0, 17, \infty_1, 16, 1, 9],\ [\infty_2, 8, 11, 1, 0, \infty_1, 17, 2, 10],\\
&[\infty_2, 9, 12, 2, 1, \infty_1, 0, 3, 11],\ [\infty_2, 10, 13, 3, 2, \infty_1, 1, 4, 12],\\
&[\infty_2, 11, 14, 4, 3, \infty_1, 2, 5, 13]\big\}.
\end{aligned}
$$

Then an $LC^{(4)}_3$-decomposition of $L^{(4)}_{2,9,9}$ consists of the $LC^{(4)}_3$-blocks in $B$ under the action of the map $\infty_i \mapsto \infty_i$ and $j \mapsto j + 1 \pmod{18}$ along with the $LC^{(4)}_3$-blocks in $B'$.

**Example 2.12.** Let $V\left(K^{(4)}_{15} \setminus K^{(4)}_6\right) = \mathbb{Z}_9 \cup \{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5, \infty_6\}$ with $\infty_1, \ldots,$ $\infty_6$ being the vertices in the hole and let

$$
\begin{aligned}
B = \big\{&[3, \infty_1, 1, 0, \infty_2, 4, 6, \infty_3, 2],\ [6, \infty_1, 2, 0, \infty_2, 8, 3, \infty_3, 4],\ [3, \infty_1, 4, 0, \infty_2, 7, 6, \infty_3, 8],\\
&[6, \infty_1, 5, 0, \infty_2, 2, 3, \infty_3, 1],\ [3, \infty_1, 7, 0, \infty_2, 1, 6, \infty_3, 5],\ [6, \infty_1, 8, 0, \infty_2, 5, 3, \infty_3, 7],\\
&[3, \infty_4, 1, 0, \infty_5, 4, 6, \infty_6, 2],\ [6, \infty_4, 2, 0, \infty_5, 8, 3, \infty_6, 4],\ [3, \infty_4, 4, 0, \infty_5, 7, 6, \infty_6, 8],\\
&[6, \infty_4, 5, 0, \infty_5, 2, 3, \infty_6, 1],\ [3, \infty_4, 7, 0, \infty_5, 1, 6, \infty_6, 5],\ [6, \infty_4, 8, 0, \infty_5, 5, 3, \infty_6, 7],
\end{aligned}
$$

$$
\begin{aligned}
&[1,\infty_2,\infty_1,0,\infty_3,\infty_4,5,\infty_5,\infty_6],\ [2,\infty_2,\infty_1,0,\infty_3,\infty_4,1,\infty_5,\infty_6],\\
&[4,\infty_2,\infty_1,0,\infty_3,\infty_4,2,\infty_5,\infty_6],\ [1,\infty_3,\infty_2,0,\infty_4,\infty_5,5,\infty_1,\infty_6],\\
&[2,\infty_3,\infty_2,0,\infty_4,\infty_5,1,\infty_1,\infty_6],\ [4,\infty_3,\infty_2,0,\infty_4,\infty_5,2,\infty_1,\infty_6],\\
&[1,\infty_1,\infty_4,0,\infty_2,\infty_6,5,\infty_3,\infty_5],\ [2,\infty_1,\infty_4,0,\infty_2,\infty_6,1,\infty_3,\infty_5],\\
&[4,\infty_1,\infty_4,0,\infty_2,\infty_6,2,\infty_3,\infty_5],\ [1,\infty_2,\infty_5,0,\infty_4,\infty_6,5,\infty_3,\infty_1],\\
&[2,\infty_2,\infty_5,0,\infty_4,\infty_6,1,\infty_3,\infty_1],\ [4,\infty_2,\infty_5,0,\infty_4,\infty_6,2,\infty_3,\infty_1],\\
&[1,\infty_3,\infty_6,0,\infty_5,\infty_1,5,\infty_4,\infty_2],\ [2,\infty_3,\infty_6,0,\infty_5,\infty_1,1,\infty_4,\infty_2],\\
&[4,\infty_3,\infty_6,0,\infty_5,\infty_1,2,\infty_4,\infty_2],\ [0,\infty_2,\infty_1,3,\infty_3,\infty_4,6,\infty_5,\infty_6],\\
&[0,\infty_3,\infty_2,3,\infty_4,\infty_5,6,\infty_1,\infty_6],\ [0,\infty_1,\infty_4,3,\infty_2,\infty_6,6,\infty_3,\infty_5],\\
&[0,\infty_2,\infty_5,3,\infty_4,\infty_6,6,\infty_3,\infty_1],\ [0,\infty_3,\infty_6,3,\infty_5,\infty_1,6,\infty_4,\infty_2],\\
&[\infty_5,0,1,2,\infty_6,6,4,8,3],\ [\infty_6,0,1,2,\infty_5,6,4,8,3],\ [\infty_1,\infty_2,\infty_6,0,1,5,\infty_4,2,\infty_3],\\
&[\infty_2,\infty_6,0,\infty_3,1,3,5,\infty_5,\infty_1],\ [\infty_3,\infty_6,3,\infty_4,0,1,2,\infty_1,\infty_2],\\
&[\infty_4,\infty_5,\infty_6,0,1,5,\infty_3,2,\infty_2],\ [\infty_5,\infty_6,2,\infty_1,0,1,5,\infty_3,\infty_4],\\
&[\infty_1,\infty_3,\infty_6,2,1,0,\infty_2,3,\infty_4],\ [\infty_2,\infty_4,\infty_6,0,1,2,7,\infty_3,\infty_5],\\
&[\infty_5,\infty_3,\infty_6,0,2,4,\infty_1,\infty_4,1],\ [\infty_4,\infty_1,\infty_6,0,1,5,\infty_2,\infty_5,2],\\
&[\infty_5,\infty_2,\infty_6,0,1,2,4,\infty_3,\infty_1],\ [0,1,3,6,7,2,5,4,8],\ [1,0,2,3,4,6,7,8,5],\\
&[2,4,7,0,8,5,3,6,1],\ [1,0,4,7,3,5,2,6,8]\big\},
\end{aligned}
$$

$$
\begin{aligned}
B' = \big\{&[0,\infty_1,3,6,\infty_2,4,2,1,\infty_3],\ [1,\infty_1,4,7,\infty_2,5,3,2,\infty_3],\ [2,\infty_1,5,8,\infty_2,6,4,3,\infty_3],\\
&[3,\infty_4,6,0,\infty_2,7,5,4,\infty_3],\ [4,\infty_4,7,1,\infty_2,8,6,5,\infty_3],\ [5,\infty_4,8,2,\infty_2,0,7,6,\infty_3],\\
&[6,\infty_5,0,3,\infty_2,1,8,7,\infty_3],\ [7,\infty_5,1,4,\infty_2,2,0,8,\infty_3],\ [8,\infty_5,2,5,\infty_2,3,1,0,\infty_3],\\
&[3,\infty_3,6,0,\infty_4,7,5,4,\infty_1],\ [4,\infty_3,7,1,\infty_4,8,6,5,\infty_1],\ [5,\infty_3,8,2,\infty_4,0,7,6,\infty_1],\\
&[6,\infty_6,0,3,\infty_4,1,8,7,\infty_1],\ [7,\infty_6,1,4,\infty_4,2,0,8,\infty_1],\ [8,\infty_6,2,5,\infty_4,3,1,0,\infty_1],\\
&[0,\infty_2,3,6,\infty_4,4,2,1,\infty_1],\ [1,\infty_2,4,7,\infty_4,5,3,2,\infty_1],\ [2,\infty_2,5,8,\infty_4,6,4,3,\infty_1]\big\}.
\end{aligned}
$$

Then an $LC_3^{(4)}$-decomposition of $K_{15}^{(4)} \setminus K_6^{(4)}$ consists of the $LC_3^{(4)}$-blocks in $B$ under the action of the map $\infty_i \mapsto \infty_i$ and $j \mapsto j+1 \pmod 9$ along with the $LC_3^{(4)}$-blocks in $B'$.

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## References

[1] P. Adams, D. Bryant and M. Buchanan, A survey on the existence of $G$-designs, *J. Combin. Des.* **16** (2008), 373–410.

[2] M. Akin, R. C. Bunge, S. I. El-Zanati, J. Hamilton, B. Kolle, S. Lehmann and L. Neiburger, On tight 6-cycle decompositions of complete 3-uniform hypergraphs, *Discrete Math.* **345** (2022), no. 2, Paper No. 112676, 8 pp.

[3] Zs. Baranyai, On the factorization of the complete uniform hypergraph, in: Infinite and finite sets, *Colloq. Math. Soc. János Bolyai* **10**, North-Holland, Amsterdam, 1975, 91–108.

[4] R. F. Bailey and B. Stevens, Hamiltonian decompositions of complete $k$-uniform hypergraphs, *Discrete Math.* **310** (2010), 3088–3095.

[5] J.-C. Bermond, A. Germa and D. Sotteau, Hypergraph-designs, *Ars Combin.* **3** (1977), 47–66.

[6] D. Bryant, S. Herke, B. Maenhaut and W. Wannasit, Decompositions of complete 3-uniform hypergraphs into small 3-uniform hypergraphs, *Australas. J. Combin.* **60** (2014), 227–254.

[7] D. E. Bryant and T. A. McCourt, Existence results for $G$-designs, `http://wiki.smp.uq.edu.au/G-designs/`.

[8] R. C. Bunge, D. Collins, D. Conko-Camel, S. I. El-Zanati, R. Liebrecht and A. Vasquez, Maximum packings of the $\lambda$-fold complete 3-uniform hypergraph with loose 3-cycles, *Opuscula Math.* **40** (2020), 209–225.

[9] R. C. Bunge, B. D. Darrow, S. I. El-Zanati, K. P. Hadaway, M. K. Pryor, A. J. Romer, A. Squires and A. C. Stover, On tight 9-cycle decompositions of complete 3-uniform hypergraphs, *Australas. J. Combin.* **80** (2021), 233–240.

[10] R. C. Bunge, S. I. El-Zanati, L. Haman, C. Hatzer, K. Koe and K. Spornberger, On loose 4-cycle decompositions of complete 3-uniform hypergraphs, *Bull. Inst. Combin. Appl.* **87** (2019), 75–84.

[11] R. C. Bunge, S. I. El-Zanati, J. Jeffries and C. Vanden Eynden, Edge orbits and cyclic and $r$-pyramidal decompositions of complete uniform hypergraphs, *Discrete Math.* **341** (2018), 3348–3354.

[12] R. C. Bunge, S. I. El-Zanati, J. Jetton, M. Juarez, A. J. Netz, D. Roberts and P. Ward, On loose 5-cycle decompositions, packings, and coverings of complete $\lambda$-fold 3-uniform hypergraphs, *Ars Combin.* (to appear).

[13] C. J. Colbourn and R. Mathon, Steiner systems, in *The CRC Handbook of Combinatorial Designs*, 2nd edition, (Eds. C. J. Colbourn and J. H. Dinitz), CRC Press, Boca Raton (2007), 102–110.

[14] S. Glock, D. Kühn, A. Lo and D. Osthus, The existence of designs via iterative absorption, arXiv:1611.06827v2, (2017), 63 pp.

[15] S. Glock, D. Kühn, A. Lo and D. Osthus, Hypergraph $F$-designs for arbitrary $F$, arXiv:1706.01800, (2017), 72 pp.

[16] H. Hanani, On quadruple systems, *Canad. J. Math.* **12** (1960), 145–157.

[17] H. Hanani, Decomposition of hypergraphs into octahedra, Second International Conference on Combinatorial Mathematics (New York, 1978), pp. 260–264, *Ann. New York Acad. Sci.*, 319, New York Acad. Sci., New York, 1979.

[18] H. Jordon and G. Newkirk, 4-cycle decompositions of complete 3-uniform hypergraphs, *Australas. J. Combin.* **71** (2018), 312–323.

[19] P. Keevash, The existence of designs, arXiv:1401.3665v2, (2018), 39 pp.

[20] G. B. Khosrovshahi and R. Laue, $t$-designs with $t \geq 3$, in *The CRC Handbook of Combinatorial Designs*, 2nd edition, (Eds. C. J. Colbourn and J. H. Dinitz), CRC Press, Boca Raton (2007), 79–101.

[21] J. Kuhl and M. W. Schroeder, Hamilton cycle decompositions of $k$-uniform $k$-partite hypergraphs, *Australas. J. Combin.* **56** (2013), 23–37.

[22] D. Kühn and D. Osthus, Decompositions of complete uniform hypergraphs into Hamilton Berge cycles, *J. Combin. Theory Ser. A* **126** (2014), 128–135.

[23] M. Meszka and A. Rosa, Decomposing complete 3-uniform hypergraphs into Hamiltonian cycles, *Australas. J. Combin.* **45** (2009), 291–302.

[24] M. W. Schroeder, On Hamilton cycle decompositions of $r$-uniform $r$-partite hypergraphs, *Discrete Math.* **315** (2014), 1–8.

[25] R. M. Wilson, Decompositions of Complete Graphs into Subgraphs Isomorphic to a Given Graph, in "Proc. Fifth British Combinatorial Conference" (C. St. J. A. Nash-Williams and J. Sheehan, Eds.), pp. 647–659, *Congr. Numer.* XV, 1975.

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