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Tournaments and the Erdős-Hajnal Conjecture

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Abstract

The celebrated Erdős-Hajnal conjecture states that for every undirected graph H there exists $\epsilon(H) > 0$ such that every undirected graph on n vertices that does not contain H as an induced subgraph contains a clique or a stable set of size at least $n^{\epsilon(H)}$. This conjecture has a directed equivalent version stating that for every tournament H there exists $\epsilon(H) > 0$ such that every H-free n-vertex tournament T contains a transitive subtournament of order at least $n^{\epsilon(H)}$. This conjecture is proved for few infinite families of tournaments. In this paper we construct a new infinite family of tournaments — the family of so-called flotilla-galaxies, and we prove the correctness of the conjecture for every flotilla-galaxy tournament.

1 Introduction

Let G be an undirected graph. We denote by $V(G)$ the set of its vertices and by $E(G)$ the set of its edges. We call $|G| = |V(G)|$ the *order* of G. Let $X \subseteq V(G)$.

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The *subgraph of G induced by X* is denoted by $G|X$, that is, the graph with vertex set X, in which $x, y \in X$ are adjacent if and only if they are adjacent in G. A *clique* in G is a set of pairwise adjacent vertices and a *stable set* in G is a set of pairwise nonadjacent vertices. For an undirected graph H, we say that G is *H-free* if no induced subgraph of G is isomorphic to H. A *digraph* is a pair $D = (V, A)$ of sets such that $A \subset V \times V$, and such that for every $(x, y) \in A$ we must have $(y, x) \notin A$. In particular if $(x, y) \in A$, then $x \neq y$. Here A is the arc set and V is the vertex set and they are denoted by $A(D)$ and $V(D)$ respectively. We say that D' is a *subdigraph* of a digraph D if $V(D') \subseteq V(D)$ and $A(D') \subseteq A(D)$. We say that D contains a copy of D' if D' is isomorphic to a subdigraph of D. A path P is a graph whose vertex set and edge set are given by $V(P) = \{x_1, x_2, \ldots, x_n\}$, and $E(P) = \{x_i x_{i+1} : 1 \leq i \leq n-1\}$ respectively. A 4-vertex path is a path on four vertices. A *cycle* C is a graph such that $V(C) = \{x_1, x_2, \ldots, x_n\}$ and $E(C) = \{x_i x_{i+1} : 1 \leq i \leq n-1\} \cup \{x_n x_1\}$. A *directed cycle* is a digraph whose vertex set and arc set are given by $V(C) = \{x_1, x_2, \ldots, x_n\}$, and $A(C) = \{(x_i, x_{i+1}) : 1 \leq i \leq n-1\} \cup \{(x_n, x_1)\}$ respectively.

A *tournament* is a directed graph (digraph) such that for every pair u and v of distinct vertices, exactly one of the arcs (u, v) or (v, u) exists. A tournament is *transitive* if it contains no directed cycle. A *triangle* is a transitive tournament on three vertices. Let T be a tournament. We denote its vertex set by $V(T)$ and its arc set by $A(T)$, and we write $|T|$ for $|V(T)|$. The *reverse* of T, denoted by $\overline{T}$, is the tournament obtained from T by reversing the directions of all the arcs of T. Let $X \subseteq V(T)$. The *subtournament of T induced by X* is denoted by $T|X$, that is, the tournament with vertex set X, such that for $x, y \in X$, $(x, y) \in A(T|X)$ if and only if $(x, y) \in A(T)$. If $(u, v) \in A(T)$, then we say that u is *adjacent to* v (alternatively: v is an *outneighbor* of u), and we write $u \to v$. We also say that v is *adjacent from* u (alternatively: u is an *inneighbor* of v), and we write $v \leftarrow u$. For two disjoint subsets V_1 and V_2 of $V(T)$, we say that V_1 is *complete to* (respectively *from*) V_2 if every vertex of V_1 is adjacent to (respectively from) every vertex of V_2, and we write $V_1 \to V_2$ (respectively $V_1 \leftarrow V_2$). We say that a vertex v is complete to (respectively from) a set V if $\{v\}$ is complete to (respectively from) V and we write $v \to V$ (respectively $v \leftarrow V$). Given a tournament H, we say that T *contains* H if H is isomorphic to $T|X$ for some $X \subseteq V(T)$. If T does not contain H, we say that T is *H-free*.

Erdős and Hajnal proposed the following conjecture [3] (EHC):

Conjecture 1 *For any undirected graph H there exists $\epsilon(H) > 0$ such that any H-free undirected graph with n vertices contains a clique or a stable set of size at least $n^{\epsilon(H)}$.*

The following conjecture is the directed version of Conjecture 1, where graphs are replaced by tournaments, and cliques and stable sets are replaced by transitive subtournaments.

Conjecture 2 *For any tournament H there exists $\epsilon(H) > 0$ such that every H-free tournament with n vertices contains a transitive subtournament of order at least $n^{\epsilon(H)}$.*

Alon et al. proved [1] that Conjectures 1 and 2 are equivalent.

A tournament H *satisfies the Erdős-Hajnal Conjecture (EHC)* (equivalently: H has the *Erdős-Hajnal property*) if there exists $\epsilon(H) > 0$ such that every H-free tournament T with n vertices contains a transitive subtournament of order at least $n^{\epsilon(H)}$.

The Erdős-Hajnal property is a *hereditary property* [7]; that is, if a tournament H has the Erdős-Hajnal property, then all its subtournaments also have the Erdős-Hajnal property. EHC is known for all tournaments on at most six vertices except one [2, 4], and for a few infinite classes of tournaments [1, 2, 5, 9]. Also, instead of forbidding just one tournament, one can state the analogous conjecture where we forbid two tournaments. The only results in this setting are in [6, 8, 10].

Let $\theta = (v_1, \dots, v_n)$ be an ordering of the vertex set $V(D)$ of an n-vertex digraph D. An arc $(v_i, v_j) \in A(D)$ is a *backward arc of D under θ* if $i > j$. We say that a vertex v_j is *between* two vertices v_i, v_k under $\theta = (v_1, \dots, v_n)$ if $i < j < k$ or $k < j < i$. The graph of backward arcs under θ, denoted by $B(D, \theta)$, is the undirected graph that has vertex set $V(D)$, and $v_iv_j \in E(B(D, \theta))$ if and only if (v_i, v_j) or (v_j, v_i) is a backward arc of D under θ. The set of backward arcs of D under θ is denoted by $A_\theta(D)$.

Let $\theta = (v_1, \dots, v_n)$ be an ordering of the vertex set $V(T)$ of an n-vertex tournament T. We say that $V(T)$ is the *disjoint union of $X_1, \dots, X_t$ under θ* if $V(T)$ is the disjoint union of $X_1, \dots, X_t$, and $E(B(T, \theta)) = \bigcup_{i=1}^{t} E(B(T|X_i, \theta_i))$, where θ_i is the restriction of θ to X_i. A tournament S on p vertices with $V(S) = \{u_1, u_2, \dots, u_p\}$ is a *right star* (respectively, *left star*; *middle star*) if there exists an ordering $\theta^* = (u_1, u_2, \dots, u_p)$ of its vertices, such that the backward arcs of S under θ^* are (u_p, u_i) for $i = 1, \dots, p-1$ (respectively, (u_i, u_1) for $i = 2, \dots, p$; (u_i, u_r) for $i = r+1, \dots, p$ and (u_r, u_i) for $i = 1, \dots, r-1$, where $2 \leq r \leq p-1$). In this case we write $S = \{u_1, u_2, \dots, u_p\}$, and we call $\theta^* = (u_1, u_2, \dots, u_p)$ a *right star ordering* (respectively, *left star ordering*; *middle star ordering*) of S, u_p (respectively, u_1; u_r) the *center of S*, and $u_1, \dots, u_{p-1}$ (respectively, $u_2, \dots, u_p$; $u_1, \dots, u_{r-1}, u_{r+1}, \dots, u_p$) the *leaves of S*. A *star* is a left star or a right star or a middle star. A *star ordering* is a left star ordering or a right star ordering or a middle star ordering. Note that in the case $p = 2$ we may choose arbitrarily any one of the two vertices to be the center of the star, and the other vertex is then considered to be the leaf. A *frontier star* is a left star or a right star (note that a frontier star is not a middle star; a frontier star is either left or right). A *star $S = \{v_{i_1}, \dots, v_{i_t}\}$ of D under θ* (where $i_1 < \cdots < i_t$) is the subdigraph of D induced by $\{v_{i_1}, \dots, v_{i_t}\}$, such that S is a star and S has the star ordering $(v_{i_1}, \dots, v_{i_t})$ under θ (i.e. $(v_{i_1}, \dots, v_{i_t})$ is the restriction of θ to $V(S)$ and $(v_{i_1}, \dots, v_{i_t})$ is a star ordering of S).

A tournament T is a *galaxy* if there exists an ordering θ of its vertices such that $V(T)$ is the disjoint union of $V(Q_1), \dots, V(Q_l), X$ under θ, where $Q_1, \dots, Q_l$ are the frontier stars of T under θ, and for every $x \in X$, $\{x\}$ is a singleton component of $B(T, \theta)$, and no center of a star is between leaves of another star under θ. In this case we also say that T *is a galaxy under θ*. If $X = \emptyset$, we say that T is a *regular* *galaxy under* θ (see Figure 1).

Theorem 1.1 *[2] Every galaxy satisfies the Erdős-Hajnal conjecture.*

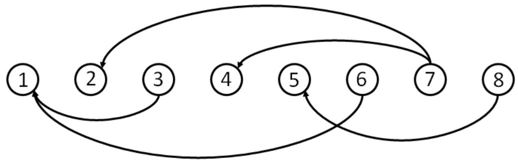

Figure 1: Galaxy under $(1, 2, \ldots, 8)$ consisting of one left star and two right stars. All the non-drawn arcs are forward.

In 2015 Choromanski [5] extended the family of galaxies to *constellations* by making the condition concerning the centers of the stars much weaker than that in galaxies, but not allowing middle stars (constellations are fully characterized in [5]):

Theorem 1.2 *[5] Every constellation satisfies the Erdős-Hajnal conjecture.*

More recently we extended the family of galaxies to *galaxies with spiders* [9] in which we allowed middle stars to exist under some conditions and we replaced the condition concerning centers of stars by a weaker one:

Theorem 1.3 *[9] Every galaxy with spiders satisfies the Erdős-Hajnal conjecture.*

A tournament T is a *nebula* if there exists an ordering θ of its vertices such that $V(T)$ is the disjoint union of $V(Q_1), \ldots, V(Q_l), X$ under θ, where Q_i is a star of T under θ (Q_i may be a middle star) for $i = 1, \ldots, l$, and for every $x \in X$, $\{x\}$ is a singleton component of $B(T, \theta)$ (note that there is no condition concerning the location of the centers of the stars and middle stars). In this case say that θ is a *nebula ordering of* T.

Unfortunately, showing that every nebula satisfies the Erdős-Hajnal conjecture is still a wide open problem and considered very hard. The only known results concerning nebulas are for galaxies, constellations and galaxies with spiders.

On the other hand, there exist infinitely many tournaments with no nebula ordering i which are not known to satisfy EHC. That motivates us to work on a new configuration of backward arcs. Our first result concerning tournaments with no nebula ordering is for an infinite class of tournaments — the so-called *asterisms* [9]. To prove EHC for asterisms we introduced a very powerful tool — the so-called *"Corresponding Digraph"* that turned out to be very useful in *flotilla-galaxies*, the infinite class treated in this paper. Flotilla-galaxy tournaments have no nebula ordering; instead, a flotilla-galaxy has a special backward arc configuration consisting of a disjoint group of 4-vertex paths and stars (note that middle stars on three vertices are allowed). Middle stars and 4-vertex paths are considered of special interest and are very hard to treat. That motivates us to work on backward arc configurations consisting of such structures.

The main result of this paper is the following:

Theorem 1.4 *Every flotilla-galaxy satisfies the Erdős-Hajnal conjecture.*

This paper is organized as follows:

- In Section 2 we formally define flotilla-galaxies.
- In Section 3 we give some properties of ϵ-critical tournaments needed in the proof of the main result in this paper.
- In Section 4 we introduce the tools that play a central role in the proof of the main result, and we prove Theorem 1.4.

2 Flotilla-Galaxy Tournaments

Our paper addresses the problem of middle stars and substructures called *boats*, and proves the conjecture for infinitely many tournaments having boats and middle stars on three vertices under some conditions that we explain in this section. A *boat* B is a 4-vertex tournament with $V(B) = \{x, u, v, y\}$ and $A(B) = \{(y, x), (y, u), (v, x), (x, u), (u, v), (v, y)\}$.

In order to define formally the infinite family of flotilla-galaxies, we need to define four special tournaments on seven vertices obtained from a boat $B = \{1, 2, 3, 4\}$. These tournaments are called *generalized boats* or *γ-boats*.

The *left γ_1-boat* is the tournament obtained from B by adding three extra vertices 5, 6 and 7, and making 5 adjacent to $\{3, 4, 7\}$, 6 adjacent to $\{4, 5\}$, and 7 adjacent to 6. The *left γ_2-boat* is the tournament obtained from the left γ_1-boat by reversing the direction of the arc $(4, 7)$. The *right γ_1-boat* (respectively, *γ_2-boat*) is the reverse of left γ_1-boat (respectively, γ_2-boat).

A *left* (respectively *right*) *γ-boat* under θ is a left γ_1-boat or a left γ_2-boat (respectively, right γ_1-boat or a right γ_2-boat). A *γ-boat under θ* is a left or a right γ-boat under θ.

In what follows we define two special orderings of the vertices of a γ-boat B^γ that will be crucial in our latter analysis. The first ordering is called the *path ordering* of B^γ, and is denoted by θ_P. And the second ordering, denoted by θ_C, is called the *cyclic ordering* of B^γ (see Figure 2):

- If B^γ is a left γ_1-boat, then:
  $\theta_P = (1, 2, 5, 3, 6, 4, 7)$, such that $A_{\theta_P}(B^\gamma) = \{(3, 1), (4, 1), (4, 2), (6, 5), (7, 6)\}$,
  $\theta_C = (2, 3, 1, 6, 5, 4, 7)$, such that $A_{\theta_C}(B^\gamma) = \{(1, 2), (4, 1), (4, 2), (5, 3), (7, 6)\}$.

- If B^γ is a left γ_2-boat, then:
  $\theta_P = (1, 2, 5, 3, 6, 7, 4)$, such that $A_{\theta_P}(B^\gamma) = \{(3, 1), (4, 1), (4, 2), (6, 5), (7, 6)\}$,
  $\theta_C = (2, 3, 1, 6, 5, 7, 4)$, such that $A_{\theta_C}(B^\gamma) = \{(1, 2), (4, 1), (4, 2), (5, 3), (7, 6)\}$.

- If B^γ is a right γ_1-boat, then:
  $\theta_P = (7, 4, 6, 3, 5, 2, 1)$, such that $A_{\theta_P}(B^\gamma) = \{(1, 3), (1, 4), (2, 4), (5, 6), (6, 7)\}$,
  $\theta_C = (7, 4, 5, 6, 1, 3, 2)$, such that $A_{\theta_C}(B^\gamma) = \{(2, 1), (1, 4), (2, 4), (3, 5), (6, 7)\}$.

- If B^γ is a right γ_2-boat, then:
  $\theta_P = (4, 7, 6, 3, 5, 2, 1)$, such that $A_{\theta_P}(B^\gamma) = \{(1, 3), (1, 4), (2, 4), (5, 6), (6, 7)\}$,
  $\theta_C = (4, 7, 5, 6, 1, 3, 2)$, such that $A_{\theta_C}(B^\gamma) = \{(2, 1), (1, 4), (2, 4), (3, 5), (6, 7)\}$.

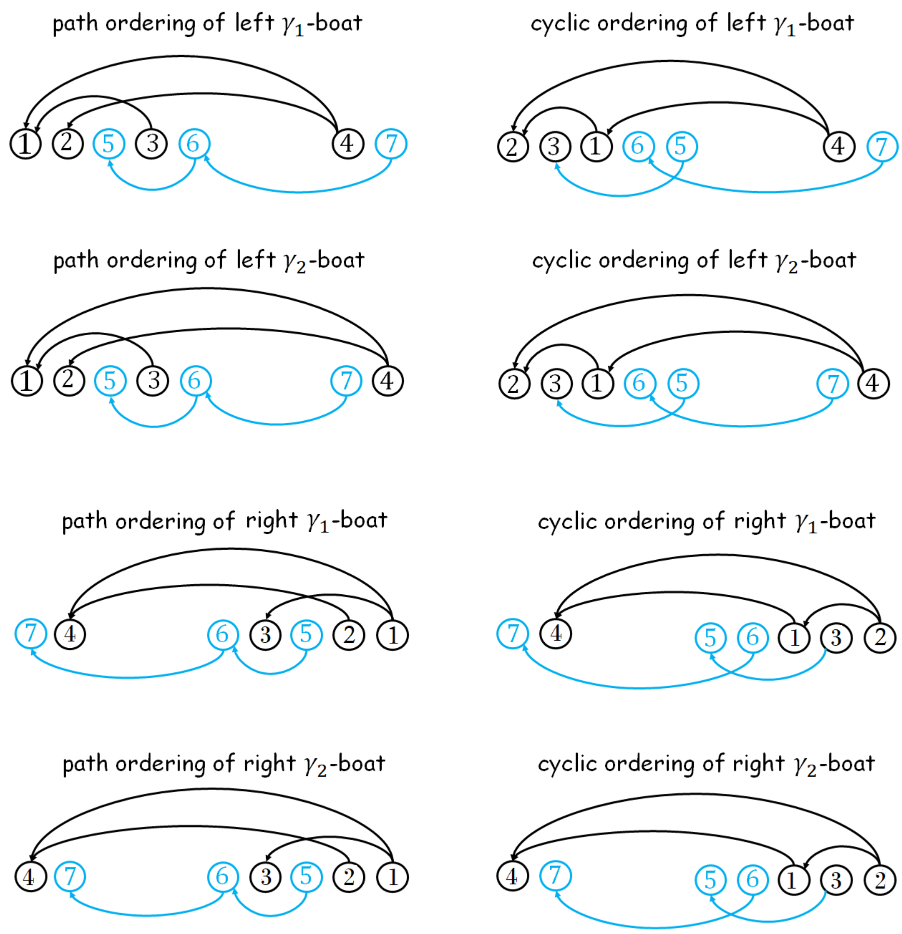

Figure 2: Crucial orderings of the vertices of left and right γ_1-boat, left and right γ_2-boat. All the non-drawn arcs are forward.

Let $\theta = (v_1, \dots, v_n)$ be an ordering of the vertex set $V(T)$ of an n-vertex tournament T. A *left γ-boat* (respectively, *right γ-boat*) $B^\gamma = \{v_{i_1}, v_{i_2}, v_{i_3}, v_{i_4}, v_{i_5}, v_{i_6}, v_{i_7}\}$ *of T under θ* is an induced subtournament of T, such that B^γ is a left γ-boat (respectively, right γ-boat), B^γ has its path ordering $(v_{i_1}, v_{i_2}, v_{i_3}, v_{i_4}, v_{i_5}, v_{i_6}, v_{i_7})$ under θ, and $v_{i_1}, \dots, v_{i_5}$ are consecutive under θ and v_{i_6}, v_{i_7} are consecutive under θ (respectively, v_{i_1}, v_{i_2} are consecutive under θ and $v_{i_3}, \dots, v_{i_7}$ are consecutive under θ).

We are ready to define formally the infinite family of flotilla-galaxies.

A tournament T is a *flotilla-galaxy* if there exists an ordering θ of its vertices such that $V(T)$ is the disjoint union of $V(B_1^\gamma), \dots, V(B_l^\gamma), X$ under θ, where $B_1^\gamma, \dots, B_l^\gamma$ are the γ-boats of T under θ, $T|X$ is a galaxy under $\overline{\theta}$ ($\overline{\theta}$ is the restriction of θ to X), and no vertex of a γ-boat appears in the ordering θ between leaves of a star of $T|X$. We also say that T *is a flotilla-galaxy under* θ, and θ is called a *flotilla-galaxy* *ordering of* T (see Figure 3). If $T|X$ is a regular galaxy under $\overline{\theta}$ and the number of the frontier stars of $T|X$ under $\overline{\theta}$ is l, then T is called a *regular flotilla-galaxy under* θ.

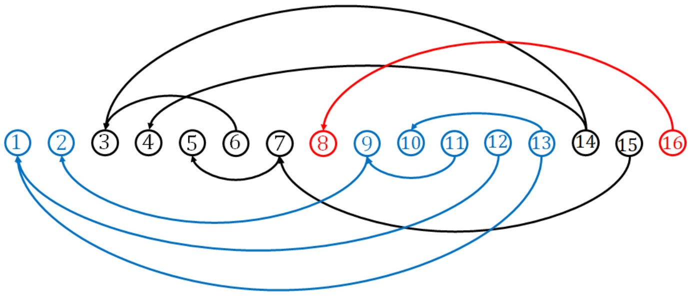

Figure 3: Flotilla-galaxy under $\theta = (1, \ldots, 16)$ consisting of one left γ_1-boat, one right γ_2-boat, and one right star. All the non-drawn arcs are forward.

3 ϵ-critical tournaments

Denote by $tr(T)$ the largest order of a transitive subtournament of a tournament T. For $X \subseteq V(T)$, write $tr(X)$ for $tr(T|X)$. Let $X, Y \subseteq V(T)$ be disjoint. Denote by $e_{X,Y}$ the number of directed arcs (x, y), where $x \in X$ and $y \in Y$. The *directed density* from X to Y is defined as $d(X, Y) = \frac{e_{X,Y}}{|X|.|Y|}$.

We say that T is ϵ-*critical* for $\epsilon > 0$ if $tr(T) < |T|^\epsilon$ but for every proper subtournament S of T, we have: $tr(S) \geq |S|^\epsilon$. The following are some properties of ϵ-critical tournaments that we borrow from [2, 4, 8, 9]. For the reader's convenience we include the proof of some lemmas (the proofs are stated exactly as in [2, 8, 9]).

Lemma 3.1 *[2] For every $N > 0$, there exists $\epsilon(N) > 0$ such that for every $0 < \epsilon < \epsilon(N)$, every ϵ-critical tournament T satisfies $|T| \geq N$.*

Proof. Since every tournament contains a transitive subtournament of order 2, it suffices to take $\epsilon(N) = \log_N(2)$. □

Lemma 3.2 *[2] Let T be an ϵ-critical tournament with $|T| = n$ and $\epsilon, c, f > 0$ be constants such that $\epsilon < \log_c(1 - f)$. Then for every $A \subseteq V(T)$ with $|A| \geq cn$ and every transitive subtournament G of T with $|G| \geq f.tr(T)$ and $V(G) \cap A = \emptyset$, we have: A is not complete from $V(G)$ and A is not complete to $V(G)$.*

Lemma 3.3 *[2] Let T be an ϵ-critical tournament with $|T| = n$ and $\epsilon, c > 0$ be constants such that $\epsilon < \log_{\frac{c}{2}}(\frac{1}{2})$. Then for every two disjoint subsets $X, Y \subseteq V(T)$ with $|X| \geq cn, |Y| \geq cn$, there exists an integer $k \geq \frac{cn}{2}$, and vertices $x_1, \ldots, x_k \in X$ and $y_1, \ldots, y_k \in Y$ such that y_i is adjacent to x_i for $i = 1, \ldots, k$.*

Lemma 3.4 *[9] Let $f_1,\dots,f_m,c,\epsilon>0$ be constants, where $0<f_1,\dots,f_m,c<1$ and $0<\epsilon<\log_{\frac{c}{2m}}(1-f_i)$ for $i=1,\dots,m$. Let T be an ϵ-critical tournament with $|T|=n$, and let $S_1,\dots,S_m$ be m disjoint transitive subtournaments of T with $|S_i| \geq f_i.tr(T)$ for $i=1,\dots,m$. Let $A\subseteq V(T)\backslash(\bigcup_{i=1}^m V(S_i))$ with $|A|\geq cn$. Then there exist vertices $s_1,\dots,s_m,a$ such that $a\in A$, $s_i\in S_i$ for $i=1,\dots,m$, and $\{a\}$ is complete to $\{s_1,\dots,s_m\}$. Similarly there exist vertices $u_1,\dots,u_m,b$ such that $b\in A$, $u_i\in S_i$ for $i=1,\dots,m$, and $\{b\}$ is complete from $\{u_1,\dots,u_m\}$.*

Proof. Let $A_i\subseteq A$ such that A_i is complete from S_i for $i=1,\dots,m$. i Let $1\leq j\leq m$. If $|A_j|\geq\frac{|A|}{2m}\geq\frac{c}{2m}n$, then this will contradict Lemma 3.2 since $|S_j|\geq f_jtr(T)$ and $\epsilon<\log_{\frac{c}{2m}}(1-f_j)$. Then for all $i\in\{1,\dots,m\}$, $|A_i|<\frac{|A|}{2m}$. Let $A^*=A\backslash(\bigcup_{i=1}^m A_i)$; then $|A^*|>|A|-m.\frac{|A|}{2m}\geq\frac{|A|}{2}$. Then $A^*\neq\emptyset$. Fix $a\in A^*$. So there exist vertices $s_1,\dots,s_m$, such that $s_i\in S_i$ for $i=1,\dots,m$, and $\{a\}$ is complete to $\{s_1,\dots,s_m\}$.

Analogously we can prove the last sentence stated in Lemma 3.4. □

The proof of the following lemma is completely analogous to the proof of Lemma 3.4.

Lemma 3.5 *[8] Let $f_1,f_2,c,\epsilon>0$ be constants, where $0<f_1,f_2,c<1$ and $0<\epsilon<\min\{\log_{\frac{c}{4}}(1-f_1),\log_{\frac{c}{4}}(1-f_2)\}$. Let T be an ϵ-critical tournament with $|T|=n$, and let S_1,S_2 be two disjoint transitive subtournaments of T with $|S_1|\geq f_1.tr(T)$ and $|S_2|\geq f_2.tr(T)$. Let $A\subseteq V(T)\backslash(V(S_1)\cup V(S_2))$ with $|A|\geq cn$. Then there exist vertices a,s_1,s_2 such that $a\in A, s_1\in S_1, s_2\in S_2$ and $s_1\leftarrow a\leftarrow s_2$.*

Proof. Let A_1 be the set of vertices of A that are complete from S_1, and let A_2 be the set of vertices of A that are complete to S_2. Assume that $|A_1|\geq\frac{|A|}{4}\geq\frac{c}{4}n$. Since $\epsilon<\log_{\frac{c}{4}}(1-f_1)$, it follows that Lemma 3.2 implies that S_1 is not complete to A_1, a contradiction. Then $|A_1|<\frac{|A|}{4}$. Similarly we prove that $|A_2|<\frac{|A|}{4}$. Now let $A^*=A\backslash(A_1\cup A_2)$; then $|A^*|>\frac{|A|}{2}$. Then $A^*\neq\emptyset$. Fix $a\in A^*$. So there exists $s_1\in S_1$ and $s_2\in S_2$, such that $s_1\leftarrow a\leftarrow s_2$. □ i

Lemma 3.6 *[8] Let $f,c,\epsilon>0$ be constants, where $0<f,c<1$ and $0<\epsilon<\min\{\log_{\frac{c}{2}}(1-f),\log_{\frac{c}{4}}(\frac{1}{2})\}$. Let T be an ϵ-critical tournament with $|T|=n$, and let S_1,S_2 be two disjoint transitive subtournaments of T with $|S_1|\geq f.tr(T)$ and $|S_2| \geq f.tr(T)$. Let A_1,A_2 be two disjoint subsets of $V(T)$ with $|A_1|\geq cn$, $|A_2|\geq cn$, and $A_1,A_2\subseteq V(T)\backslash(V(S_1)\cup V(S_2))$. Then there exist vertices a,x,s_1,s_2 such that $a\in A_1,x\in A_2,s_1\in S_1,s_2\in S_2$, $\{a,s_1\}\leftarrow x$, and $a\leftarrow s_2$.*

Proof. Let $A_1^*=\{a\in A_1 :$ there exists $s\in S_2$ such that $a\leftarrow s\}$ and let $A_2^*=\{x\in A_2 :$ there exists $v\in S_1$ such that $v\leftarrow x\}$. Then $A_1\backslash A_1^*$ is complete to S_2 and $A_2\backslash A_2^*$ is complete from S_1. Now assume that $|A_1^*|<\frac{|A_1|}{2}$, then $|A_1\backslash A_1^*|\geq\frac{|A_1|}{2}\geq\frac{c}{2}n$. Since $|S_2|\geq f.tr(T)$ and since $\epsilon<\log_{\frac{c}{2}}(1-f)$, then Lemma 3.2 implies that $A_1\backslash A_1^*$ is not complete to S_2, a contradiction. Then $|A_1^*|\geq\frac{|A_1|}{2}\geq\frac{c}{2}n$. Similarly we prove that

$|A_2^*| \geq \frac{c}{2}n$. Now since $\epsilon < \log_{\frac{c}{4}}(\frac{1}{2})$, then Lemma 3.3 implies that there exist $k \geq \frac{c}{4}n$, $a_1, \ldots, a_k \in A_1^*$, and $x_1, \ldots, x_k \in A_2^*$, such that $a_i \leftarrow x_i$ for $i = 1, \ldots, k$. So there exists $s_1 \in S_1$ and $s_2 \in S_2$, such that $\{a_1, s_1\} \leftarrow x_1$, and $a_1 \leftarrow s_2$. $\square$

Lemma 3.7 *[2] Let A_1, A_2 be two disjoint sets such that $d(A_1, A_2) \geq 1 - \lambda$ and let $0 < \eta_1, \eta_2 \leq 1$. Let $\widehat{\lambda} = \frac{\lambda}{\eta_1 \eta_2}$. Let $X \subseteq A_1, Y \subseteq A_2$ be such that $|X| \geq \eta_1 |A_1|$ and $|Y| \geq \eta_2 |A_2|$. Then $d(X, Y) \geq 1 - \widehat{\lambda}$.*

The following is introduced in [4].
Let $c > 0, 0 < \lambda < 1$ be constants, and let w be a $\{0, 1\}$-vector of length $|w|$. Let T be a tournament with $|T| = n$. A sequence of disjoint subsets $\chi = (S_1, S_2, \ldots, S_{|w|})$ of $V(T)$ is a *smooth* (c, λ, w)*-structure* if:
- whenever $w_i = 0$ we have $|S_i| \geq cn$ (we say that S_i is a *linear set*);
- whenever $w_i = 1$ the tournament $T|S_i$ is transitive and $|S_i| \geq c.tr(T)$ (we say that S_i is a *transitive set*);
- $d(\{v\}, S_j) \geq 1 - \lambda$ for $v \in S_i$ and $d(S_i, \{v\}) \geq 1 - \lambda$ for $v \in S_j, i < j$ (we say that χ is *smooth*).

Theorem 3.8 *[4] Let S be a tournament, let w be a $\{0, 1\}$-vector, and let $0 < \lambda_0 < \frac{1}{2}$ be a constant. Then there exist $\epsilon_0, c_0 > 0$ such that for every $0 < \epsilon < \epsilon_0$, every S-free ϵ-critical tournament contains a smooth (c_0, λ_0, w)-structure.*

Let $(S_1, \ldots, S_{|w|})$ be a smooth (c, λ, w)-structure of a tournament T, let $i \in \{1, \ldots, |w|\}$, and let $v \in S_i$. For $j \in \{1, 2, \ldots, |w|\} \backslash \{i\}$, denote by $S_{j,v}$ the set of the vertices of S_j adjacent from v for $j > i$ and adjacent to v for $j < i$.

Lemma 3.9 *[8] Let $0 < \lambda < 1$, $0 < \gamma \leq 1$ be constants and let w be a $\{0, 1\}$-vector. Let $(S_1, \ldots, S_{|w|})$ be a smooth (c, λ, w)-structure of a tournament T for some $c > 0$. Let $j \in \{1, \ldots, |w|\}$. Let $S_j^* \subseteq S_j$ such that $|S_j^*| \geq \gamma |S_j|$ and let $A = \{x_1, \ldots, x_k\} \subseteq \bigcup_{i \neq j} S_i$ for some positive integer k. Then $|\bigcap_{x \in A} S_{j,x}^*| \geq (1 - k\frac{\lambda}{\gamma}) |S_j^*|$. In particular $|\bigcap_{x \in A} S_{j,x}| \geq (1 - k\lambda) |S_j|$.*

Proof. The proof is by induction on k. Without loss of generality, assume that $x_1 \in S_i$ and $j < i$. Since $|S_j^*| \geq \gamma |S_j|$ then by Lemma 3.7, $d(S_j^*, \{x_1\}) \geq 1 - \frac{\lambda}{\gamma}$. So $1 - \frac{\lambda}{\gamma} \leq d(S_j^*, \{x_1\}) = \frac{|S_{j,x_1}^*|}{|S_j^*|}$. Then $|S_{j,x_1}^*| \geq (1 - \frac{\lambda}{\gamma})|S_j^*|$ and so true for $k = 1$. Suppose the statement is true for $k - 1$. Here,

$$\begin{aligned}
|\bigcap_{x \in A} S_{j,x}^*| &= |(\bigcap_{x \in A \backslash \{x_1\}} S_{j,x}^*) \cap S_{j,x_1}^*| \\
&= |\bigcap_{x \in A \backslash \{x_1\}} S_{j,x}^*| + |S_{j,x_1}^*| - |(\bigcap_{x \in A \backslash \{x_1\}} S_{j,x}^*) \cup S_{j,x_1}^*| \\
&\geq (1 - (k-1)\frac{\lambda}{\gamma})|S_j^*| + (1 - \frac{\lambda}{\gamma})|S_j^*| - |S_j^*| \\
&= (1 - k\frac{\lambda}{\gamma})|S_j^*|.
\end{aligned}$$

$\square$

4 EHC for flotilla-galaxies

In the following subsection we introduce some tools and technical definitions used to prove EHC for flotilla-galaxies.

4.1 Definitions and tools

In order to make the proof of Theorem 1.4 easy to follow, for a given flotilla-galaxy G, we define below some of its vertex orderings that are obtained from the flotilla-galaxy ordering under performing permutation for some of the vertices. In Theorem 4.2, we prove that the corresponding digraph of G is contained in an ϵ-critical tournament, for some $\epsilon > 0$. In order to make the proof easier, we define in Section 4.1.2, the corresponding digraph of G that is constructed using the backward arc configurations of all the orderings we define in Section 4.1.1. Our goal is to use the corresponding digraph to prove that G is contained in an ϵ-critical tournament for some $\epsilon > 0$.

4.1.1 Crucial orderings of a flotilla-galaxy

Let D be a tournament with seven vertices $v_1, \dots, v_7$ and let $\theta_1 = (v_1, v_2, v_3, v_4, v_5, v_6, v_7)$ be an ordering of $V(D)$. Let *operation* 1 be the permutation of the vertices $v_1, \dots, v_7$ that converts the ordering θ_1 to the ordering $\theta_2 = (v_2, v_4, v_1, v_5, v_3, v_6, v_7)$ of $V(D)$, and let *operation* 2 be the permutation of the vertices $v_1, \dots, v_7$ that converts the ordering θ_1 to the ordering $\theta_3 = (v_1, v_2, v_5, v_3, v_7, v_4, v_6)$ of $V(D)$. Let H be a flotilla-galaxy under an ordering $\theta = (v_1, \dots, v_h)$ of its vertices with $|H| = h$. Let $B_1^\gamma, \dots, B_l^\gamma$ be the γ-boats of H under θ. Let $i \in \{1, \dots, l\}$. If B_i^γ is a left γ-boat (respectively, right γ-boat), let θ_i (respectively, θ^i) be the restriction of θ to the vertices of B_i^γ. Define $\Theta_H(\theta) = \{\theta'$ an ordering of $V(H)$; θ' is obtained from θ by performing operation 1 to k θ_i's, and operation 2 to t θ^i's, with $0 \leq k, t \leq l$ $\}$. Notice that when $k = t = 0$, θ' is exactly the ordering θ. Here $|\Theta_H(\theta)| = 2^l$. Clearly one can notice that when applying such operations to the vertices of γ-boats, the 4-vertex path and the 3-vertex middle star will be transformed into a triangle and two 2-vertex stars, which is very interesting (see Figure 2).

4.1.2 Corresponding digraph

Unlike galaxies and constellations, in flotilla-galaxies, the flotilla-galaxy ordering alone failed to make the proof for flotilla-galaxies work. To this end we started thinking about other crucial orderings that give different backward arc configurations like triangles and stars. The corresponding digraph is a tool we introduced in [9] for which we construct, starting from a tournament H, a new larger digraph following all backward arc configurations of H under different crucial orderings of its vertex set. Note that in all cases discussed in the proof of Corollary 4.3, H can be extracted from an ϵ-critical tournament T using its corresponding digraph.

Let H be a regular flotilla-galaxy under an ordering $\theta = (v_1, \ldots, v_h)$ of its vertices with $|H| = h$. Let $B_1^\gamma, \ldots, B_l^\gamma$ be the γ-boats of H under θ. In what follows we explain how we constructed the *corresponding digraph of H under θ*. We call this digraph the *helping digraph* (or *key digraph*) due to its impact on the proof of our result. For $i = 1, \ldots, l$, let α_i be the restriction of θ to $V(B_i^\gamma)$.

$*$ If $B_i^\gamma = \{v_{s_i}, \ldots, v_{s_i+4}, v_{q_i}, v_{q_i+1}\}$ is a left γ_1-boat, let $B_i^{\gamma+}$ be the tournament obtained from B_i^γ, such that $E(B(B_i^{\gamma+}, \hat{\alpha})) = E(B(B_i^\gamma, \alpha)) \cup \{x_i g_i, w_i m_i, r_i y_i\}$, where $\hat{\alpha} := (v_{s_i}, x_i, v_{s_i+1}, v_{s_i+2}, v_{s_i+3}, w_i, v_{s_i+4}, r_i, g_i, m_i, v_{q_i}, v_{q_i+1}, y_i)$. Let $\widehat{B_i^\gamma}$ be the digraph obtained from $B_i^{\gamma+}$ after deleting the arcs $(v_{s_i+2}, v_{q_i+1}), (v_{s_i+3}, v_{q_i}), (v_{s_i}, v_{s_i+1})$. We write $\widehat{B_i^\gamma} = \{v_{s_i}, x_i, v_{s_i+1}, v_{s_i+2}, v_{s_i+3}, w_i, v_{s_i+4}, r_i, g_i, m_i, v_{q_i}, v_{q_i+1}, y_i\}$, and we call $\hat{\alpha}$ the *forest ordering of* $\widehat{B_i^\gamma}$ and $x_i, v_{s_i+1}, v_{s_i+2}, v_{s_i+3}, w_i, r_i, v_{q_i+1}$ the *leaves of* $\widehat{B_i^\gamma}$. Here $\widehat{B_i^\gamma}$ is called the *mutant left γ_1-boat* (see Figure 4).

$*$ If $B_i^\gamma = \{v_{s_i}, \ldots, v_{s_i+4}, v_{q_i-1}, v_{q_i}\}$ is a left γ_2-boat, let $B_i^{\gamma+}$ be the 13-vertex tournament obtained from B_i^γ, such that $E(B(B_i^{\gamma+}, \hat{\alpha})) = E(B(B_i^\gamma, \alpha)) \cup \{x_i g_i, w_i m_i, r_i y_i\}$, where $\hat{\alpha} := (v_{s_i}, x_i, v_{s_i+1}, v_{s_i+2}, v_{s_i+3}, w_i, v_{s_i+4}, r_i, g_i, m_i, y_i, v_{q_i-1}, v_{q_i})$. Let $\widehat{B_i^\gamma}$ be the digraph obtained from $B_i^{\gamma+}$ by deleting the arcs (v_{s_i+2}, v_{q_i-1}), (v_{s_i+3}, v_{q_i}), (v_{s_i}, v_{s_i+1}). We write $\widehat{B_i^\gamma} = \{v_{s_i}, x_i, v_{s_i+1}, v_{s_i+2}, v_{s_i+3}, w_i, v_{s_i+4}, r_i, g_i, m_i, y_i, v_{q_i-1}, v_{q_i}\}$, and we call $\hat{\alpha}$ the *forest ordering of* $\widehat{B_i^\gamma}$ and $x_i, v_{s_i+1}, v_{s_i+2}, v_{s_i+3}, w_i, r_i, v_{q_i-1}$ the *leaves of* $\widehat{B_i^\gamma}$. Here we call $\widehat{B_i^\gamma}$ the *mutant left γ_2-boat* (see Figure 4).

$*$ If $B_i^\gamma = \{v_{q_i-1}, v_{q_i}, v_{s_i}, \ldots, v_{s_i+4}\}$ is a right γ_1-boat, let $B_i^{\gamma+}$ be the tournament obtained from B_i^γ, such that $E(B(B_i^{\gamma+}, \hat{\alpha})) = E(B(B_i^\gamma, \alpha)) \cup \{x_i g_i, w_i m_i, r_i y_i\}$, where $\hat{\alpha} := (y_i, v_{q_i-1}, v_{q_i}, m_i, g_i, r_i, v_{s_i}, w_i, v_{s_i+1}, v_{s_i+2}, v_{s_i+3}, x_i, v_{s_i+4})$. Let $\widehat{B_i^\gamma}$ be the digraph obtained from $B_i^{\gamma+}$ by deleting the arcs $(v_{q_i-1}, v_{s_i+2}), (v_{q_i}, v_{s_i+1})$, (v_{s_i+3}, v_{s_i+4}). We write $\widehat{B_i^\gamma} = \{y_i, v_{q_i-1}, v_{q_i}, m_i, g_i, r_i, v_{s_i}, w_i, v_{s_i+1}, v_{s_i+2}, v_{s_i+3}, x_i, v_{s_i+4}\}$, and we call $\hat{\alpha}$ the *forest ordering of* $\widehat{B_i^\gamma}$ and $x_i, v_{s_i+1}, v_{s_i+2}, v_{s_i+3}, w_i, r_i, v_{q_i-1}$ the *leaves of* $\widehat{B_i^\gamma}$. We say that $\widehat{B_i^\gamma}$ is the *mutant right γ_1-boat* (see Figure 4).

$*$ If $B_i^\gamma = \{v_{q_i}, v_{q_i+1}, v_{s_i}, \ldots, v_{s_i+4}\}$ is a right γ_2-boat, let $B_i^{\gamma+}$ be the tournament obtained from B_i^γ, such that $E(B(B_i^{\gamma+}, \hat{\alpha})) = E(B(B_i^\gamma, \alpha)) \cup \{x_i g_i, w_i m_i, r_i y_i\}$, where $\hat{\alpha} := (v_{q_i}, v_{q_i+1}, y_i, m_i, g_i, r_i, v_{s_i}, w_i, v_{s_i+1}, v_{s_i+2}, v_{s_i+3}, x_i, v_{s_i+4})$. Let $\widehat{B_i^\gamma}$ be the digraph obtained from $B_i^{\gamma+}$ by deleting the arcs $(v_{q_i+1}, v_{s_i+2}), (v_{q_i}, v_{s_i+1})$, (v_{s_i+3}, v_{s_i+4}). We write $\widehat{B_i^\gamma} = \{v_{q_i}, v_{q_i+1}, y_i, m_i, g_i, r_i, v_{s_i}, w_i, v_{s_i+1}, v_{s_i+2}, v_{s_i+3}, x_i, v_{s_i+4}\}$, and we call $\hat{\alpha}$ the *forest ordering of* $\widehat{B_i^\gamma}$ and $x_i, v_{s_i+1}, v_{s_i+2}, v_{s_i+3}, w_i, r_i, v_{q_i+1}$ the *leaves of* $\widehat{B_i^\gamma}$. We say that $\widehat{B_i^\gamma}$ is the *mutant right γ_2-boat* (see Figure 4).

We are ready now to define the *corresponding digraph of a flotilla-galaxy.*

Let $\widehat{H}$ be the digraph obtained from H by replacing B_i^γ by its corresponding digraph $\widehat{B_i^\gamma}$ for $i = 1, \ldots, l$, and let $\widehat{\theta}$ be the obtained ordering of $\widehat{H}$ (i.e. $\widehat{\theta}$ is obtained from θ by replacing the vertices of B_i^γ by the vertices of $\widehat{B_i^\gamma}$ for $i = 1, \ldots, l$, such that for all $1 \leq i \leq l$, $\widehat{B_i^\gamma}$ has its forest ordering under $\widehat{\theta}$, and such that:

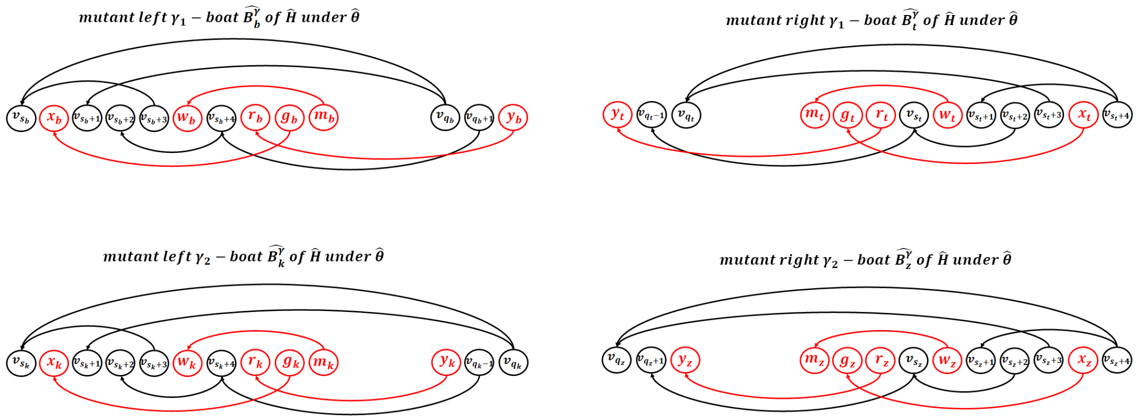

Figure 4: Mutant left γ_1-boat $\widehat{B_b^\gamma}$, mutant right γ_1-boat $\widehat{B_t^\gamma}$, mutant left γ_2-boat $\widehat{B_k^\gamma}$, and mutant right γ_2-boat $\widehat{B_z^\gamma}$. All the backward arcs are drawn. All the non-drawn arcs are forward except that the arcs (v_{s_b+2}, v_{q_b+1}), (v_{s_b+3}, v_{q_b}), $(v_{s_b}, v_{s_b+1}) \notin A(\widehat{B_b^\gamma})$, (v_{s_k+2}, v_{q_k-1}), (v_{s_k+3}, v_{q_k}), $(v_{s_k}, v_{s_k+1}) \notin A(\widehat{B_k^\gamma})$, (v_{q_t-1}, v_{s_t+2}), (v_{s_t+3}, v_{s_t+4}), $(v_{q_t}, v_{s_t+1}) \notin A(\widehat{B_t^\gamma})$, (v_{q_z+1}, v_{s_z+2}), (v_{s_z+3}, v_{s_z+4}), $(v_{q_z}, v_{s_z+1}) \notin A(\widehat{B_z^\gamma})$.

- if B_i^γ is a left γ_1-boat, then $v_{s_i}, x_i, v_{s_i+1}, v_{s_i+2}, v_{s_i+3}, w_i, v_{s_i+4}, r_i, g_i, m_i$ are consecutive under $\widehat{\theta}$, and v_{q_i}, v_{q_i+1}, y_i are consecutive under $\widehat{\theta}$;
- if B_i^γ is a left γ_2-boat, then $v_{s_i}, x_i, v_{s_i+1}, v_{s_i+2}, v_{s_i+3}, w_i, v_{s_i+4}, r_i, g_i, m_i$ are consecutive under $\widehat{\theta}$, and y_i, v_{q_i-1}, v_{q_i} are consecutive under $\widehat{\theta}$;
- if B_i^γ is a right γ_1-boat, then y_i, v_{q_i-1}, v_{q_i} are consecutive under $\widehat{\theta}$, and $m_i, g_i, r_i, v_{s_i}, w_i, v_{s_i+1}, v_{s_i+2}, v_{s_i+3}, x_i, v_{s_i+4}$ are consecutive under $\widehat{\theta}$;
- if B_i^γ is a right γ_2-boat, then v_{q_i}, v_{q_i+1}, y_i are consecutive under $\widehat{\theta}$, and $m_i, g_i, r_i, v_{s_i}, w_i, v_{s_i+1}, v_{s_i+2}, v_{s_i+3}, x_i, v_{s_i+4}$ are consecutive under $\widehat{\theta}$).

We have $V(\widehat{H}) = V(H) \cup (\bigcup_{i=1}^{l}\{x_i, w_i, r_i, g_i, m_i, y_i\})$ and

$$E(B(\widehat{H}, \widehat{\theta})) = E(B(H, \theta)) \cup \{x_i g_i, w_i m_i, r_i y_i : i = 1, \ldots, l\}.$$

In what follows we give a detailed description of $A(\widehat{H})$:

- Since $\widehat{H}$ is obtained from H by replacing B_i^γ by $\widehat{B_i^\gamma}$ for $i = 1, .., l$, then the arcs of $\bigcup_{i=1}^{l} B_i^\gamma$ are removed, and the arcs of $\bigcup_{i=1}^{l} \widehat{B_i^\gamma}$ are added to $A(\widehat{H})$. Thus, $A(\widehat{H})$ contains the arcs $A := (A(H) \setminus \bigcup_{i=1}^{l} A(B_i^\gamma)) \cup \bigcup_{i=1}^{l} A(\widehat{B_i^\gamma})$.
- Let $i \in \{1, \ldots, l\}$ and let $V_i := V(\widehat{B_i^\gamma})$. After replacing B_i^γ by $\widehat{B_i^\gamma}$, it remains to describe the orientation of the arcs connecting V_i with $V(\widehat{H}) \setminus V_i$. Let $p \in V_i$ and let $x \in V(\widehat{H}) \setminus V_i$. Simply we can say that the arc connecting x and p is a forward arc under $\widehat{\theta}$. More formally $(x, p) \in A(\widehat{H})$ if $x <_{\widehat{\theta}} p$, and $(p, x) \in A(\widehat{H})$

if $p <_{\widehat{\theta}} x$. Thus, $A(\widehat{H})$ contains the arcs $B := \bigcup_{i=1}^{l}[\bigcup_{p \in V_i}(\{(x,p) : x <_{\widehat{\theta}} p$ and $x \in V(\widehat{H}) \backslash V_i\} \cup \{(p,x) : p <_{\widehat{\theta}} x$ and $x \in V(\widehat{H}) \backslash V_i\})]$.

Now according to the first two bullet points, $A(\widehat{H}) = A \cup B$.

We say that $\widehat{H}$ is the *digraph corresponding to H under θ*, and $\widehat{\theta}$ is the *ordering of $V(\widehat{H})$ corresponding to θ.*

4.1.3 Corresponding smooth (c, λ, w)-structure

Let $\chi := (A_1, \dots, A_{|w|})$ be a smooth (c, λ, w)-structure of a tournament T, where c and λ are positive constants. Let A_i be a transitive set of χ. Let $A_i^1, \dots, A_i^r$ be disjoint subsets of A_i with approximately the same size. $\{A_i^1, \dots, A_i^r\}$ is called a *transitive partition* of A_i if for all $j \in \{1, \dots, r-1\}$, A_i^j is complete to A_i^t, for all $t \in \{j+1, \dots, r\}$.

Let s be a $\{0,1\}$-vector and let s_c be the vector obtained from s by replacing every subsequence of consecutive 1's by a single 1. In other words, s_c is obtained from s by contracting every subsequence of more than one consecutive 1's by just a single 1 (the c in s_c stands for contraction and it is not a number). Let $z := s_c$ and let i be such that $z_i = 1$. Let j be such that $s_j = 1$. We say that s_j *corresponds to* z_i if s_j belongs to the subsequence of consecutive 1's that is replaced by the entry z_i. Below we explain the importance of s_c in defining corresponding smooth structures.

Let H be a regular flotilla-galaxy under an ordering θ of its vertices. Let $B_1^\gamma, \dots, B_l^\gamma$ be the γ-boats of H under θ. In the proof we showed that we can construct a copy of $\widehat{H}$ in a tournament T. Specifically, we constructed this copy in some smooth (c, λ, w)-structure in T. Denote this structure by χ. It is done in a way that consecutive leaves are constructed in the same transitive set of χ. Recall that each 1 in w corresponds to a transitive set in χ. Here the role of s and s_c appears. s and s_c are used to encode the structure of $\widehat{H}$ under $\widehat{\theta}$. Each 1 in s corresponds to a leaf of one of the stars or one of the $\widehat{B_j^\gamma}$'s of $\widehat{H}$ under $\widehat{\theta}$. Consecutive 1's correspond to consecutive leaves. Since we constructed consecutive leaves in the same transitive set, we replaced every subsequence of consecutive 1's in s by only a single 1. So by this way instead of choosing $w = s$ and constructing consecutive leaves in distinct consecutive transitive sets, we take $w = s_c$ and construct consecutive leaves in the same transitive set. This is done by using a transitive partition of the transitive set to get for free the right type of adjacency between leaves constructed in the same transitive set. In this case we say that $\chi := (A_1, \dots, A_{|s_c|})$ is a corresponding smooth (c, λ, s_c)-structure of $\widehat{H}$ under $\widehat{\theta}$ (see Figure 5).

Let $1 \leq i \leq l$ and let $\Sigma_i := \widehat{B_i^\gamma} \cup Q_i$. In the proof, we constructed Σ_i's one by one in a way that we can merge a copy of some Σ_i by the previous constructed Σ_i's, and so constructing a copy of $\widehat{H}$ in χ. To this end, we used induction. We would like to give a brief description of how the induction works, shed the light on an issue that we may face when applying the induction hypothesis, and how we will deal with such an issue (this will explain why dealing with s_c only is not sufficient, and so this is

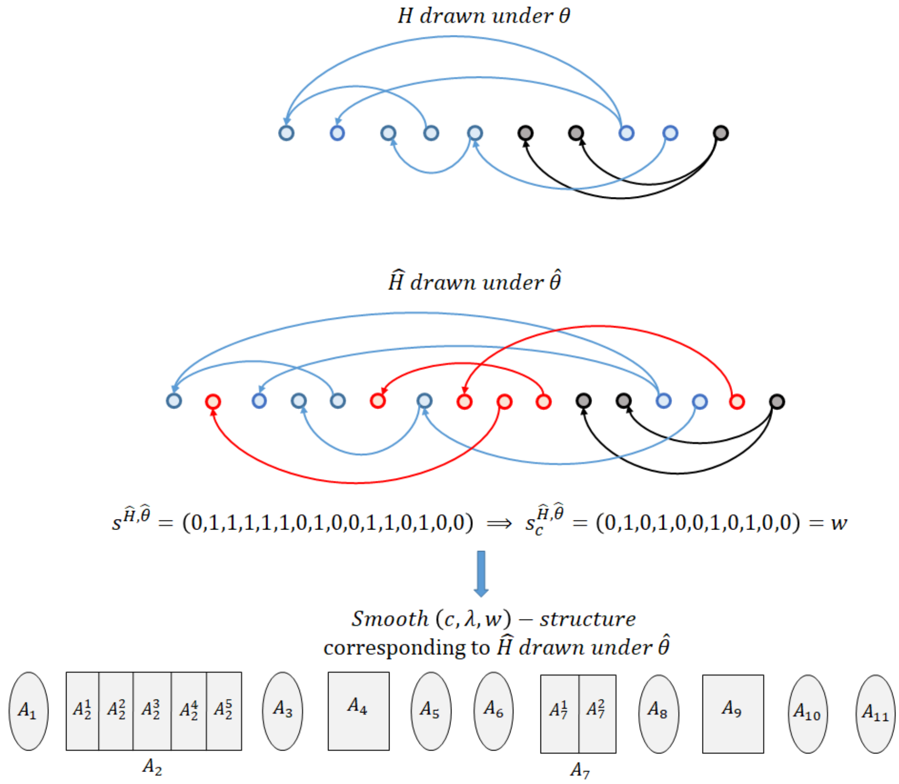

Figure 5: Flotilla-galaxy tournament H drawn under its flotilla-galaxy ordering θ. H consists of one γ-boat (in blue) and one right star (in black) under θ. The figure shows the digraph $\widehat{H}$ corresponding to H under θ. Here $\widehat{H}$ is drawn under $\widehat{\theta}$, the ordering of $\widehat{H}$ corresponding to θ. The smooth (c, λ, w)-structure corresponding to $\widehat{H}$ under $\widehat{\theta}$ is drawn below $\widehat{H}$. This structure consists of seven linear sets: A_1, A_3, A_5, A_6, A_8, A_{10}, A_{11}, and four transitive sets: A_2, A_4, A_7, and A_9. Note that only the backward arcs are drawn.

the purpose behind the definitions below). For $i \in \{0, \dots, l\}$ define $\widehat{H^i} = \widehat{H}|\bigcup_{j=1}^{i} \Sigma_i$ where $\widehat{H^l} = \widehat{H}$, and $\widehat{H^0}$ is the empty digraph.

- Let $\chi := (A_1, \dots, A_{|s_c|})$ be a smooth (c, λ, s_c)-structure corresponding to $\widehat{H}$ under $\widehat{\theta}$.

- For $l = 0$, it is trivial that χ contains $\widehat{H}$.

- Our goal is to extract from χ the sets assigned to the vertices of $\widehat{H^{l-1}}$ to form a corresponding structure for $\widehat{H^{l-1}}$ and apply the induction hypothesis. But in this case we may get two or more consecutive transitive sets in the extracted smooth structure (see Figure 6). This contradicts the definition of the corresponding structure, since corresponding structure as defined above, has no consecutive transitive sets (every sequence of consecutive 1's is replaced by a single 1 and then each single 1 represents a transitive set). This case will occur when there is a 0 corresponding to one of the vertices of Σ_l that appears between two subsequences of consecutive 1's corresponding to the vertices of $\widehat{H^{l-1}}$ (see Figure 6).

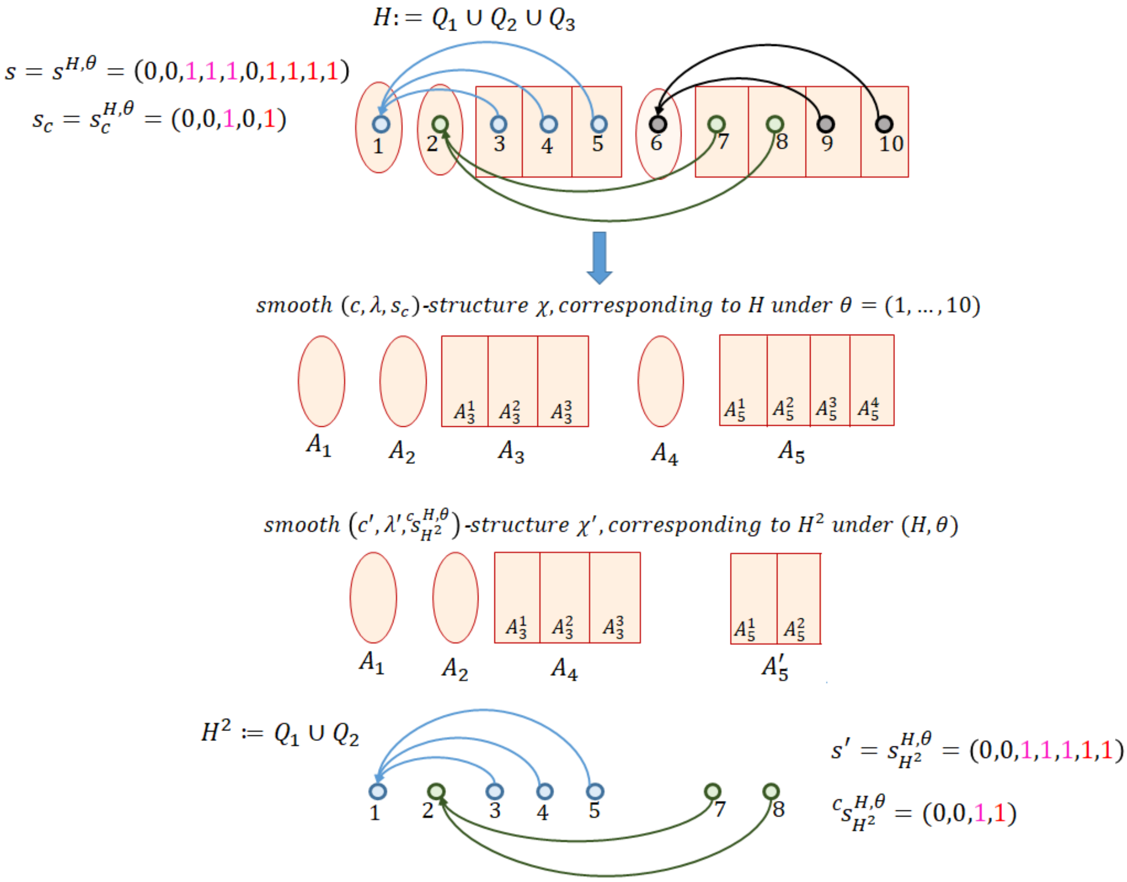

Figure 6: Flotilla-galaxy tournament H under an ordering $\theta = (1, \ldots, 10)$ of its vertices. H consists of only three stars: Q_1, Q_2, and Q_3 under θ (here $H = \widehat{H}$). The figure shows the smooth (c, λ, s_c)-structure corresponding to H under θ. The structure χ consists of three linear sets: A_1, A_2, A_4, and two transitive sets A_3 and A_5 that are divided respectively into three and four transitive chunks. One can see the form of the smooth structure (χ') that consists of the sets of χ corresponding to the vertices of H^2. The $\{0,1\}$-vector that encodes the nature of the sets of χ' is exactly ${}^c s_{H^2}^{H,\theta}$, then χ' is considered a smooth structure corresponding to H^2 under (H, θ). Note that all the non-drawn arcs are forward.

- Since the extracted smooth structure may have consecutive transitive sets, in order to apply the induction hypothesis, we need to modify the definition of the corresponding smooth structure when $1 \leq k \leq l-1$. Let s' be the restriction of s to the 0's and 1's corresponding to $V(\widehat{H^k})$. The difference now is in the way we replace subsequences of consecutive 1's by a single 1. Instead of replacing each maximal subsequence of consecutive 1's by a single 1 as in s_c, we replace every maximal subsequence of 1's in s' that corresponds to the same 1 in s_c by a single 1 (Figure 6 is a clear illustration of this point, where every maximal subsequence of 1's in s' that corresponds to the same 1 in s_c have the same color).

Remark 4.1 The tournament presented in Figure 6 contains only stars. Since choosing a tournament that contains also γ-boats will require dealing with a large order tournament in order to show the issue faced when extracting a smooth structure for $\widehat{H^k}$, with $k < l - 1$.

We are now ready to state formally some technical definitions that play a central role in the proof.

Let H be a regular flotilla-galaxy under an ordering θ of its vertices with $|H| = h$. Let $B_1^\gamma, \ldots, B_l^\gamma$ be the γ-boats of H under θ, and let $Q_1, \ldots, Q_l$ be the frontier stars of $H|X$ under $\overline{\theta}$. Let $\widehat{\theta} = (u_1, \ldots, u_{h+6l})$ be the ordering of $V(\widehat{H})$ corresponding to θ. For $i \in \{0, \ldots, l\}$, define $\widehat{H^i} = \widehat{H}|\bigcup_{j=1}^{i}(V(\widehat{B_j^\gamma}) \cup V(Q_j))$, where $\widehat{H^l} = \widehat{H}$, and $\widehat{H^0}$ is the empty digraph. Let $s^{\widehat{H},\widehat{\theta}}$ be the $\{0,1\}$-vector such that $s^{\widehat{H},\widehat{\theta}}(i) = 1$ if and only if u_i is a leaf of one of the stars or one of the $\widehat{B_j^\gamma}$'s of $\widehat{H}$ under $\widehat{\theta}$ for $j = 1, \ldots, l$. For $k \in \{1, \ldots, l\}$ let $\widehat{\theta}_k = (u_{k_1}, \ldots, u_{k_{t_k}})$ with $t_k = |\widehat{H^k}|$ be the restriction of $\widehat{\theta}$ to $V(\widehat{H}^k)$. Let $s_{\widehat{H}^k}^{\widehat{H},\widehat{\theta}}$ be the restriction of $s^{\widehat{H},\widehat{\theta}}$ to the 0's and 1's corresponding to $V(\widehat{H}^k)$ (notice that $s_{\widehat{H}^k}^{\widehat{H},\widehat{\theta}} = s^{\widehat{H}^k,\widehat{\theta}_k}$), and let ${}^c s_{\widehat{H}^k}^{\widehat{H},\widehat{\theta}}$ be the vector obtained from $s_{\widehat{H}^k}^{\widehat{H},\widehat{\theta}}$ by replacing every subsequence of consecutive 1's corresponding to the same entry of $s_c^{\widehat{H},\widehat{\theta}}$ by a single 1. We say that a smooth (c, λ, w)-structure of a tournament T *corresponds to* $\widehat{H}^k$ *under* $(\widehat{H}, \widehat{\theta})$ if $w = {}^c s_{\widehat{H}^k}^{\widehat{H},\widehat{\theta}}$. Notice that $s_{\widehat{H}^l}^{\widehat{H},\widehat{\theta}} = s^{\widehat{H},\widehat{\theta}}$ and ${}^c s_{\widehat{H}^l}^{\widehat{H},\widehat{\theta}} = s_c^{\widehat{H},\widehat{\theta}}$.

Let $\nu = {}^c s_{\widehat{H}^k}^{\widehat{H},\widehat{\theta}}$. Let $\delta^\nu : \{j : \nu_j = 1\} \to \mathbb{N}$ be a function that assigns to every nonzero entry of ν the number of consecutive 1's of $s_{\widehat{H}^k}^{\widehat{H},\widehat{\theta}}$ replaced by that entry of ν (δ^ν is used to know the size of the transitive partition of each transitive set). Fix $k \in \{0, \ldots, l\}$. Let $(S_1, \ldots, S_{|w|})$ be a smooth (c, λ, w)-structure corresponding to $\widehat{H}^k$ under $(\widehat{H}, \widehat{\theta})$. Let i be such that $w(i) = 1$. Assume that $S_i = \{s_i^1, \ldots, s_i^{|S_i|}\}$ and $(s_i^1, \ldots, s_i^{|S_i|})$ is a transitive ordering. Write $m(i) = \lfloor \frac{|S_i|}{\delta^w(i)} \rfloor$. Let $S_i^j = \{s_i^{(j-1)m(i)+1}, \ldots, s_i^{jm(i)}\}$ for $j \in \{1, \ldots, \delta^w(i)\}$. For every $v \in S_i^j$, let

$$\xi(v) := (|\ \{k < i : w(k) = 0\}\ | + \sum_{k<i:w(k)=1} \delta^w(k)) + j.$$

Thus $\xi(v)$ is the index of the vertex of $\widehat{H}^k$ (under $\widehat{\theta}_k$) associated with the set S_i^j obtained by unfolding the contraction introduced by ν. For every $v \in S_i$ such that $w(i) = 0$, let $\xi(v) := (|\ \{k < i : w(k) = 0\}\ | + \sum_{k<i:w(k)=1} \delta^w(k)) + 1$. We say that $\widehat{H}^k$

is *well-contained in* $(S_1, \dots, S_{|w|})$ that corresponds to $\widehat{H}^k$ under $(\widehat{H}, \widehat{\theta})$ if there is an injective homomorphism f of $\widehat{H}^k$ into $T \mid \bigcup_{i=1}^{|w|} S_i$ such that $\xi(f(u_{k_j})) = j$ for every $j \in \{1, \dots, t_k\}$, where $\widehat{\theta}_k = (u_{k_1}, \dots, u_{k_{t_k}})$.

4.2 An Overview of the Proof

The proof that every flotilla-galaxy satisfies the Erdős-Hajnal conjecture is very technical. To this end, before going into details of the formal proof, we would like to give an intuition as to how the proof works, explain the techniques that allowed the breakthrough in flotilla-galaxies, and explain the key steps of the proof.

Let H be a flotilla-galaxy under an ordering θ of its vertices. Assume that H has l γ-boats and l frontier stars under θ. The proof is by contradiction. Assume that EHC is not true for H. Then by taking $\epsilon > 0$ small enough, we can assume the existence of an H-free ϵ-critical tournament T. Now by Theorem 3.8, we can assume that there exists in T a smooth (c, λ, w)-structure χ corresponding to $\widehat{H}$ under $\widehat{\theta}$, where c and λ are positive constants. Our goal is to find a copy of H in T.

Obviously, Lemmas 3.5 and 3.6 are insufficient to construct a copy of a γ-boat in χ. So we cannot proceed here by constructing the γ-boats one by one following the ordering θ to get a copy of H. This simply means that following the ordering θ alone is insufficient and ineffective. At this point, we began to consider other orderings for H that provided different configurations of the backward arcs from the ones produced under the ordering θ. Hence, we created the set of orderings denoted by $\Theta_H(\theta)$, that contains orderings of H, such that every connected component of the graph of backward arcs of H under $\theta' \in \Theta_H(\theta)$ is one of a triangle, a 4-vertex path or a star.

Our goal now is to construct a copy of H in T in a way that this copy is viewed under some ordering $\theta' \in \Theta$. To do this, we constructed a copy of the corresponding digraph of H in χ. We created this copy by constructing mutant γ-boat and star couples one by one in a way that we can merge all the constructed couples together to get a copy of $\widehat{H}$. Note that this copy of $\widehat{H}$ will be viewed in χ under the ordering $\widehat{\theta}$. Let us now order the vertices of the $\widehat{H}$ copy according to their appearance in χ. Denote this ordering by α (α is exactly the ordering $\widehat{\theta}$). Denote by $\Gamma_1,\dots,\Gamma_l$, the copies of mutant γ-boats constructed in χ, and by $Q_1,\dots,Q_l$ the constructed copies of the frontier stars. Observe that for all $i \in \{1, \dots, l\}$, we do not know how the arcs in $A(T|\bigcup_i^l V(\Gamma_i))\backslash A(\bigcup_i^l \Gamma_i)$ are oriented in T. To this purpose, we completed the proof based on how the arcs in $A(T|\bigcup_{i=1}^l V(\Gamma_i))\backslash A(\bigcup_i^l \Gamma_i)$ are possibly oriented. Fix $i \in \{1, \dots, l\}$. If all the arcs in $A(T|V(\Gamma_i)\backslash A(\Gamma_i)$ are forward under α, then we can extract from $T|V(\Gamma_i)$ a γ-boat. Otherwise, there exists a backward arc under α among $A(T|V(\Gamma_i)\backslash A(\Gamma_i)$, and so we can extract a triangle from $T|V(\Gamma_i)$ that is formed by backward arcs under α. It is the time to extract a copy of H from the tournament in T induced by the vertex set of the copy of $\widehat{H}$. So, for all i, we can extract either a γ-boat or a triangle. If the former holds, we extract the γ-boat with the frontier star Q_i. If the latter holds, we extract the triangle, Q_i, and the two stars that form together with the triangle the cyclic ordering of the γ-boat. Finally, we delete from α all the vertices of the nonextracted substructures. The outcome is an ordering $\theta^{'} \in \Theta$. We are done.

4.3 Proof

Theorem 4.2 *Let H be a regular flotilla-galaxy under an ordering θ of its vertices with $|H| = h$. Let $B_1^{\gamma}, \dots, B_l^{\gamma}$ be the γ-boats of H under θ and let $Q_1, \dots, Q_l$ be the frontier stars of $H|X$ under $\overline{\theta}$. Let $0 < \lambda < \frac{1}{(4h)^{h+4}}$, $c > 0$ be constants, and w be a $\{0,1\}$-vector. Fix $k \in \{0, \dots, l\}$ and let $\widehat{\lambda} = (2h)^{l-k}\lambda$ and $\widehat{c} = \frac{c}{(2h)^{l-k}}$. There exist $\epsilon_k > 0$ such that for all $0 < \epsilon < \epsilon_k$, for every ϵ-critical tournament T with $|T| = n$ containing $\chi = (S_1, \dots, S_{|w|})$ as a smooth $(\widehat{c}, \widehat{\lambda}, w)$-structure corresponding to $\widehat{H^k}$ under $(\widehat{H}, \widehat{\theta})$, we have $\widehat{H^k}$ is well-contained in χ.*

Proof. The proof is by induction on k. For $k = 0$ the statement is obvious since $\widehat{H^0}$ is the empty digraph. Suppose that $\chi = (S_1, \dots, S_{|w|})$ is a smooth $(\widehat{c}, \widehat{\lambda}, w)$-structure in T corresponding to $\widehat{H^k}$ under $(\hat{H}, \hat{\theta})$, with $\hat{\theta} = (h_1, \dots, h_{h+6l})$. Let $\widehat{\theta}_k = (h_{k_1}, \dots, h_{k_s})$ be the restriction of $\widehat{\theta}$ to $V(\widehat{H^k})$, where $s = |\widehat{H^k}|$.

Let $\widehat{B_k^{\gamma}} = \{h_{k_{q_1}}, \dots, h_{k_{q_{13}}}\}$. Let $h_{k_{p_0}}$ be the center of Q_k and $h_{k_{p_1}}, \dots, h_{k_{p_q}}$ be its leaves for some integer $q > 0$. Our goal is to find a copy of $\widehat{H}|(V(\widehat{B_k^{\gamma}}) \cup V(Q_k))$ in χ.

- Let $D_i = \{v \in \bigcup_{j=1}^{|w|} S_j;\ \xi(v) = q_i\}$ for $i = 1, \dots, 13$. To construct a copy of $\widehat{H}|V(\widehat{B_k^{\gamma}})$ in χ, we are going to to find vertices $x_i \in D_i$ for $i = 1, \dots, 13$, such that $T|\{x_1, \dots, x_{13}\}$ contains a copy of $\widehat{H}|V(\widehat{B_k^{\gamma}})$.
  Assume that B_k^{γ} is a left γ_1-boat (otherwise, the argument is similar, and we omit it). Then there exist $1 \leq y \leq |w|$ and $1 \leq b \leq |w|$, such that $y + 5 < b$ and $D_1 = S_y$, $D_2 = S_{y+1}^1$, $D_3 = S_{y+1}^2$, $D_4 = S_{y+1}^3$, $D_5 = S_{y+1}^4$, $D_6 = S_{y+1}^5$, $D_7 = S_{y+2}$, $D_8 = S_{y+3}$, $D_9 = S_{y+4}$, $D_{10} = S_{y+5}$, $D_{11} = S_b$, $D_{12} = S_{b+1}$, $D_{13} = S_{b+2}$ with $w(y) = w(y+2) = w(y+4) = w(y+5) = w(b) = w(b+2) = 0$ and $w(y+1) = w(y+3) = w(b+1) = 1$. By Lemma 3.6, since T is ϵ-critical and $\epsilon < \min\{\log_{\frac{\widehat{c}}{4}}(\frac{1}{2}), \log_{\frac{\widehat{c}}{2}}(1 - \frac{\widehat{c}}{6})\}$, there exist vertices $x_1 \in D_1$, $x_3 \in D_3$, $x_5 \in D_5$, $x_{11} \in D_{11}$, such that $\{x_1, x_3\} \leftarrow x_{11}$ and $x_1 \leftarrow x_5$. Let $D_4^* = \{x_4 \in D_4; x_1 \rightarrow x_4 \rightarrow x_{11}\}$, $D_7^* = \{x_7 \in D_7; \{x_1, x_3, x_5\} \rightarrow x_7 \rightarrow x_{11}\}$, and $D_{12}^* = \{x_{12} \in D_{12}; \{x_1, x_3, x_5, x_{11}\} \rightarrow x_{12}\}$. Then by Lemma 3.9, $|D_7^*| \geq (1-4\widehat{\lambda})\widehat{c}n \geq \frac{\widehat{c}}{2}n$ since $\widehat{\lambda} \leq \frac{1}{8}$, $|D_{12}^*| \geq (1-4\widehat{\lambda})\widehat{c}tr(T) \geq \frac{\widehat{c}}{2}tr(T)$ since $\widehat{\lambda} \leq \frac{1}{8}$, and $|D_4^*| \geq \frac{1-12\widehat{\lambda}}{6}\widehat{c}tr(T) \geq \frac{\widehat{c}}{12}tr(T)$ since $\widehat{\lambda} \leq \frac{1}{24}$. Since we can assume that $\epsilon < \log_{\frac{\widehat{c}}{8}}(1 - \frac{\widehat{c}}{12})$, then Lemma 3.5 implies that there exist vertices $x_4 \in D_4^*$, $x_7 \in D_7^*$, and $x_{12} \in D_{12}^*$ such that $x_4 \leftarrow x_7 \leftarrow x_{12}$. Let $D_2^* = \{x_2 \in D_2; x_1 \rightarrow x_2 \rightarrow \{x_7, x_{11}, x_{12}\}\}$ and $D_9^* = \{x_9 \in D_9; \{x_1, x_3, x_4, x_5, x_7\} \rightarrow x_9 \rightarrow \{x_{11}, x_{12}\}\}$. Then by Lemma 3.9, $|D_2^*| \geq \frac{1-24\widehat{\lambda}}{6}\widehat{c}tr(T) \geq \frac{\widehat{c}}{12}tr(T)$ since $\widehat{\lambda} \leq \frac{1}{48}$ and $|D_9^*| \geq (1 - 7\widehat{\lambda})\widehat{c}n \geq \frac{\widehat{c}}{2}n$ since $\widehat{\lambda} \leq \frac{1}{14}$. Since $\epsilon < \log_{\frac{\widehat{c}}{2}}(1 - \frac{\widehat{c}}{12})$, then Lemma 3.2 implies that there exist

vertices $x_2 \in D_2^*$ and $x_9 \in D_9^*$ such that $x_2 \leftarrow x_9$. Let $D_6^* = \{x_6 \in D_6; x_1 \rightarrow x_6 \rightarrow \{x_7, x_9, x_{11}, x_{12}\}\}$ and $D_{10}^* = \{x_{10} \in D_{10}; \{x_1, \dots, x_5, x_7, x_9\} \rightarrow x_{10} \rightarrow \{x_{11}, x_{12}\}\}$. Then by Lemma 3.9, $|D_6^*| \geq \frac{1-30\widehat{\lambda}}{6}\widehat{c}tr(T) \geq \frac{\widehat{c}}{12}tr(T)$ since $\widehat{\lambda} \leq \frac{1}{60}$ and $|D_{10}^*| \geq (1-9\widehat{\lambda})\widehat{c}n \geq \frac{\widehat{c}}{2}n$ since $\widehat{\lambda} \leq \frac{1}{18}$. Since $\epsilon < \log_{\frac{\widehat{c}}{2}}(1-\frac{\widehat{c}}{12})$, then Lemma 3.2 implies that there exist vertices $x_6 \in D_6^*$ and $x_{10} \in D_{10}^*$ such that $x_6 \leftarrow x_{10}$. Let $D_8^* = \{x_8 \in D_8; \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\} \rightarrow x_8 \rightarrow \{x_9, x_{10}, x_{11}, x_{12}\}\}$ and $D_{13}^* = \{x_{13} \in D_{13}; \{x_1, \dots, x_7, x_9, \dots, x_{12}\} \rightarrow x_{13}\}$. Then by Lemma 3.9, $|D_8^*| \geq (1-11\widehat{\lambda})\widehat{c}tr(T) \geq \frac{\widehat{c}}{2}tr(T)$ since $\widehat{\lambda} \leq \frac{1}{22}$ and $|D_{13}^*| \geq (1-11\widehat{\lambda})\widehat{c}n \geq \frac{\widehat{c}}{2}n$ since $\widehat{\lambda} \leq \frac{1}{22}$. Since $\epsilon < \log_{\frac{\widehat{c}}{2}}(1-\frac{\widehat{c}}{2})$, then Lemma 3.2 implies that there exist vertices $x_8 \in D_8^*$ and $x_{13} \in D_{13}^*$ such that $x_8 \leftarrow x_{13}$. Then $T|\{x_1, \dots, x_{13}\}$ contains a copy of $\widehat{H^k}|V(\widehat{B_k^\gamma})$ where $(x_1, \dots, x_{13})$ is its forest ordering. Denote this copy by W.

- Now we will construct a copy of $\widehat{H}|V(Q_k)$ in χ. To this end, for all $0 \leq i \leq q$, let $R_i = \{v \in \bigcup_{j=1}^{|w|} S_j; \xi(v) = p_i\}$ and let $R_i^* = \bigcap_{x \in V(W)} R_{i,x}$. More precisely, we are going to find vertices $r_0, r_1, \dots, r_q$ that induces a copy of $\widehat{H}|V(Q_k)$, such that $r_i \in R_i^*$ for $i = 0, 1, \dots, q$. The definition of R_i^* for $i = 0, 1, \dots, q$ implies that there exist $m, f \in \{1, \dots, |w|\} \backslash \{y, \dots, y+5, b, b+1, b+2\}$ with $w(m) = 0$ and $w(f) = 1$, such that $R_0 = S_m$ and for all $1 \leq i \leq q$, $R_i \subseteq S_f$. Then by Lemma 3.9, $|R_0^*| \geq (1-13\widehat{\lambda})\ |R_0| \geq \frac{|R_0|}{2} \geq \frac{\widehat{c}}{2}n$ since $\widehat{\lambda} \leq \frac{1}{26}$, and $|R_i^*| \geq \frac{1-13h\widehat{\lambda}}{h}\ |S_f| \geq \frac{\widehat{c}}{2h}tr(T)$ since $\widehat{\lambda} \leq \frac{1}{26h}$. Since we can assume that $\epsilon < \log_{\frac{\widehat{c}}{4h}}(1-\frac{\widehat{c}}{2h})$, then Lemma 3.4 implies that there exist vertices $r_0, r_1, \dots, r_q$ such that $r_i \in R_i^*$ for $i = 0, 1, \dots, q$ and
  - $r_1, \dots, r_q$ are all adjacent from r_0 if $m > f$;
  - $r_1, \dots, r_q$ are all adjacent to r_0 if $m < f$.

So $T|\{x_1, \dots, x_{13}, r_0, r_1, \dots, r_q\}$ contains a copy of $\widehat{H^k}|(V(\widehat{B_k^\gamma}) \cup V(Q_k))$. Denote this copy by Y.

In what follows we are going to extract from χ a smooth structure corresponding to $\widehat{H^{k-1}}$ under $(\widehat{H}, \widehat{\theta})$, and then apply the induction hypothesis to complete the proof. For all $i \in \{1, \dots, |w|\} \backslash \{y, \dots, y+5, b, b+1, b+2, m, f\}$, let $S_i^* = \bigcap_{x \in V(Y)} S_{i,x}$. Then by Lemma 3.9, $|S_i^*| \geq (1-|Y|\ \widehat{\lambda})\ |S_i| \geq (1-(h+6)\widehat{\lambda})\ |S_i| \geq \frac{|S_i|}{2h}$ since $\widehat{\lambda} \leq \frac{2h-1}{2h(h+6)}$. Write $\mathcal{H} = \{1, \dots, s\} \backslash \{q_1, \dots, q_{13}, p_0, \dots, p_q\}$. If $\{v \in S_f : \xi(v) \in \mathcal{H}\} \neq \emptyset$, then define $J_f = \{\eta \in \mathcal{H} :$ there exists $v \in S_f$ with $\xi(v) = \eta\}$. Now for all $\eta \in J_f$, let $S_f^{*\eta} = \{v \in S_f : \xi(v) = \eta$ and $v \in \bigcap_{x \in V(Y) \backslash \{r_1, \dots, r_q\}} S_{f,x}\}$. Then by Lemma 3.9, for all $\eta \in J_f$, we have $|S_f^{*\eta}| \geq \frac{1-14h\widehat{\lambda}}{h}\ |S_f| \geq \frac{|S_f|}{2h}$ since $\widehat{\lambda} \leq \frac{1}{28h}$. Now for all $\eta \in J_f$, select arbitrary $\lceil \frac{|S_f|}{2h} \rceil$ vertices of $S_f^{*\eta}$ and denote the union of these $|J_f|$ sets by S_f^*. So we have defined t sets $S_1^*, \dots, S_t^*$, where $t = |w| - 10$ if S_f^* is defined and $t = |w| - 11$ if S_f^* is not defined. We have $|S_i^*| \geq \frac{\widehat{c}}{2h}tr(T)$ for every defined S_i^* with $w(i) = 1$, and $|S_i^*| \geq \frac{\widehat{c}}{2h}n$ for every defined S_i^* with $w(i) = 0$. Now Lemma 3.7 implies that $\chi^* = (S_1^*, \dots, S_t^*)$ form a smooth $(\frac{\widehat{c}}{2h}, 2h\widehat{\lambda}, w^*)$-structure of T corresponding to $\widehat{H^{k-1}}$

under $(\hat{H}, \hat{\theta})$, where $\frac{\widehat{c}}{2h} = \frac{c}{(2h)^{l-(k-1)}}, 2h\widehat{\lambda} = (2h)^{l-(k-1)}\lambda$, and w^* is an appropriate $\{0,1\}$-vector. Now take $\epsilon_k < \min\{\epsilon_{k-1}, \log_{\frac{\widehat{c}}{4}}(\frac{1}{2}), \log_{\frac{\widehat{c}}{8}}(1-\frac{\widehat{c}}{16}), \log_{\frac{\widehat{c}}{4h}}(1-\frac{\widehat{c}}{2h})\}$. So by the induction hypothesis $\widehat{H^{k-1}}$ is well-contained in χ^*. Now by merging the well-contained copy of $\widehat{H^{k-1}}$ and Y we get a copy of $\widehat{H^k}$. $\square$

In the following corollary we introduce a rule that uses the corresponding digraph of a flotilla-galaxy H to find H as an induced copy in T.

Corollary 4.3 *Let H be a regular flotilla-galaxy under an ordering θ of its vertices. Let $B_1^\gamma, \dots, B_l^\gamma$ be the γ-boats of H under θ, and let $Q_1, \dots, Q_l$ be the frontier stars of $H|(V(H)\setminus \bigcup_{i=1}^{l} V(B_i^\gamma))$ under $\overline{\theta}$. Let $\lambda > 0$ (λ is small enough), $c > 0$ be constants, and let w be a $\{0,1\}$-vector. Suppose that $\chi = (S_1, \dots, S_{|w|})$ is a smooth (c, λ, w)-structure of an ϵ-critical tournament T (ϵ is small enough) corresponding to $\widehat{H}$ under $(\widehat{H}, \widehat{\theta})$. Then T contains H.*

Proof. Now $\widehat{H} = \widehat{H^l}$ is well-contained in χ by the previous theorem when taking $k = l$. For all $1 \leq i \leq l$, let $\widetilde{B_i^\gamma} = \{x_i, d_i, b_i, u_i, n_i, r_i, p_i, q_i, z_i, s_i, f_i, a_i, t_i\}$ be the copy of $\widehat{B_i^\gamma}$ in T, and let $\tilde{Q}_i$ be the copy of Q_i in T. Let θ' be the ordering of $A = \bigcup_{i=1}^{l}(V(\widetilde{B_i^\gamma}) \cup V(\tilde{Q}_i))$ according to their appearance in $(S_1, \dots, S_{|w|})$ (that is, if $a, b \in A$ and $a \in S_i$, $b \in S_j$ with $i < j$, then a precedes b in θ', and if $a \in S_j^m, b \in S_j^r$ with $m < r$ then a precedes b in θ').

Let $1 \leq i \leq l$ such that B_i^γ is a left γ_1-boat. If $u_i \leftarrow a_i$, then we remove $x_i, d_i, b_i, n_i, z_i, f_i$ from θ'. Otherwise, if $x_i \leftarrow b_i$, then we remove $u_i, n_i, r_i, p_i, s_i, a_i$ from θ'. Otherwise, if $n_i \leftarrow f_i$, then we remove $b_i, u_i, r_i, p_i, s_i, a_i$ from θ'. Otherwise, $u_i \rightarrow a_i$, $x_i \rightarrow b_i$ and $n_i \rightarrow f_i$; in this case we remove $d_i, r_i, q_i, z_i, s_i, t_i$ from θ'. Note that in the first three cases we obtain the cyclic ordering of the left γ_1-boat and in the last case we obtain the forest ordering of the left γ_1-boat.

Let $1 \leq i \leq l$ such that B_i^γ is a left γ_2-boat. If $u_i \leftarrow a_i$, then we remove $x_i, d_i, b_i, n_i, z_i, t_i$ from θ'. Otherwise, if $x_i \leftarrow b_i$, then we remove $u_i, n_i, r_i, p_i, s_i, a_i$ from θ'. Otherwise, if $n_i \leftarrow t_i$, then we remove $b_i, u_i, r_i, p_i, s_i, a_i$ from θ'. Otherwise, $u_i \rightarrow a_i$, $x_i \rightarrow b_i$ and $n_i \rightarrow t_i$; in this case we remove $d_i, r_i, q_i, z_i, s_i, f_i$ from θ'. Note that in the first three cases we obtain the cyclic ordering of the left γ_2-boat and in the last case we obtain the forest ordering of the left γ_2-boat.

Let $1 \leq i \leq l$ such that B_i^γ is a right γ_1-boat. If $d_i \leftarrow s_i$, then we remove $b_i, n_i, z_i, f_i, a_i, t_i$ from θ'. Otherwise, if $b_i \leftarrow z_i$, then we remove $d_i, u_i, p_i, q_i, s_i, f_i$ from θ'. Otherwise, if $f_i \leftarrow t_i$, then we remove $d_i, u_i, p_i, q_i, z_i, s_i$ from θ'. Otherwise, $d_i \rightarrow s_i$, $b_i \rightarrow z_i$ and $f_i \rightarrow t_i$; in this case we remove $x_i, u_i, n_i, r_i, q_i, a_i$ from θ'. Note that in the first three cases we obtain the cyclic ordering of the right γ_1-boat and in the last case we obtain the forest ordering of the right γ_1-boat.

Let $1 \leq i \leq l$ such that B_i^γ is a right γ_2-boat. If $d_i \leftarrow s_i$, then we remove $x_i, n_i, z_i, f_i, a_i, t_i$ from θ'. Otherwise, if $x_i \leftarrow z_i$, then we remove $d_i, u_i, p_i, q_i, s_i, f_i$ from θ'. Otherwise, if $f_i \leftarrow t_i$, then we remove $d_i, u_i, p_i, q_i, z_i, s_i$ from θ'. Otherwise, $d_i \rightarrow s_i$, $x_i \rightarrow z_i$ and $f_i \rightarrow t_i$; in this case we remove $b_i, u_i, n_i, r_i, q_i, a_i$ from θ'. Note that in the first three cases we obtain the cyclic ordering of the right γ_2-boat and in the last case we obtain the forest ordering of the right γ_2-boat. We apply this rule for all $1 \leq i \leq l$. We obtain one of the orderings in $\Theta_\theta(H)$. So T contains H. □

We are ready to prove Theorem 1.4:

Proof. Let H be a flotilla-galaxy under θ. We may assume that H is a regular flotilla-galaxy since every flotilla-galaxy is a subtournament of a regular flotilla-galaxy. Let $B_1^\gamma, \dots, B_l^\gamma$ be the γ-boats of H under θ, and let $Q_1, \dots, Q_l$ be the frontier stars of $H|(V(H) \setminus \bigcup_{i=1}^{l} V(B_i^\gamma))$ under $\overline{\theta}$. Let $\epsilon > 0$ be small enough and let $\lambda > 0$ be small enough. Assume that H does not satisfy EHC, then there exists an H-free ϵ-critical tournament T. By Theorem 3.8, T contains a smooth (c, λ, w)-structure $(S_1, \dots, S_{|w|})$ corresponding to $\widehat{H}$ under $(\widehat{H}, \widehat{\theta})$ for some $c > 0$ and appropriate $\{0, 1\}$-vector w. Then by the previous corollary, T contains H, a contradiction. □

Theorem 4.4 *If H is a subtournament of a flotilla-galaxy, then H has the Erdős-Hajnal property.*

Proof. The result follows from Theorem 1.4 and the fact that the Erdős-Hajnal property is a hereditary property. □

References

[1] N. Alon, J. Pach and J. Solymosi, Ramsey-type theorems with forbidden subgraphs, *Combinatorica* 21 (2001), 155–170.

[2] E. Berger, K. Choromanski and M. Chudnovsky, Forcing large transitive subtournaments, *J. Combin. Theory Ser. B.* 112 (2015), 1–17.

[3] P. Erdős and A. Hajnal, Ramsey-type theorems, *Discrete Appl. Math.* 25 (1989), 37–52.

[4] E. Berger, K. Choromanski and M. Chudnovsky, On the Erdős-Hajnal conjecture for six-vertex tournaments, *European J. Combin.* 75 (2019), 113–122.

[5] K. Choromanski, EH-suprema of tournaments with no nontrivial homogeneous sets, *J. Combin. Theory Ser. B.* 114 (2015), 97–123.

[6] K. Choromanski, Excluding pairs of tournaments, *J. Graph Theory* 89 (2018), 266–287.

[7] K. Choromanski, Tournaments with forbidden substructures and the Erdős-Hajnal Conjecture, PhD Thesis, Columbia University, New York, 2013.

[8] S. Zayat and S. Ghazal, About the Erdős-Hajnal conjecture for seven-vertex tournaments, (submitted); `arXiv:2010.12331`.

[9] S. Zayat and S. Ghazal, Erdős-Hajnal Conjecture for New Infinite Families of Tournaments, *J. Graph Theory* 102(2) (2022), 388–417. https://doi.org/10.1002/jgt.22877.

[10] S. Zayat and S. Ghazal, Forbidding Couples of Tournaments and the Erdős-Hajnal Conjecture, *Graphs Combin.* 39 (2023), Art. 41. https://doi.org/10.1007/s00373-023-02643-x.

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