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# Kempe equivalence of almost bipartite graphs

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**Abstract**

Two vertex colorings of a graph are Kempe equivalent if they can be transformed into each other by a sequence of Kempe changes which interchange the colors used on a component of the subgraph induced by two color classes. It is PSPACE-complete to determine whether two given vertex $k$-colorings of a graph are Kempe equivalent for any fixed $k \geq 3$, and it is easy to see that every two vertex colorings of any bipartite graph are Kempe equivalent. In this paper, we consider Kempe equivalence of *almost* bipartite graphs which can be obtained from a bipartite graph by adding several edges, each connecting two vertices in the same partite set. We give a conjecture of Kempe equivalence of such graphs, and we prove several partial solutions and best possibility of the conjecture, but it was recently proved by Cranston and Feghali that this conjecture is false in general. This is a short version of the original paper [arXiv:2207.13244].

## 1 Introduction

A *Kempe change* was essentially introduced by Alfred Kempe in his false proof of the four color theorem. Each Kempe change interchanges the colors used on a component of the subgraph induced by two color classes. Even today, this remains one

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of the fundamental and most powerful tools in graph coloring theory. Two colorings are *Kempe equivalent* if they are transformed into each other by a sequence of Kempe changes. The Kempe equivalence of vertex colorings of graphs is deeply and widely studied, since it is not only an important research subject in combinatorial reconfiguration but also is related to many other subjects in graph theory; for example, random coloring [16], perfectly contractile graph [2], a recoloring version of Hadwiger's conjecture [4, 13] and so on. Moreover, Kempe change has many applications to various research fields; e.g., scheduling [10] and statistical physics [15]. On the other hand, it is PSPACE-complete to determine whether two given $k$-colorings of a graph $G$ are Kempe equivalent for any fixed $k \geq 3$ [3], where a *$k$-coloring* is a proper vertex coloring with $k$ colors. It is also PSPACE-complete even if $k = 3$ and $G$ is a planar graph with maximum degree 6. So it is hard in general to transform a given $k$-coloring into another $k$-coloring by Kempe changes.

Mohar [14] surveyed the study of Kempe changes, and showed several fundamental results on Kempe equivalence of vertex/edge colorings. He also proposed many interesting problems, one of which concerns Kempe equivalence of regular graphs and is completely solved in [1]. Here we introduce two results described in [14] below, which are used to prove our results (the second is originally proved in [13]).

**Proposition 1.1** ([14]). *Let $G$ be a bipartite graph. Every two $k$-colorings of $G$ with $k \geq 2$ are Kempe equivalent.*

A graph $G$ is *$d$-degenerate* if every subgraph of $G$ contains a vertex of degree at most $d$.

**Proposition 1.2** ([13, 14]). *Let $G$ be a $d$-degenerate graph with $d \geq 1$. For any integer $k > d$, every two $k$-colorings of $G$ are Kempe equivalent.*

By these results, it is an important problem to consider Kempe equivalence of two colorings of non-bipartite graphs without small maximum degree. Then we focus on *almost* bipartite graphs which can be obtained from a bipartite graph by adding several edges, each connecting two vertices in the same partite set. Such a graph is often investigated in various other contexts; for example, see [8, 12]. In this paper, we give an interesting conjecture on Kempe equivalence of almost bipartite graphs and show partial solutions of it. We also verify that the conjecture is best possible in some sense. Moreover, we mention Kempe equivalence on color-critical graphs.

In the rest of this section, we precisely define the terminology used in this paper, and then introduce our main conjecture and results.

### 1.1 Definitions

A graph $G$ is *$k$-colorable* if there exists a map $c : V(G) \to \{1, 2, \ldots, k\}$ with $c(u) \neq c(v)$ for any $uv \in E(G)$, where $V(G)$ and $E(G)$ denote the vertex and edge set of $G$, respectively, and such a map is called a *$k$-coloring*. The chromatic number of $G$, denoted by $\chi(G)$, is the minimum number $k$ such that $G$ is $k$-colorable. A graph $G$

is *k-chromatic* if $\chi(G) = k$. A graph $G$ is *k-critical* if it is $k$-chromatic but every proper subgraph of $G$ is $(k-1)$-colorable. An *odd wheel* is the graph obtained from an odd cycle $C$ by adding a vertex $v$ and joining $v$ to every vertex of $C$. Note that every odd wheel is 4-chromatic and the smallest odd wheel is $K_4$, where $K_n$ denotes the complete graph with $n$ vertices.

For a graph $G$ and a subset $S \subseteq V(G)$, $G[S]$ denotes the subgraph induced by $S$. For a vertex colored graph $G$, $G(i,j)$ denotes the subgraph induced by all vertices colored with $i$ or $j$. Every connected component $D$ of $G(i,j)$ is called a *Kempe component* (or K-*component*), and such a component $D$ is also called an $(i,j)$-*component*. An edge $e \in E(G(i,j))$ is called an $(i,j)$-*edge*. By switching the colors $i$ and $j$ on $D$, a new $k$-coloring can be obtained. This operation is called a *Kempe change* (or K-*change*). In particular, we call a K-change on a component of $G(i,j)$ an $(i,j)$-*change*. If a K-change is applied on a component containing a vertex $v$, then such a K-change is also called a K-change *concerning* $v$. Two $k$-colorings $c_1$ and $c_2$ of a graph $G$ are *Kempe equivalent* (or K-*equivalent*), denoted by $c_1 \sim_k c_2$, if $c_1$ can be obtained from $c_2$ by a sequence of Kempe changes, possibly involving more than one pair of colors in successive Kempe changes. Let $\mathcal{C}_k = \mathcal{C}_k(G)$ be the set of all $k$-colorings of $G$. The equivalence classes $\mathcal{C}_k / \sim_k$ are called the $\mathrm{K}^k$-*classes*. The number of $\mathrm{K}^k$-classes of $G$ is denoted by $\mathrm{Kc}(G,k)$.

For an integer $\ell \geq 0$, a graph is called a *$B + E_\ell$ graph* (respectively, a *$B + M_\ell$ graph*) if it is obtained from a bipartite graph $B$ by adding some $\ell$ edges (respectively, a matching of size $\ell$), where a *matching* is a set of edges sharing no vertices with each other. For such graphs, the bipartite graph $B$ to which several edges are added is called a *base* bipartite graph, and $E_\ell$ denotes the set of $\ell$ edges added.

For fundamental terminology and notation undefined in this paper, we refer the reader to [5].

### 1.2 Results & Problems

We start with the following observation.

**Observation 1.3.** Let $G$ be a $k$-colorable $B + E_\ell$ graph. If neither $S$ nor $T$ has an $(i,j)$-edge, then the $k$-coloring of $G$ is Kempe equivalent to a $k$-coloring such that $i \notin C(S)$ and $j \notin C(T)$.

*Proof.* Suppose $i, j \in C(S)$. Since $G(i,j)$ has no edge in $G[S]$, two vertices $u, v \in S$ colored by $i$ and $j$ respectively, belong to distinct components in $G(i,j)$. Let $F_u$ be a component in $G(i,j)$ containing $u$. By applying a $(i,j)$-change to $F_u$, we reduce the number of vertices with color $i$ in $S$. Therefore, by repeating such K-changes, we obtain $i \notin C(S)$. Similarly, we can also obtain $j \notin C(T)$ (preserving $i \notin C(S)$). $\square$

For a $k$-colorable graph $G$, if $|E(G)| < \binom{k}{2}$, then $G(i,j)$ is disconnected for some distinct colors $i,j$. In addition, if $G$ is a $B + E_\ell$ graph, then there is no edge in $E_\ell \cap E(G(i,j))$ for some distinct colors $i,j$. By Observation 1.3, we can obtain

$i \notin C(S)$ and $j \notin C(T)$ for such a $B + E_\ell$ graph $G$. In such a case, we have one more observation as follows.

**Observation 1.4.** Let $G$ be a $k$-colorable $B + E_\ell$ graph with $\ell < \binom{k}{2}$, and $S$ and $T$ be partite sets of the base bipartite graph. Suppose that $G$ is already colored by $k$ colors so that $i \notin C(S)$ and $j \notin C(T)$. Then the $k$-coloring of $G$ can be transformed into a $k$-coloring by K-changes such that each component of $G[S]$ (respectively, $G[T]$) has a vertex of color $j$ (respectively, $i$).

*Proof.* Suppose that a component $H$ in $G[T]$ contains no vertex of color $i$. Let $r$ be a color used on a vertex of $H$. Since $i \notin C(S)$, an $(r, i)$-change on $G(r, i) \cap H$ can make $H$ have a vertex of color $i$. By repeating such operations, we have the desired $k$-coloring of $G$. □

By these observations, if the number of added edges is less than $\binom{k}{2}$ for a $k$-colorable $B + E_\ell$ graph, then we can have a $k$-coloring desired in Observation 1.4 by K-changes. Such a $k$-coloring may help us to show the K-equivalence of every two colorings of a given graph. Therefore, we conjecture the following, but this is recently disproved for $k \geq 8$ by Cranston and Feghali [7].

**Conjecture 1.5** (False for $k \geq 8$)**.** *Let $G$ be a $(k-1)$-colorable $B + E_\ell$ graph with $k \geq 4$ and $\ell < \binom{k}{2}$. Then* $\mathrm{Kc}(G, k) = 1$.

Throughout this paper, we show several partial solutions (for small $k$) and the sharpness of this conjecture. (We do not consider the case when $k = 3$ in the conjecture since $G$ is bipartite in that case.)

<u>3- or 4-colorable case</u>

For Kempe equivalence of colorings of 3- or 4-colorable $B + E_\ell$ or $B + M_\ell$ graphs, we have the following results. Note that $B + M_\ell$ is 4-colorable for any $\ell \geq 0$ and that Theorem 1.7 proves Conjecture 1.5 when $k = 4$.

**Theorem 1.6.** *Let $G$ be a $B + M_\ell$ graph with $k \geq 4$ and $\ell < \binom{k}{2}$. Then* $\mathrm{Kc}(G, k) = 1$.

**Theorem 1.7.** *Let $G$ be a 3-colorable $B + E_\ell$ graph with $\ell \leq 5$. Then* $\mathrm{Kc}(G, 4) = 1$.

Theorem 1.6 (respectively, Theorem 1.7) is best possible by the following result (respectively, by the $k = 4$ case of Proposition 1.9).

**Proposition 1.8.** *For any $k \geq 3$, there are infinitely many $B + M_\ell$ graphs $G$ with $\ell = \binom{k}{2}$ and* $\mathrm{Kc}(G, k) \geq 2$.

<u>General case</u>

If the answer of Conjecture 1.5 is yes, then the statement is best possible by the following proposition.

**Proposition 1.9.** *For any integer $k \geq 4$, the following hold.*

*(i) There are infinitely many $(k-1)$-colorable $B+E_\ell$ graphs $G$ with $\ell = \binom{k}{2}$ and* $\mathrm{Kc}(G,k) \geq 2$.

*(ii) There are infinitely many $k$-chromatic $B+E_\ell$ graphs $G$ with $\ell = \binom{k}{2} - 1$ and* $\mathrm{Kc}(G,k) \geq 2$.

We also give a partial positive solution of Conjecture 1.5, as follows.

**Theorem 1.10.** *Let $G$ be a $(k-1)$-colorable $B+E_\ell$ graph with $\ell < \binom{k}{2}$ and $k \geq 4$. If every component $H$ induced by $\ell$ edges added is a path, a cycle of length at least* 4 *or a complete bipartite graph, then* $\mathrm{Kc}(G,k) = 1$.

**Remark 1.11.** The proofs of Theorems 1.7 and 1.10 are just (long) case-by-case arguments, and so we omit the proofs in this paper. The full version of this paper is posted on arXiv; see [11].

A special case

A graph $G$ is *$k$-critical* (or *edge $k$-critical*) if $\chi(G) = k$ and every proper subgraph of $G$ is $(k-1)$-colorable. It is easy to see that for $k = 1, 2$ and 3, the only $k$-critical graphs are $K_1$, $K_2$, and odd cycles, respectively. So, it is worthwhile to consider $k$-critical graphs for $k \geq 4$.

Chen et al. [6] gave a construction producing a $B+M_\ell$ graph from an arbitrary given graph. The construction consists of two steps (see Figure 1):

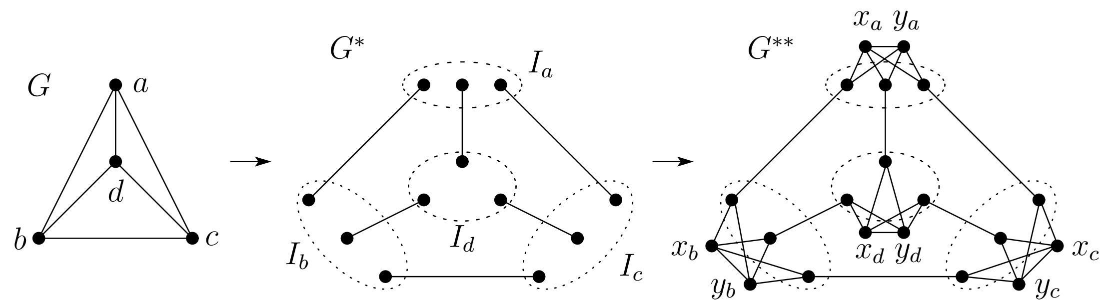


Figure 1: Construction of $G^{**}$

First, for each vertex $v$ of $G$, we construct an independent set $I_v$ of size $\deg_G(v)$ so that $I_v \cap I_u = \emptyset$ if $u \neq v$. For each edge $uv$ of $G$, place an edge between $I_u$ and $I_v$ so that these edges are vertex disjoint. Let $G^*$ be the resulting graph. Note that $G^*$ consists of $|E(G)|$ disjoint edges. Second, we add two new vertices $x_v$ and $y_v$ to each set $I_v$ joining them to each other and to all vertices of $I_v$. We denote by $G^{**}$ the resulting graph. Note that $G^{**}$ is a connected 4-colorable $B+M_\ell$ graph and that it is 4-critical if $G$ is 4-critical [6]. Moreover, since $G^{**}$ is 3-degenerate, the following holds by Proposition 1.2.

**Corollary 1.12.** *For every connected graph $G$,* $\mathrm{Kc}(G^{**}, 4) = 1$.

This corollary is a partial solution for our conjecture described later (Conjecture 1.13). Moreover, Corollary 1.12 implies that if a $B + M_\ell$ graph is isomorphic to $G^{**}$ for some given graph $G$, then we can arbitrarily increase $\ell$ in Theorem 1.6, though $\ell < 6$ in general by Theorem 1.6.

Recently, Feghali [9] proved that for every 4-critical planar graph $G$, $\mathrm{Kc}(G, 4) = 1$ (which is conjectured in [14]). Moreover, it is not known whether a 4-critical graph $G$ with $\mathrm{Kc}(G, 4) \geq 2$ exists though there are infinitely many 3-colorable graphs $H$ with $\mathrm{Kc}(H, 4) \geq 2$ (by Propositions 1.8 and 1.9), and from our various considerations, we are convinced that for every 4-critical $B + M_\ell$ graph $G$, $\mathrm{Kc}(G, 4) = 1$. Thus, we propose the following conjecture.

**Conjecture 1.13.** *For every* 4*-critical graph* $G$*,* $\mathrm{Kc}(G, 4) = 1$.

More generally, we also propose the following problem.

**Problem 1.14.** Is it true that for every $k$-critical graph $G$, $\mathrm{Kc}(G, k) = 1$?

### 1.3 Organization of the paper

In Section 2, we introduce notation and several lemmas. In Section 3, we prove Propositions 1.8 and 1.9. In Section 4, we prove Theorem 1.6. In the final section, Section 5, we give a conclusion.

## 2 Notation and Lemmas

### 2.1 Notation

For a graph $G$ and a vertex $v \in V(G)$, $N_G(v)$ denotes the set of neighbors of $v$. For a $B + E_\ell$ graph $G$, let $S$ and $T$ denote the partite sets of a base bipartite graph $B$. For a vertex subset $R$, $C(R)$ denotes the set of colors used on vertices in $R$. For the sake of simplicity, let $E$ be the set of $\ell$ added edges and let $E_S$ (respectively, $E_T$) be the subset of $E$ in which each edge joins two vertices in $S$ (respectively, $T$). If $E$ is a matching, i.e., $G$ is a $B + M_\ell$ graph, then we denote $E, E_S$ and $E_T$ by $M, M_S$ and $M_T$, respectively.

In several figures referenced in the proofs, we omit isolated vertices in $G[S]$ and $G[T]$. Moreover, bold lines in those figures denote edges added to a base bipartite graph.

### 2.2 Lemmas

We first prepare several lemmas to show our main results. Throughout this subsection, $G$ denotes a $k$-colorable $B + E_\ell$ graph and is already colored by $k$ colors.

**Lemma 2.1.** *The following hold:*

*(i) If $i \notin C(S)$ and $E_T$ has no $(i,j)$-edge, then the $k$-coloring of $G$ is Kempe equivalent to a $k$-coloring such that $j \notin C(T)$.*

*(ii) If a vertex $v$ in $T$ with color $j$ is not adjacent to any vertex with $i$, then the $k$-coloring $f$ of $G$ is Kempe equivalent to a $k$-coloring $g$ such that $j = f(v) \neq g(v) = i$ and $f(u) = g(u)$ for any other vertex $u \neq v$.*

*Proof.* (i) This immediately follows from Observation 1.3.
(ii) We clearly have the desired coloring by a $(i,j)$-change concerning $v$. □

**Lemma 2.2.** *Suppose $C(S) \cap C(T) = \emptyset$. If each component in $G[E]$ is either a bipartite graph or an odd cycle, then every two colorings of $S$ (or $T$) are K-equivalent.*

*Proof.* If $G[S]$ is bipartite, then the lemma holds by Proposition 1.1. If $G[S]$ is an odd cycle, then it receives at least three colors, so the lemma holds by Proposition 1.2. □

**Lemma 2.3.** *Let $c$ and $c'$ be two distinct $k$-colorings of $G$, and suppose that every K-component in $c$ is connected and that there is a K-component $H$ in $c'$ which is not isomorphic to any K-component in $c$. Then $c$ and $c'$ are not K-equivalent.*

*Proof.* It is clear that any sequence of K-changes cannot transform a K-component in $c$ into $H$, since every K-component in $c$ is connected. □

Finally, we show the Kempe equivalence of 3-colorings of a graph with a few edges.

**Lemma 2.4.** *Let $H$ be a 3-colorable graph with $|E(H)| \leq 5$. Then every two 3-colorings of $H$ are Kempe equivalent.*

*Proof.* If $|V(H)| \leq 3$, $H$ is 2-degenerate by simplicity of $H$. If $|V(H)| \geq 4$, $H$ is 2-degenerate since $2|E(G)|/|V(H)| \leq 5/2 < 3$. Therefore, the lemma holds by Proposition 1.2. □

## 3 Constructions

We first prove Proposition 1.8.

*Proof of Proposition* 1.8. For any $k \geq 3$, let $G$ be the graph shown in the left of Figure 2. Form the graph $G$ from $K_{k,k(k-1)}$ as follows, where $S$ is the larger part and $T$ is the smaller part. Add a matching of size $k$ on the vertices of $S$. Use each of colors $1, ..., k$ on a single vertex of $T$, and use each distinct pair in $\{1, ..., k\}$ on the endpoints of an edge added to $S$. Finally, remove each edge between parts with endpoints colored the same.

Note that $|S| = k(k-1)$ and $|T| = k$ and that $G$ has another $k$-coloring shown in the right of Figure 2 (using only three colors). Since $G(i,j)$ is connected for any $i, j$ in the left of Figure 2 and there is a K-component in the right coloring not isomorphic to any K-component in the left one, these two $k$-colorings are not K-equivalent by Lemma 2.3. Moreover, it is easy to see that the order of $G$ can be arbitrarily large by adding isolated vertices in $S$ or $T$ suitably. Therefore, this completes the proof of the proposition. $\square$

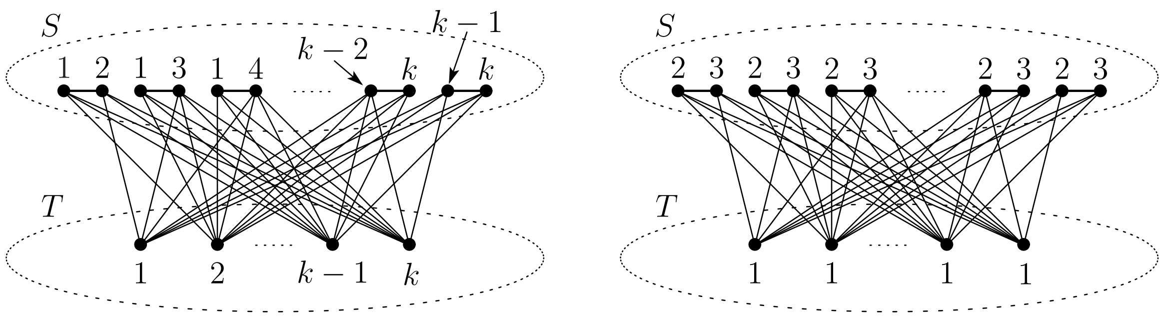


Figure 2: A $B + M_\ell$ graph $G$ with $k \geq 3$, $\ell = \binom{k}{2}$ and $\mathrm{Kc}(G, k) \geq 2$

Next we prove Proposition 1.9.

*Proof of Proposition* 1.9. We first construct the graph for $(i)$ shown in Figure 3. Prepare a complete bipartite graph $B$ with partite sets $S$ and $T$, where $|S| = (k - 2) + 2 \cdot (2k - 3) = 5k - 8$ and $|T| = k$. We add $\binom{k}{2}$ edges to $S$ so that those edges induce $K_{k-2} \cup (2k - 3)K_2$, and so $T$ consists of (exactly) $k$ isolated vertices. Then we give a $k$-coloring of the graph as shown in the left of Figure 3: We color vertices in $T$ by colors $1, 2, \ldots, k$, color $2k - 3$ independent edges in $S$ by $(1, k - 1), (2, k - 1), \ldots, (k-2, k-1), (1, k), (2, k), \ldots, (k-1, k)$, and color $K_{k-2}$ by colors $1, 2, \ldots, k-2$. (This is possible since $k \geq 4$.) According to the above coloring, we remove all edges between $S$ and $T$ joining two vertices with the same color. The resulting graph is a $(k - 1)$-colorable $B + E_\ell$ graph with $\ell = \binom{k}{2}$; see the left of Figure 3. It is easy to check that every K-component of the coloring is connected, but there is another $k$-coloring of the graph shown in the right of Figure 3, and hence, those $k$-colorings are not K-equivalent by Lemma 2.3. This completes the proof of $(i)$, since the order of the above graph can be arbitrarily large as in the proof of Proposition 1.8.

Next we construct the graph for $(ii)$ shown in Figure 4. Prepare a complete bipartite graph $B$ with partite sets $S$ and $T$, where $|S| = k$ and $|T| = 3$. We add an edge to $T$ joining two vertices of $T$, add $\binom{k-1}{2}$ edges to $S$ forming $K_{k-1}$, and add $k - 3$ edges to $S$ joining the (unique) vertex $v$ not in the above $K_{k-1}$ and exactly $k - 3$ vertices of the $K_{k-1}$. (This is possible since $k \geq 4$.) Similarly to the above case, we first give a $k$-coloring of the graph and then determine the base bipartite graph according to the given coloring (see the left of Figure 4): Since $T$ contains exactly one edge $xy$ and an isolated vertex $z$, we color $x$ (respectively, $y, z$) by color $k - 1$ (respectively, $k$), respectively. Then we color vertices of $K_{k-1}$ in $S$ by colors $1, 2, \ldots, k - 1$ and $v$ by color $k - 2$, and hence, we set vertices in $S$ adjacent to $v$ be colored by colors $1, 2, \ldots, k - 3$. Finally, we remove two edges between $S$ and $T$; one of them joins $x$ and a vertex in $S$ with color $k - 1$ and the other joins $y$ and a vertex with color $k - 2$ which is not $v$. The resulting graph is a $k$-chromatic

$B + E_\ell$ graph with $\ell = \binom{k}{2} - 1$; see the left of Figure 4. Every K-component of the coloring is connected, but there is another $k$-coloring of the graph shown in the right of Figure 4, and hence those $k$-colorings are not K-equivalent by Lemma 2.3. As in (i), the order of the constructed graph can be arbitrarily large, and hence this completes the proof of $(ii)$. □

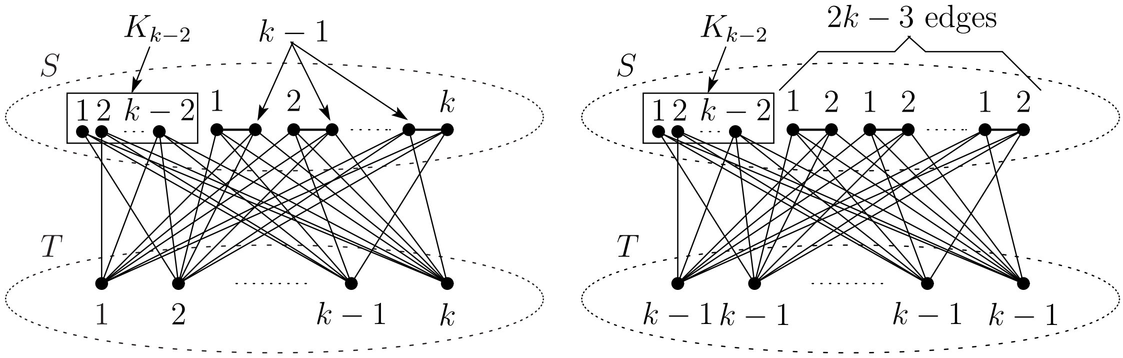


Figure 3: Two non-K-equivalent $k$-colorings of a $(k-1)$-colorable $B + E_\ell$ graph with $\ell = \binom{k}{2}$

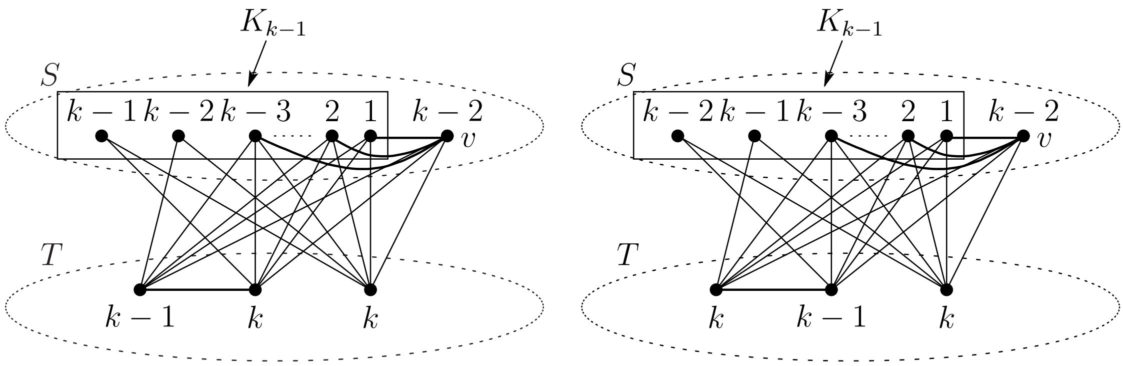


Figure 4: Two non-K-equivalent $k$-colorings of a $k$-chromatic $B+E_\ell$ graph with $\ell = \binom{k}{2} - 1$

## 4 Proof of Theorem 1.6

Let $G$ be a $B + M$ graph with $k \geq 4$ and $|M| < \binom{k}{2}$. Suppose that $G$ is colored by $k$ colors $1, 2, \ldots, k$. In this proof, we show that every $k$-coloring of $G$ can be transformed into a 4-coloring of $G$ using colors in $\{1, 2, 3, 4\}$ such that $C(S) \cap C(T) = \emptyset$. This 4-coloring is called a *standard 4-coloring* of $G$. Note that every two standard 4-colorings are K-equivalent by Lemma 2.2.

We may assume that $G(1, k)$ contains no edge in $M$, and so we have $C(S) \subseteq \{1, 2, \ldots, k-1\}$ and $C(T) \subseteq \{2, 3, \ldots, k\}$ by Observation 1.3. By repeatedly applying Lemma 2.1(ii), we can make every edge in $G[T]$ have color $k$. We divide the remaining proof into two cases.

**Case 1.** $M_T$ has no $(i, k)$-edge for some $i \in \{2, 3, \dots, k-1\}$.

Without loss of generality, we set $i = 2$. By Lemma 2.1(i), we have $1, 2 \notin C(T)$. Then we can obtain $C(S) \subseteq \{1, 2\}$ by $(1, i)$- and $(2, i)$-changes for $i \in \{3, \dots, k-1\}$. Therefore, since we can also obtain $C(T) \subseteq \{3, 4\}$ by Lemma 2.1(ii) again, the resulting coloring is a standard 4-coloring.

**Case 2.** $M_T$ has an $(i, k)$-edge for every $i \in \{2, 3, \dots, k-1\}$.

If $M_S$ contains a $(2, 3)$-edge $uv$ where $u$ is colored with 2, then the color of $u$ can be changed to color 1 by a $(1, 2)$-change concerning $u$ since $1 \notin C(T)$. Thus, we can make $M_S$ contain no $(2, 3)$-edge by repeated application of such $(1, 2)$-changes. Moreover, since every edge in $G[T]$ has color $k$, there is no $(2, 3)$-edge in $M_T$, either. Thus, by Observation 1.3, we can make $1, 2 \notin C(T)$ by $(2, 3)$-changes, and hence, this case can be deduced to Case 1. □

## 5 Conclusion

The Kempe equivalence of colorings of graphs is an important concept in many ways. Since every two colorings of any bipartite graph are Kempe equivalent, it is natural to ask whether or not two given colorings of an *almost* bipartite graph are Kempe equivalent. So we consider the Kempe equivalence of $B+E_\ell$ graphs, and in particular, we propose a conjecture of interest stating that every two $k$-colorings of any $(k-1)$-colorable $B+E_\ell$ graph with $k \geq 4$ and $\ell < \binom{k}{2}$ are Kempe equivalent (Conjecture 1.5). This conjecture is recently disproved for $k \geq 8$ by Cranston and Feghali [7], but the conjecture is possibly true for small $k$. Throughout this paper, we show several non-trivial partial solutions for small $k$ and the sharpness of the conjecture. In particular, the assumption of Theorem 1.10 needs to prove the theorem by induction on $k$. To prove Conjecture 1.5 for $k \in \{6, 7\}$, we probably need to find a new proof method without induction on $k$. Moreover, we also conjecture that every two 4-colorings of any 4-critical graph are Kempe equivalent (Conjecture 1.13), and our results are also partial solutions of this conjecture. However, neither of our proofs uses the 4-criticality of graphs. Therefore, we strongly believe that we can prove the Kempe equivalence of 4-colorings of 4-critical graphs as the planar case by combining the 4-criticality and several ideas that we have not found yet.

## References


[1] M. Bonamy, N. Bousquet and C. Feghali and M. Johnson. On a conjecture of mohar concerning kempe equivalence of regular graphs. *J. Combin. Theory Ser. B* **135** (2019), 179–199.

[2] M. E. Bertschi, Perfectly contractile graphs, *J. Combin. Theory Ser. B* **50(2)** (1990), 222–230.

[3] M. Bonamy, M. Heinrich, T. Ito, Y. Kobayashi, H. Mizuta, M. Mühlenthaler, A. Suzuki and K. Wasa, Diameter of colorings under kempe changes, *Theoret. Comput. Sci.* **838** (2020), 45–57.

[4] M. Bonamy, M. Heinrich, C. Legrand-Duchesne and J. Narboni, On a recolouring version of Hadwiger's conjecture, arXiv preprint, arXiv:2103.10684 (2021).

[5] J. A. Bondy and U. S. R. Murty, Graph theory with applications, *Macmillan London* (1976).

[6] G. Chen, P. Erdős, A. Gyárfás and R. H. Schelp, A class of edge critical 4-chromatic graphs, *Graphs Combin.* **13(2)** (1997), 139–146.

[7] D. W. Cranston and C. Feghali, Kempe classes and almost bipartite graphs, arXiv preprint, arXiv:2303.09365 (2023).

[8] P. Damaschke, Linear-time recognition of bipartite graphs plus two edges, *Discrete Math.* **262** (2003), 99–112.

[9] C. Feghali, Kempe equivalence of 4-critical planar graphs, arXiv preprint, arXiv:2101.04065 (2021).

[10] F. Gardi, Mutual exclusion scheduling with interval graphs or related classes, I, *Discrete Appl. Math.* **157(1)** (2009), 19–35.

[11] A. Higashitani and N. Matsumoto, Kempe equivalence of almost bipartite graphs, arXiv preprint, arXiv:2207.13244 (2022).

[12] A. V. Kostochka and B. M. Reiniger, The minimum number of edges in a 4-critical graph that is bipartite plus 3 edges, *European J. Combin.* **46** (2015), 89–94.

[13] M. Las Vergnas and H. Meyniel, Kempe classes and the hadwiger conjecture. *J. Combin. Theory Ser. B* **31(1)** (1981), 95–104.

[14] B. Mohar, Kempe equivalence of colorings, *Graph Theory in Paris*, Springer (2006), 287–297.

[15] B. Mohar and J. Salas, A new kempe invariant and the (non)-ergodicity of the wang-swendsen-koteckỳ algorithm, *J. Phys. A* **42(22)** (2009), 225204.

[16] E. Vigoda, Improved bounds for sampling colorings, *J. Math. Phys.* **41(3)** (2000), 1555–1569.

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